Today: Introduction. Birth processes
> Q&A: October 5
Next: PK 6.2-6.3

Week 0/1:
- visit course web site
- homework 0 (due Wednesday October 7)
- homework 1 (due Friday October 9)
- join Piazza
Stochastic (random) processes

Def. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space. A stochastic process is a collection \( (X_t : t \in T) \) of random variables \( X_t : \Omega \to S \subseteq \mathbb{R} \) (all defined on the same probability space)

- often \( t \) represents time, but can be 1-D space
- \( T \) is called the index set, \( S \) is called the state space
- \( X : \Omega \times T \to S \) \( (X_t(\omega) \in S) \)
- for any fixed \( \omega \), we get a realization of all random variables \( (X_t(\omega) : t \in T) \) ← sample path trajectory
  \[ X_\cdot(\omega) : T \to S \] ← function in \( t \)
- stochastic process = random function
Stochastic processes. Classification

Questions:
- What is T?
- What is S?
- Relations between $X_{t_1}$ and $X_{t_2}$ for $t_1 \neq t_2$?
- Properties of the trajectory?

Discrete time
$T = \mathbb{N}, \mathbb{Z}$, finite set

Continuous time
$T = \mathbb{R}, [0, +\infty), [0, 1]$,

Random vector

Real-valued $S = \mathbb{R}$

Integer-valued $S = \mathbb{Z}$

Nonnegative $S \subseteq [0, +\infty)$

Continuous, right-continuous (cadlag) sample path
Examples of stochastic processes

- Gaussian processes: for any \( teT \), \( X_t \) has normal distrib.
- Stationary processes: distribution doesn't change in time
- Processes with stationary /independent increments (Lévy)
- Poisson process: increments are indep. and Poisson (.)
- Markov processes: "distribution in the future depends only on the current state, but does not depend on the past"
Examples of stochastic processes

- **martingales**: \( \mathbb{E}[X_{n+1} | X_n, X_{n-1}, \ldots, X_0] = X_n \) ("fair game")

- **Brownian motion (Wiener process)** is continuous-time s.p. Gaussian, martingale, has stationary and independent increments, Markov, \( \text{Var}[W_t] = t \), \( \text{Cov}(W_t, W_s) = \min\{s, t\} \), its sample path is everywhere continuous and nowhere differentiable.

- **diffusion processes** (stochastic differential equations)
Continuous time MC
**Continuous Time Markov Chains**

**Def (Discrete-time Markov chain)**

Let \((X_n)_{n \geq 0}\) be a discrete time stochastic process taking values in \(\mathbb{Z}_+ = \{0, 1, 2, \ldots\}\) (for convenience). \((X_n)_{n \geq 0}\) is called Markov chain if for any \(n \in \mathbb{N}\) and \(i, i_0, i_1, \ldots, i_{n-1}, i, j \in \mathbb{Z}_+\)

\[
P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i) = P(X_{n+1} = j \mid X_n = i)
\]

**Def (Continuous-time Markov chain)**

Let \((X_t)_{t \geq 0} = (X_t : 0 \leq t < \infty)\) be a continuous time process taking values in \(\mathbb{Z}_+\). \((X_t)_{t \geq 0}\) is called Markov chain if for any \(n \in \mathbb{N}, 0 \leq t_0 < t_1 < \cdots < t_{n-1} < s, t > 0, i_0, i_1, \ldots, i_{n-1}, i, j \in \mathbb{Z}_+\)

\[
P(X_{s+t} = j \mid X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}, X_s = i) = P(X_{s+t} = j \mid X_s = i)
\]

Markov property
Example: Poisson process as a continuous time MC

Is Poisson process a continuous time MC?

Poisson process:
- ✓ continuous time
- ✓ discrete state
- ✓ Markov property

Let $(X_t)_{t \geq 0}$ be a Poisson process, let $i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i \leq j$

$$P(X_{s+t} = j \mid X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{tn-1} = i_{n-1}, X_s = i)$$

$$= \frac{P(X_{t_0} = i_0, X_{t_1} - X_{t_0} = i_1 - i_0, \ldots, X_s - X_{tn-1} = i - (n-1), X_{s+t} - X_s = j - i \mid X_{t_0} = i_0, X_{t_1} - X_{t_0} = i_1 - i_0, \ldots, X_s - X_{tn-1} = i - (n-1))}{P(X_{t_0} = i_0, X_{t_1} - X_{t_0} = i_1 - i_0, \ldots, X_s - X_{tn-1} = i - (n-1))}$$

$$= P(X_{s+t} - X_s = j - i \mid X_s = i)$$

$$= P(X_{s+t} - X_s = j - i \mid X_s = i) = P(X_{s+t} = j \mid X_s = i) \quad \text{(by the Markov property)}$$
Transition probability function

One way of describing a continuous time MC is by using the transition probability functions.

