Today: Strong Markov property.
Hitting probabilities
> Q&A: October 12
Next: PK 6.6, Durrett 4.1

Week 2:

- No homework!
- Quiz 1 on Wednesday, October 14
Alternative (jump and hold) characterization

Sojourn times $S_k$ are independent.

Each transition has two parts
- Wait in state $i$ for time $\sim \text{Exp}(\lambda_i + \mu_i)$
- Then choose where to go:
  - Go to $i+1$ with probability $\frac{\lambda_i}{\lambda_i + \mu_i}$
  - Go to $i-1$ with probability $\frac{\mu_i}{\lambda_i + \mu_i}$
Def. (Informal). Let \((X_t)_{t \geq 0}\) be a stochastic process and let \(T \geq 0\) be a random variable. We call \(T\) a stopping time if the event
\[
\{T \leq t\}
\]
can be determined from the knowledge of the process up to time \(t\) (i.e., from \(\{X_s : 0 \leq s \leq t\}\)).

Examples: Let \((X_t)_{t \geq 0}\) be right-continuous
1. \(\min\{t \geq 0 : X_t = i\}\) is a stopping time
2. \(W_k\) is a stopping time
3. \(\sup\{t \geq 0 : X_t = i\}\) is not a stopping time
Stopping times

\[ \{ T \leq t \} \]
Strong Markov property

Theorem (no proof)
Let \((X_t)_{t \geq 0}\) be a MC, let \(T\) be a stopping time of \((X_t)_{t \geq 0}\). Then, conditional on \(T < \infty\) and \(X_T = i\),
\[(X_{T+t})_{t \geq 0}\]

(i) is independent of \(\{X_s, 0 \leq s \leq T\}\)
(ii) has the same distribution as \((X_t)_{t \geq 0}\) starting from \(i\).

Example
\[(X_{W+t})_{t \geq 0}\] has the same distribution as \((X_t)_{t \geq 0}\) conditioned on \(X_0 = i\) and is indep. of what happened before
Alternative (jump and hold) characterization

"Proof"

Denote \( G_i(t) := P( S_k > t \mid X_{W_k} = i) \)

\( G_i(t+h) = P( S_k > t+h \mid X_{W_k} = i) \)

S-Markov = \( P \left( \text{no jumps on } [0, t+h] \mid X_0 = i \right) \) stopping time

Markov

= \( P \left( \text{no jumps on } [0, t] \mid X_0 = i \right) P \left( \text{no jumps on } [0, h] \mid X_0 = i \right) \)

= \( P \left( S_0 > t \mid X_0 = i \right) P \left( S_0 > h \mid X_0 = i \right) = G_i(t) \left( 1 - (\lambda_i + \mu_i) h + o(h) \right) \)

= \( G_i(t) - (\lambda_i + \mu_i) G_i(t) h + G_i(t) o(h) \)

\( G_i'(t) = - (\lambda_i + \mu_i) G_i(t) , \ G_i(0) = 1 \)
Alternative (jump and hold) characterization

"Proof" cont.

\[ G_i'(t) = -(\lambda_i + \mu_i) G_i(t), \quad G_i(0) = 1 \]

\[ \Rightarrow G_i(t) = e^{-(\lambda_i + \mu_i) t} = P(S_k > t | X_{W_k} = i) \]

\[ \Rightarrow S_k \sim \text{Exp}(\lambda_i + \mu_i) \text{ (given that the process sojourns in } i) \]

Suppose the process waits \( \text{Exp}(\lambda_i + \mu_i) \), then jumps to \( i+1 \) with probability \( \frac{\lambda_i}{\lambda_i + \mu_i} \)

to \( i-1 \) with probability \( \frac{\mu_i}{\lambda_i + \mu_i} \)

\[ P_{i,i+1}(h) = P(S_k \leq h | X_{W_k} = i) P(\text{jump to } i+1) \]
\[ = (1 - e^{-(\lambda_i + \mu_i) h}) \frac{\lambda_i}{\lambda_i + \mu_i} = (\lambda_i + \mu_i) h + o(h)) \frac{\lambda_i}{\lambda_i + \mu_i} = \lambda_i h + o(h) \]

\[ P_{i,i-1}(h) = P(S_k \leq h | X_{W_k} = i) P(\text{jump to } i-1) = (\lambda_i + \mu_i) h + o(h)) \frac{\mu_i}{\lambda_i + \mu_i} = \mu_i h + o(h) \]
Related discrete time MC.

