Exercise Show ARE of sample mean and median for unimodal model $\geq 1/3$.

Without loss of generality we assume the mean and median as 0 and the distribution is symmetry. Then we see that $f(0)$ is the largest value of the density function. The median has an asymptotic variance of $1/(4f(0)^2)$ and mean has variance $\sigma^2$. The target is to show that

$$\sigma^2 \geq \frac{1}{3} \cdot \frac{1}{4f(0)^2}$$

or

$$f(0)^2 \int x^2 f(x) dx \geq \frac{1}{12}$$

This can be done by considering another distribution $g$ which is a uniform distribution on $[-1/(2f(0)), 1/(2f(0))]$. First argue that $g$ has a smaller variance of $f$, and replace the target integration with $g$.

Exercise Unstability of median.

The target is to show that the asymptotic variance of the sample mean of the distribution $f(x) = |x|$ on $[-1, 1]$ is infinity. We finish this by showing the second moment, multiplied by $n$, is infinity.

First, for $n$ even, try to show the density of the sample mean is

$$f_n(x) = C_n \left(\frac{1}{4} - \frac{|x|^4}{4}\right)^{n/2} |x| dx$$

Second, write out the expression of the second moment as

$$n \mathbb{E}(|X|^2) = 2n \int_0^1 C_n \left(\frac{1}{4} - \frac{|x|^4}{4}\right)^{n/2} |x|^3 dx = 2nC_n \int_0^{1/4} (\frac{1}{4} - t)^{n/2} dt = 2 \frac{n}{n/2 + 1} C_n(1/4)^{n/2+1}$$

To hold this finite, we must have $C_n(1/4)^{n/2}$ asymptotically finite. But then the density function will break down to zero. Try to argue this.