Today: Pure death processes.
Birth and death processes
> Q&A: October 9

Next: PK 6.5

Week 1:

- homework 1 (due Friday October 9)
- join Piazza
Pure death processes

Pure birth process

What if the chain moves in the opposite direction?

Pure death process:
- exponential sojourn times with rates $\mu_i$
- only negative jumps of magnitude 1 allowed
Pure death processes

Infinitesimal description:

Pure death process \((X_t)_{t \geq 0}\) of rates \((\mu_k)_{k=1}^{\infty}\) is a continuous time MC taking values in \(\{0, 1, 2, \ldots, N-1, N\}\) (state 0 is absorbing) with stationary infinitesimal transition probability functions

(a) \(P_{k,k-1}(h) = \mu_k h, \quad k = 1, \ldots, N\)

(b) \(P_{k,k}(h) = 1 - e^{-\mu_k h}, \quad k = 1, \ldots, N\)

(c) \(P_{k,j}(h) = 0 \text{ for } j > k\).

State 0 is absorbing \((\mu_0 = 0)\)
Pure death process

\[ S_k \sim \text{Exp}(\mu_k) \]

Sojourn time/jump description:

Pure death process of rates \((\mu_k)_{k=1}^N\) is a nonincreasing right-continuous process taking values in \(\{0, 1, \ldots, N\}\).

- with sojourn times \(S_1, S_2, S_3, \ldots, S_N\) being independent exponential r.v.s of rates \(\mu_1, \mu_2, \ldots, \mu_N\) and
- jumps \(X_{W_i+1} - X_{W_i} = -1\) of magnitude 1
Differential equations for pure birth processes

Define \( P_n(t) = P(X_t = n | X_0 = N) \) distribution of \( X_t \) starting in state \( N \)

(a), (b), (c) implies (check)

\[
\begin{align*}
P_n'(t) &= \mu_n \left( P_{n-1}(t) + P_{n-2}(t) + \cdots + P_0(t) \right) \\
0 &= P_N'(t)
\end{align*}
\]

for \( n = 0 \ldots N-1 \)

(note that \( \mu_0 = 0 \))

Initial conditions:

Solve recursively: \( P_N(t) \rightarrow P_{N-1}(t) \rightarrow \cdots \rightarrow P_0(t) \)

General solution (assume \( \mu_i \neq \mu_j \))

\[
P_n(t) = \mu_{n+1} \cdots \mu_N \left( A_{n,1} e^{\mu_1 t} + \cdots + A_{n,N} e^{\mu_N t} \right), \quad A_{kn} = \prod_{\ell=n}^{N} \frac{1}{\mu_{\ell+1} - \mu_k}
\]
**Linear death process**

Similar to Yule process:

death rate is proportional to the size of the population

Compute $P_n(t)$:

1. $\mu_{n+1} \ldots \mu_n = \alpha^n \frac{N^n}{n!}$
2. $A_{kn} = \prod_{e=n}^{N-k} \frac{1}{\mu_e - \mu_k} = \frac{1}{a^{N-n} (-1)^{n-k} (k-n)! (N-k)!}$

\[ P_n(t) = a^n \frac{N!}{n!} \cdot \frac{1}{a^{N-n}} \sum_{k=n}^{\infty} \frac{1}{(-1)^{k-n} (k-n)! (N-k)!} \cdot e^{-kt} \]

\[ = \frac{N!}{n!} \sum_{j=0}^{n-2} (-1)^j \frac{1}{j! (N-n-j)!} \cdot e^{-j+n+1} \]

\[ = \frac{N!}{n!} e^{-nat} \sum_{j=0}^{n-2} \frac{1}{j! (N-n-j)!} (-e^{-t})^j \]

\[ = \frac{N!}{n! (N-n)!} e^{-nat} \left( 1 - e^{-t} \right)^{N-n} \]
Interpretation of $X_t \sim \text{Bin}(n, e^{-dt})$

Consider the following process: Let $\xi_i$, $i = 1 \ldots N$, be i.i.d. r.v.s, $\xi_i \sim \text{Exp}(d)$. Denote by $X_t$ the number of $\xi_i$'s that are bigger than $t$ ($\xi_i$ is the lifetime of an individual, $X_t =$ size of the population at $t$), $X_0 = N$.

Then: $S_k \sim \text{Bin}(N', \text{independent})$, where ($X_t)_{t \geq 0}$ is a pure death process.

Probability that an individual survives to time $t$ is

Probability that exactly $n$ individuals survive to time $t$ is

$$(N) \cdot e^{-x t} \cdot (1-x t)^{N-n} = P(X_t = n).$$
Example: Cable

If a fiber fails, then this increases the load on the remaining fibers, which results in a shorter lifetime.

$L$ = pure death process.
Birth and death processes

Birth rate

Death rate

Combine both
Infinitesimal definition

Def. Let \((X_t)_{t \geq 0}\) be a continuous time MC, \(X_t \in \{0, 1, 2, \ldots\}\) with stationary transition probabilities. Then \((X_t)_{t \geq 0}\) is called a birth and death process with birth rates \((\lambda_i)\) and death rates \((\mu_k)\) if

1. \(P_{i, i+1}(h) = \)
2. \(P_{i, i-1}(h) = \)
3. \(P_{i, i}(h) = \)
4. \(P_{ij}(0) = \left( P(X_0 = j \mid X_0 = i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \right) \)
5. \(\mu_0 = 0, \lambda_0 > 0, \lambda_i, \mu_i > 0 \)
Example: Linear growth with immigration

Dynamics of a certain population is described by the following principles:

- during any small period of time of length $h$
  - each individual gives birth to one new member with probability independently of other members;
  - each individual dies with probability independently of other members;
  - one external member joins the population with probability

Can be modeled as a Markov process
Example: Linear growth with immigration

Let \((X_t)_{t \geq 0}\) denote the size of the population. Using a similar argument as for the Yule/pure death models:

- \(P_{n,n+1}(h) = \)
- \(P_{n,n-1}(h) = \)
- \(P_{n,n}(h) = \)

Let birth and death process with

\[ \lambda_n = \]
\[ \mu_n = \]
Alternative (jump and hold) characterization

Sojourn times $S_k$ are independent.

Each transition has two parts
- wait in state $i$ for time $\sim$
- then choose where to go:
  - go to $i+1$ with probability $\lambda_{i-1}$
  - go to $i-1$ with probability $\mu_i$