Def. Let \((X_t)_{t \geq 0}\) be a MC. We call

\[ P(X_{s+t} = j | X_s = i), \quad i, j \in \{0, 1, \ldots\}, \quad s \geq 0, \quad t > 0 \]

the transition probability function for \((X_t)_{t \geq 0}\).

If \( P(X_{s+t} = j | X_s = i) \) does not depend on \( s \), we say that \((X_t)_{t \geq 0}\) has stationary transition probabilities and we define

\[ P_{ij}(t) := P(X_{s+t} = j | X_s = i) \quad (= P(X_t = j | X_0 = i)) \]

[compare with \( n \)-step transition probabilities]
Characterization of the Poisson process

Experiment: count events occurring along $[0, +\infty)$ for 1-D space $\mathbb{R}$.

Denote by $N((a, b])$ the number of events that occur on $(a, b]$.

Assumptions:

1. Number of events happening in disjoint intervals are independent.

2. For any $t > 0$ and $h > 0$, the distribution of $N((t, t+h])$ does not depend on $t$ (only on $h$, the length of the interval).

3. There exists $\lambda > 0$ s.t. $P(N((t,t+h]) \geq 1) = \lambda h + o(h)$ as $h \to 0$ (rare events).

4. Simultaneous events are not possible: $P(N((t,t+h]) \geq 2) = o(h), h \to 0$.

Then $X_t := N((0,t])$ is a Poisson process with rate $\lambda$. 
Transition probabilities of the Poisson process

Let \((X_t)_{t \geq 0}\) be the Poisson process.

Define the transition probability functions

\[ P_{ij}(h) := \mathbb{P}(X_{t+h} = j \mid X_t = i), \quad i, j \in \{0, 1, 2, \ldots\}, \quad t \geq 0, \quad h > 0 \]

What are the infinitesimal (small \(h\)) transition probability functions for \((X_t)_{t \geq 0}\)? As \(h \to 0\),

\[ P_{ii}(h) = \mathbb{P}(X_{t+h} = i \mid X_t = i) \]

\[ = \mathbb{P}(X_{t+h} - X_t = 0 \mid X_t = i) = \mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h) \]

\[ P_{i,i+1}(h) = \mathbb{P}(X_{t+h} = i+1 \mid X_t = i) = \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h) \]

\[ \sum_{j \in \{i,i+1\}} P_{ij}(h) = o(h) \]
Poisson process and transition probabilities

To sum up: \((X_t)_{t \geq 0}\) is a MC with (infinitesimal) transition probabilities satisfying

\[
P_{ii}(h) = 1 - \lambda h + o(h)
\]

\[
P_{i,i+1}(h) = \lambda h + o(h)
\]

\[
s_{j \neq \{i,i+1\}} P_{ij}(h) = o(h)
\]

What if we allow \(P_{ij}(h)\) depend on \(i\)?

\[
\text{birth and death processes}
\]
Pure birth processes

**Def** Let $(\lambda_k)_{k \geq 0}$ be a sequence of positive numbers. We define a pure birth process as a Markov process $(X_t)_{t \geq 0}$ whose stationary transition probabilities satisfy

1. $P_{k,k+1}(h) = \lambda_k h + o(h)$ as $h \to 0^+$
2. $P_{k,k}(h) = 1 - \lambda_k h + o(h)$
3. $P_{k,j}(h) = 0$ for $j < k$
4. $X_0 = 0$

Related model: Yule process: $\lambda_k = \beta k$ for some $\beta > 0$. Describes the growth of a population - birth rate is proportional to the size of the population
Birth processes and related differential equations

Now define \( P_n(t) = P(X_t = n) \). For small \( h > 0 \)

\[
P_n(t+h) = P(X_{t+h} = n) = \sum_{k=0}^{n} P(X_{t+h} = n \mid X_t = k) P(X_t = k)
\]

\[
= \sum_{k=0}^{n} P_{k,n}(h) \cdot P(X_t = k)
\]

\[
= P_{n,n}(h) \cdot P_n(t) + P_{n-1,n}(h) \cdot P_{n-1}(t) + \sum_{k=0}^{n-2} P_{k,n}(h) \cdot P(X_t = k)
\]

\[
= (1 - \lambda_n h) P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h)
\]

\[
= P_n(t) - \lambda_n h P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h)
\]

\[
P_n(t+h) - P_n(t) = -\lambda_n h P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h)
\]

\[
P_n'(t) = \lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h} = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)
\]
Birth processes and related differential equations

