Today: Random sampling.
Infinitely many outcomes.
Properties of probability.

Next: ASV 2.1 - 2.2

Week 1:
Homework 0 (due Wednesday October 7)
Homework 1 (due Friday October 9)
Join Piazza
Combinatorics

* selecting $k$ objects from among $n$, with replacement:
  \[ \# \text{ways} = n^k \]

* selecting $k$ objects from among $n$, without replacement; order matters:
  \[ \# \text{ways} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} \quad (k \leq n) \]

* selecting $k$ objects from among $n$, without replacement; order doesn't matter:
  \[ \# \text{ways} = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} \]
Sampling with Replacement (order doesn’t matter)

E.g. An urn contains 10 balls: \( b_1, b_2, b_3, b_4, b_5 \)
\( 2 \) blue \( b_6, b_7, b_8, b_9, b_{10} \)
\( 3 \) yellow
\( 5 \) red

Problem: 3 balls are chosen without replacement.

\[ P(2 \text{ yellow}, 1 \text{ red}) \]

\[ \Omega = \{ \{b_i, b_j, b_k\} : b_i \neq b_j \text{ if } i \neq j \} \]

\[ \# \Omega = \binom{10}{3} \]

\[ A = \{2 \text{ are yellow, } 1 \text{ red}\} \]

\[ \#A = \binom{3}{1} \cdot \binom{7}{2} \]

\[ P(A) = \frac{\binom{3}{1} \cdot \binom{7}{2}}{\binom{10}{3}} = \frac{15}{120} = \frac{1}{8} = 12.5\% \]
What if \( \#\Omega = \infty \) ?

Then we need a different notion of uniform.

E.g. A random real number is chosen in \([0,1]\).

(a) What is the probability it is \( \geq 0.7 \)?

(b) What is the probability it is \( = \frac{1}{2} \)?

\[
\begin{align*}
\Omega &= \mathbb{R}, \mathcal{F} = \mathcal{B}(\mathbb{R}), P\\
P([a,b]) &= b - a.
\end{align*}
\]

(a) \( P([0.7,1]) = 1 - 0.7 = 0.3 \)

(b) \( P\left(\left[\frac{1}{2}, \frac{3}{2}\right]\right) = \frac{1}{2} - \frac{1}{2} = 0. \)

\[
\begin{align*}
P\left(\left[0,0.3\right] \cup \left[\frac{1}{2},0.96\right]\right) &= P\left(\left[0,0.3\right]\right) + P\left(\left[\frac{1}{2},0.96\right]\right) \\
&= 0.3 + 0.46 = 0.76.
\end{align*}
\]
An archery target is a disk 50 cm in diameter. A blue disk in the center is 25 cm in diameter. A red disk in the center is 5 cm in diameter.

Given that you hit the target (randomly), what are the chances of hitting the blue disk? The red disk?

\[ \Omega = \text{target} \]
\[ \mathcal{F} = \{ \text{subsets that have "area"} \} \]
\[ P(A) = \frac{\text{Area}(\overline{A})}{\text{Area}(\Omega)} \]

\[ P(\text{bullseye}) = 1\% \]
Decompositions

E.g. A fair coin is tossed 5 times. What is the probability that at least 3 tosses come up tails?

\[ A = \{ \text{at least 3 tails} \} = A_3 \cup A_4 \cup A_5 \]

\[ A_k = \{ \text{exactly k tails} \} \]

\[
P(A) = P(A_3) + P(A_4) + P(A_5)
\]

\[ P(A_3) = \binom{5}{3} \left( \frac{1}{2} \right)^3 \]

\[ P(A_4) = \binom{5}{4} \left( \frac{1}{2} \right)^4 \]

\[ \therefore P(A) = \frac{1}{2^5} \left( \binom{5}{3} + \binom{5}{4} + \binom{5}{5} \right) = \frac{1}{2^5} (1 + 5 + 10) = \frac{16}{32} = 50\% \]
Eg. A fair die is rolled 4 times. What is the probability of at least one double?

\[ A = \{ \text{some number comes up at least two times} \} \]
\[ A_k = \{k \text{ comes up at least two times} \} \]

\[ A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \]
\[ A_k^m = \{k \text{ comes up exactly } m \text{ times} \} \]
\[ A_i = A_{i1} \cup A_{i2} \cup A_{i3} \cup A_{i4} \cup A_{i5} \cup A_{i6} \]

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**Question:** Are all these events disjoint? **NO**

\[ P(A^c) = \frac{6 \cdot 5 \cdot 4 \cdot 3}{6^4} = \frac{5}{18} \]

\[ 1 = P(\Omega) = P(A) + P(A^c) \]
\[ \therefore P(A) = 1 - P(A^c) = 1 - \frac{5}{18} = \frac{13}{18} \]
Sometimes, you can't avoid lack of disjointness so easily. You have to take intersections into account.

Notation: \( A \cap B = \{ \text{all outcomes in both } A \text{ and } B \} \)

\[
\begin{align*}
\emptyset & = AB \\
AB & = AB^c \cup AB \cup A^c B \leftarrow \text{disjoint}
\end{align*}
\]

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]
**Principle of Inclusion/Exclusion**

The probability of a union can be computed by adding the probabilities, then subtracting off the intersection(s) overcounted. If you have more sets, you have to keep going and re-add back in pieces that you over-subtracted, etc.

\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)
\]
Eg. 20% of the population own cats.
25% of the population own dogs.
5% of the population own both,

What is the probability that a random person owns neither?

\[
P(C) = 0.2 \\
P(D) = 0.25 \\
P(CD) = 0.05
\]

\[
P(C^cD^c) = P((CD)^c) \\
= 1 - P(CD) \\
= 1 - (P(C) + P(D) - P(CD)) \\
= 1 - (0.2 + 0.25 - 0.05) \\
= 0.6,
\]
Monotonicity

If \( A \subseteq B \) then \( B = A \cup A^c B \) is a disjoint union

\[
\therefore P(B) = P(A) + P(A^c B) \\
\geq P(A)
\]

Eg 90\% of your friends like the xiao long bao at Din Tai Fung.
80\% of your friends like the xiao long bao at Shanghai Saloon.

What is the smallest possible proportion of your friends who like the xiao long bao at both restaurants?