Today: General continuous time MC. Q-matrices. Matrix exponentials
> Q&A: October 16
Next: PK 6.3, 6.6, Durrett 4.2

Week 2:
- No homework!
\textbf{Q-matrices (infinitesimal generators)}

Let $S=\{0,1,\ldots,N\}$. We call $Q=(q_{ij})_{i,j=0}^N$ a $Q$-matrix if $Q$ satisfies the following conditions:

(a) $0 \leq q_{ii} < \infty$ for all $i$;
(b) $q_{ij} \geq 0$ for all $i \neq j$;
(c) $\sum q_{ij} = 0$ for all $i$.

Denote $q_i = \sum_{j \neq i} q_{ij}$ then $q_{ii} = -q_i$.

\textbf{Examples}

(a) $Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -7 & 5 \\ 0 & 2 & -2 \end{pmatrix}$

(b) \[Q = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -4 & 1 & 0 & 3 \\ 1 & 2 & -8 & 5 & 1 \\ 2 & 4 & 0 & -4 & 0 \\ 3 & 0 & 0 & 7 & -7 \end{pmatrix}\]
Matrix exponentials

Let $Q = (q_{ij})_{i,j=1}^n$ be a matrix. Then the series

$$\sum_{k=0}^{\infty} \frac{Q^k}{k!}$$

converges componentwise, and we denote

its sum $\sum_{k=0}^{\infty} \frac{Q^k}{k!} = e^Q$, the matrix exponential of $Q$.

In particular, we can define $e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}$ for $t \geq 0$.

Thm. Define $P(t) = e^{tQ}$. Then

(i) $P(t+s) = P(t) P(s)$ for all $s, t$

(ii) $(P(t))_{t \geq 0}$ is the unique solution to the equations

$$\begin{cases}
\frac{d}{dt} P(t) = P(t)Q, & \text{and} \\
P(0) = I
\end{cases}$$

$$\begin{cases}
\frac{d}{dt} P(t) = Q P(t), & \text{and} \\
P(0) = I
\end{cases}$$
Main theorem

Let \( P(t) \) be a matrix-valued function \( t \geq 0 \).
Consider the following properties

(a) \( P_{ij}(t) \geq 0 \), \( \sum_j P_{ij}(t) = 1 \) for all \( i,j,t \geq 0 \)

(b) \( P(0) = I \)

(c) \( P(t+s) = P(t)P(s) \) for all \( t,s \geq 0 \)

(d) \( \lim_{t \downarrow 0} P(t) = I \) (continuous at 0)

Theorem A. \( P(t) \) satisfies (a)-(d) if and only if
Main theorem. Remarks

This theorem establishes one-to-one correspondance between matrices $P(t)$ satisfying (a)-(d) and the $Q$-matrices of the same dimension.

Remarks

1. Conditions (a)-(d) imply that $P(t)$ is differentiable.

2. If $P(t) = e^{tQ}$, then $P(h) = \ldots$ as $h \to 0$
Q-matrices and Markov chains

Let \((X_t)_{t \geq 0}\) be a continuous time MC, \(X_t \in \{0, 1, \ldots, N\}\) with right-continuous sample paths.

Denote \(P_{ij}(t) = P(X_t = j \mid X_0 = i), \ i, j \in \{0, 1, \ldots, N\}\) stationary.

Then

- \(P_{ij}(t)\), \(\sum_{j=0}^{N} P_{ij}(t) = \sum_{j=0}^{N} P(X_t = j \mid X_0 = i))\)
- \(P_{ij}(0) = \left\{\begin{array}{ll} 1 & j = i \\ 0 & j \neq i \end{array}\right\}\)
- \(P_{ij}(t+s) = P(X_{t+s} = j \mid X_0 = i)\)
- \(\lim_{h \to 0} P(X_0 = j \mid X_0 = i) = \frac{1}{i} \)
Q-matrices and Markov chains (cont.)

$P(t)$ satisfies properties (a)-(d) from Theorem A.

$\Rightarrow$ there is a Q-matrix $Q$ such that

$P(t) = \text{ }$

In particular,

$P(h) = \text{ }$

This implies the one-to-one correspondance between Q-matrices and continuous time MC with right-continuous sample paths.

$Q$ is called the infinitesimal generator of $(X_t)_{t \geq 0}$
Infinitesimal description of cont. time MC

Let \( Q = (q_{ij})_{i,j=0}^N \) be a Q-matrix, let \( (X_t)_{t \geq 0} \) be right-continuous stochastic process, \( X_t \in \{0, 1, \ldots, N\} \). We call \( (X_t)_{t \geq 0} \) a Markov chain with generator \( Q \), if

(i) \( (X_t)_{t \geq 0} \) satisfies the Markov property

(ii) \( P(X_{t+h} = j | X_t = i) = \)

Example

Pure death process

\[ P_{i,i-1}(h) = \mu_i h + o(h) \]
\[ P_{ii}(h) = 1 - \mu_i h + o(h) \]
\[ P_{ij}(h) = o(h) \text{ for } j \neq \{i-1, i\} \]

The corresponding Q-matrix

\[
Q = \begin{pmatrix}
\ddots & \ddots & \ddots \\
\ddots & 1 - \mu_i h + o(h) & 0(h) \\
\end{pmatrix}
\]
Sojourn time description

Let $Q = (q_{ij})_{i,j=0}^\infty$ be a $Q$-matrix. Denote $q_i = \sum_{j\neq i} q_{ij}$ so that $q_i = q_{0i} = \sum_{i\neq 0} q_{0i}$.

$$Q = \begin{pmatrix} q_{01} & q_{02} & \cdots \\ q_{10} & q_{12} & \cdots \\ q_{20} & q_{21} & \cdots \end{pmatrix}$$

Denote $Y_k := X_{W_k}$ (jump chain).

Then the MC with generator matrix $Q$ has the following equivalent jump and hold description

- sojourn times $S_k$ are independent r.v.
  with $P(S_k > t \mid Y_k = i) =$

- transition probabilities $P(Y_{k+1} = j \mid Y_k = i) =$
Example

\[
\begin{bmatrix}
0 & 1 & 2 & 3 \\
0 & -4 & 1 & 0 & 3 \\
1 & 2 & -7 & 5 & 1 \\
2 & 4 & 0 & -4 & 0 \\
3 & 0 & 0 & 7 & -7
\end{bmatrix}
\]

\[
\begin{array}{c}
0 \quad 1 \quad 2 \quad 3 \\
1 \quad 2 \quad 3 \\
4 \quad 5 \quad 6 \\
7 \quad 8 \quad 9
\end{array}
\]

\[
\begin{array}{c}
i \quad j \\
\end{array}
\]

\[
= 
\]

\[
\begin{array}{c}
0 \quad 1 \quad 2 \quad 3 \\
0 \quad 1 \quad 2 \quad 3
\end{array}
\]
Example

Birth and death process on \( \{0, 1, 2, 3\} \)

\[
Q = \begin{pmatrix}
-\lambda_0 & \lambda_0 \\
\mu_1 & -\lambda_1 - (\lambda_1 + \mu_1) & \lambda_1 \\
\mu_2 & -\lambda_2 - (\lambda_2 + \mu_2) & \lambda_2 \\
\mu_3 & -\mu_3 & \mu_3
\end{pmatrix}
\]

\[\text{Exp}(\lambda_0) \quad \text{Exp}(\lambda_1 + \mu_1) \quad \text{Exp}(\lambda_2 + \mu_2) \quad \text{Exp}(\mu_3)\]