$P_n(t)$ satisfies the following system of differential eqns. with initial conditions:

\[
\begin{align*}
\frac{dP_0(t)}{dt} &= -\lambda_0 P_0(t) & P_0(0) &= 1 \\
\frac{dP_1(t)}{dt} &= -\lambda_1 P_1(t) + \lambda_0 P_0(t) & P_1(0) &= 0 \\
\frac{dP_2(t)}{dt} &= -\lambda_2 P_2(t) + \lambda_1 P_1(t) & P_2(0) &= 0 \\
&\vdots \\
\frac{dP_n(t)}{dt} &= -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) & P_n(0) &= 0 \\
&\vdots
\end{align*}
\]

\((\star)\)

Solving this system gives the p.m.f. of $X_t$ for any $t$

\[
P(X_t = k) = P_k(t)
\]
Description of the birth processes via sojourn times

\((X_t)_{t \geq 0}\)

\(W_i\) - i-th "birth time"   \(S_i\) - "time between (i-1)-th birth and i-th birth"

\[ W_i = \sum_{k=0}^{i-1} S_k \]

Alternative way of characterizing \((X_t)_{t \geq 0}\):
- describe the distribution of \((S_i)_{i \geq 0}\)
- describe the jumps \(X_{w_i-} - X_{w_i}\)
Theorem

Let \((\lambda_k)_{k \geq 0}\) be a sequence of positive numbers. Let 
\((X_t)_{t \geq 0}\) be a non-decreasing right-continuous process, \(X_0 = 0\),
 taking values in \(\{0, 1, 2, \ldots\}\), Let 
\((S_i)_{i \geq 0}\) be the sojourn times associated with 
\((X_t)_{t \geq 0}\), and define \(W_\varepsilon = \sum_{i=0}^{\varepsilon-1} S_i\).

Then conditions

(a) \(S_0, S_1, S_2, \ldots\) are independent exponential r.v.s of rates \(\lambda_0, \lambda_1, \lambda_2, \ldots\)

(b) \(X_{W_\varepsilon} = \varepsilon\) (jumps of magnitude 1)

are equivalent to

(c) \((X_t)_{t \geq 0}\) is a pure birth process with parameters \((\lambda_k)_{k \geq 0}\).
Explosion
$(X_t)_{t \geq 0}$

Thm. Let $(X_t)_{t \geq 0}$ be a pure birth process of rates $(\lambda_k)_{k \geq 0}$.

Then
- if $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} < \infty$, then $P( (X_t)_{t \geq 0} \text{ explodes}) = 1$
- if $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} = \infty$, then $P( (X_t)_{t \geq 0} \text{ does not explode}) = 1$

Hint. $E( \sum_{k=0}^{\infty} S_k ) = \sum_{k=0}^{\infty} \frac{1}{\lambda_k}$
Solving the system of differential equations (*)

\[
\begin{cases}
   P_0'(t) = -\lambda_0 P_0(t), & P_0(0) = 1 \\
   P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), & P_n(0) = 0 \quad \text{for} \quad n \geq 1
\end{cases}
\]

\( P_0(t) : \)

\[
\begin{align*}
   P_0'(t) &= -\lambda_0 P_0(t) \\
   \frac{P_0'(t)}{P_0(t)} &= -\lambda_0 \\
   g'(t) &= -\lambda_0 \\
   g(t) &= -\lambda_0 t + K = \log(P_0(t)) \\
   \Rightarrow P(t) &= e^k e^{-\lambda_0 t} = Ce^{-\lambda_0 t}, \quad C > 0 \\
   P_0(0) &= C = 1 \Rightarrow C = 1
\end{align*}
\]
Solving the system of differential equations (*)

\( P_n(t), \ n \geq 1 \)

Consider the function \( Q_n(t) = e^{\lambda_n t} P_n(t) \)

\[
(Q_n(t))' = (e^{\lambda_n t} P_n(t))' = \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} (P_n(t))'
\]

\[
= \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} (-\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t))
\]

\[
= \lambda_{n-1} e^{\lambda_n t} P_{n-1}(t)
\]

\( Q_n(t) = \int_0^t \lambda_{n-1} e^{\lambda_n s} P_{n-1}(s) \, ds \)

\( L_1 \ P_n(t) = e^{-\lambda_n t} \int_0^t \lambda_{n-1} e^{\lambda_n s} P_{n-1}(s) \, ds \leftarrow \text{apply recursively} \)

\( P_1(t) = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{\lambda_1 s} e^{-\lambda_0 s} \, ds = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{(\lambda_1-\lambda_0)s} \, ds \) (if \( \lambda_1 \neq \lambda_0 \))

\[
= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1-\lambda_0} (e^{(\lambda_1-\lambda_0)t} - 1) = \frac{\lambda_0}{\lambda_1-\lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0}{\lambda_1-\lambda_0} e^{\lambda_1 t}
\]