Def. Let \((X_t)_{t \geq 0}\) be a continuous time MC, let \(W_n, n \geq 0\), be the corresponding waiting (arrival, jump) times. Then we call \((Y_n)_{n \geq 0}\) defined by \(Y_0 = X_0\), \(Y_n = X_{W_n}\), \(n \geq 1\) the jump chain of \((X_t)_{t \geq 0}\).
Related discrete time MC.

\((X_t)_{t \geq 0}\) and its jump chain \((Y_n)_{n \geq 0}\) execute the same transitions.

Let \((X_t)_{t \geq 0}\) be a birth and death process. Then the transition probability matrix of the random walk \((Y_n)_{n \geq 0}\) is given by:

\[
P = \begin{pmatrix}
0 & \frac{\lambda_0}{\lambda_0 + \mu_0} & & & & \\
1 & \frac{\lambda_1}{\lambda_1 + \mu_1} & \frac{\lambda_1}{\lambda_1 + \mu_1} & & & \\
2 & \frac{\lambda_2}{\lambda_2 + \mu_2} & \frac{\lambda_2}{\lambda_2 + \mu_2} & \frac{\lambda_2}{\lambda_2 + \mu_2} & & \\
3 & \frac{\lambda_3}{\lambda_3 + \mu_3} & \frac{\lambda_3}{\lambda_3 + \mu_3} & \frac{\lambda_3}{\lambda_3 + \mu_3} & \frac{\lambda_3}{\lambda_3 + \mu_3} & \\
& & & & & \ddots
\end{pmatrix}
\]
Absorption probabilities for B&D processes

Let \((X_t)_{t \geq 0}\) be a birth and death process, and assume that the state 0 is absorbing, \(\lambda_0 = 0\). Then

\[
P((X_t)_{t \geq 0} \text{ gets absorbed in } 0 \mid X_0 = i) = P((Y_n)_{n \geq 0} \text{ gets absorbed in } 0 \mid Y_0 = i)
\]

Let use the first step analysis to compute the absorption probabilities for \((Y_n)_{n \geq 0}\) (and for \((X_t)_{t \geq 0}\))

Denote \(u_i = P(Y_n \text{ is absorbed in } 0 \mid Y_0 = i)\)

Then \(u_0 = 1, u_n = \frac{\mu_n}{\lambda_n + \mu_0} u_{n-1} + \frac{\lambda_n}{\lambda_n + \mu_n} u_{n+1}\)
Absorption probabilities for B&D processes

$u_0 = 1$, 

$u_n = \frac{\mu_n}{\lambda_n + \mu_n} u_{n-1} + \frac{\lambda_n}{\lambda_n + \mu_n} u_{n+1}$

Rewrite 

$(\lambda_n + \mu_n)u_n = \mu_n u_{n-1} + \lambda_n u_{n+1}$

$\lambda_n (u_{n+1} - u_n) = \mu_n (u_n - u_{n-1})$

$u_{n+1} - u_n = \frac{\mu_n}{\lambda_n} (u_n - u_{n-1})$

$= \frac{\mu_n}{\lambda_n} \cdot \frac{\mu_{n-1}}{\lambda_{n-1}} \cdot \cdots \cdot \frac{\mu_1}{\lambda_1} (u_1 - u_0)$

$\rho_n$

$(\ast) \quad u_{n+1} - u_n = \rho_n (u_1 - 1)$

Note that $\sum_{k=1}^{n-1} (u_{k+1} - u_k) = u_n - u_1 = (u_1 - 1) \sum_{n=1}^{n-1} \rho_n$

If $\sum_{n=1}^{\infty} \rho_n = \infty$, then $u_1 = 1$ and from $(\ast)$ $u_n = 1$ $\forall$ $n \geq 0$. 
Absorption probabilities for B&D processes

Let \( \sum_{k=1}^{\infty} p_k < \infty \). If we assume that \( u_n \to 0 \), \( n \to \infty \), then by taking \( n \to \infty \)

\[
\sum_{k=1}^{n-1} p_k = \frac{\sum_{k=1}^{\infty} p_k}{1 + \sum_{k=1}^{\infty} p_k}
\]

and

\[
u_n = u_1 + (u_1 - 1) \sum_{k=1}^{n-1} p_k = \frac{\sum_{k=1}^{\infty} p_k + (\sum_{k=1}^{\infty} p_k + 1 - \sum_{k=1}^{\infty} p_k) \sum_{k=1}^{n-1} p_k}{1 + \sum_{k=1}^{\infty} p_k}
\]
Mean time until absorption

Let \((X_t)_{t \geq 0}\) be a birth and death process. Denote \(T = \min\{ t \geq 0 : X_t = 0 \}\) absorption time and

Let \((Y_n)_{n \geq 0}\) be the jump chain for \((X_t)_{t \geq 0}\).

\[ N := \min\{ n \geq 0 : Y_n = 0 \} \]

Then \( T = \sum_{k=0}^{N-1} S_k \)

\[ w_i := E(T \mid X_0 = i) \]

\[ w_i = E\left( \sum_{k=0}^{N-1} S_k \mid X_0 = i \right) = \frac{1}{\lambda_i + \mu_i} + E\left( \sum_{k=1}^{N-1} S_k \mid X_0 = i \right) \]

\[ = \frac{1}{\lambda_i + \mu_i} + E\left( \sum_{k=1}^{N-1} S_k \mid X_0 = i, Y_1 = i+1 \right) P(Y_1 = i+1 \mid Y_0 = i) \]

\[ + E\left( \sum_{k=1}^{N-1} S_k \mid X_0 = i, Y_1 = i-1 \right) P(Y_1 = i-1 \mid Y_0 = i) \]
Mean time until absorption

\[
\begin{align*}
    w_i &= \frac{1}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} w_{i+1} + \frac{\mu_i}{\lambda_i + \mu_i} w_{i-1}, \\
    w_0 &= 0
\end{align*}
\]

Alternatively, \( S_k \sim \text{Exp}(\lambda_k + \mu_k) \) and one can show that

\[
E(T | X_0 = i) = E\left( \sum_{k=0}^{N-1} \frac{1}{\lambda_k + \mu_k} | Y_0 = i \right)
\]

Now apply the first step analysis for the general MC

\[
w_i = E(\sum_{k=0}^{N-1} g(Y_k) | Y_0 = i)
\]

which leads to (the same) system of equations

\[
w_i = g(i) + \sum_{j=1}^{\infty} P_{ij} w_j
\]
First step analysis for birth and death processes

Summary:
Let \((X_t)_{t \geq 0}\) be a birth and death process of rates \((\lambda_i, \mu_i)_{i \geq 0}\) with \(\lambda_0 = 0\) (state 0 absorbing).

Denote \(T = \min \{t : X_t = 0\}\), \(u_i = P(X_t \text{ gets absorbed in } 0 | X_0 = i)\), \(W_i = E(T | X_0 = i)\) and \(p_j = \frac{\mu_1 \mu_2 \ldots \mu_j}{\lambda_1 \lambda_2 \ldots \lambda_j}\). Then

\[
W_i = \begin{cases} 
  \sum_{j=1}^{\infty} \frac{1}{\lambda_j p_j}, & \text{if } \sum_{j=1}^{\infty} p_j < \infty \\
  1 + \sum_{j=1}^{\infty} p_j, & \text{if } \sum_{j=1}^{\infty} p_j = \infty 
\end{cases}
\]

\[
u_i = \begin{cases} 
  \sum_{j=1}^{\infty} \frac{1}{\lambda_j p_j}, & \text{if } \sum_{j=1}^{\infty} \frac{1}{\lambda_j p_j} < \infty \\
  \infty, & \text{if } \sum_{j=1}^{\infty} \frac{1}{\lambda_j p_j} = \infty
\end{cases}
\]