1. \( X \sim \text{Bin}(n, p) \), and \( \hat{p} = \frac{\sqrt{n}}{1 + \sqrt{n}} \left( \frac{x}{n} \right) + \frac{1}{1 + \sqrt{n}} \cdot \frac{1}{2} \), the minimax for \( p \). prove that \( \sqrt{n} (\hat{p} - p) \overset{d}{\to} N\left( \frac{1}{2} - p, p(1-p) \right) \).

\[
\sqrt{n} (\hat{p} - p) = \sqrt{n} \left( \frac{\sqrt{n}}{1 + \sqrt{n}} \left( \frac{x}{n} \right) + \frac{1}{1 + \sqrt{n}} \cdot \frac{1}{2} - p \right)
\]

\[
= \frac{\sqrt{n}}{1 + \sqrt{n}} \cdot \sqrt{n} \left( \frac{x}{n} - p \right) + \frac{\sqrt{n}}{1 + \sqrt{n}} \left( \frac{1}{2} - p \right)
\]

(I) \quad (II)

since \( \sqrt{n} \left( \frac{x}{n} - p \right) \overset{d}{\to} N(0, p(1-p)) \) by CLT,

\[
\frac{\sqrt{n}}{1 + \sqrt{n}} \to 1. \quad \text{by Slutsky's theorem,}
\]

\( (I) \overset{d}{\to} N(0, p(1-p)) \), \( (II) \to (\frac{1}{2} - p) \).

therefore

\[
\sqrt{n} (\hat{p} - p) \overset{d}{\to} N\left( \frac{1}{2} - p, p(1-p) \right).
\]
2. \( R_{\hat{\chi}}(p) = \frac{1}{n} p (1-p) \quad R_{\hat{p}}(p) = \frac{1}{4(1+\sqrt{n})^2} \)

Determine \( I_n = \{ p : R_{\hat{p}}(p) \leq R_{\hat{\chi}}(p) \} \) and its behavior when \( n \to \infty \).

Suppose \( p \neq \frac{1}{2} \), \( p(1-p) \leq \frac{1}{4} \), then

\[
\frac{R_{\hat{\chi}}(p)}{R_{\hat{p}}(p)} = \frac{p(1-p)}{1/4} \cdot \frac{(1+\sqrt{n})^2}{n} \xrightarrow{n \to \infty} 4p(1-1) < 1.
\]

Therefore, eventually \( R_{\hat{\chi}}(p) < R_{\hat{p}}(p) \).

If \( p = \frac{1}{2} \), \( R_{\hat{\chi}}(p) = R_{\hat{\chi}}(\frac{1}{2}) = \frac{1}{4n} > R_{\hat{p}}(\frac{1}{2}) = \frac{1}{4(1+\sqrt{n})^2} \)

To sum up: \( I_n \to \{ \frac{1}{2} \} \).
3. Let $X \sim \text{Bin}(n, p)$, $L(p, d) = \frac{(d-p)^2}{p(1-p)}$. Show that:

1. $\frac{X}{n}$ has constant risk

2. $\frac{X}{n}$ is the Bayes solution with respect to uniform prior.

1. $R_{\frac{X}{n}}(p) = E(L(p, \frac{X}{n})) = E\left( \frac{(\frac{X}{n} - p)^2}{p(1-p)} \right) = \frac{1}{n}$.

2. We have $\pi(p) = 1$ and $f(x|p) = \binom{n}{x} p^x (1-p)^{n-x}$

Therefore $\pi(p|x) \propto p^x (1-p)^{n-x}$

which indicates that the posterior is Beta $(x+1, n-x+1)$

To minimize the risk, which is equivalent to

$$\min_d \int_0^1 B(x+1, n-x+1) \frac{(d-p)^2}{p(1-p)} p^x (1-p)^{n-x} \, dp$$

$$= \min_d \int_0^1 B(x, n-x) (d-p)^2 p^{x-1} (1-p)^{n-x-1} \, dp. \tag{*}$$

(i) $x > 0$, $x < n$, $(*)$ is equivalent to minimize squared loss under Beta $(x, n-x)$, which is the mean of such distribution: $\frac{x}{n}$.

(ii) $x = 0$: It's easy to see that any $d$ such that $d > 0$ on a positive measure measure set will result in $(*) \to \infty$.

Therefore $d = 0 = \frac{x}{n}$.

(iii) $x = n$: Similar to (ii), $d = 1 = \frac{x}{n}$.

To sum up: $\frac{X}{n}$ is the optimal Bayes estimator.
4. Suppose $\hat{g}$ is unbiased for $g(\theta)$. Then there is $c \in (0, 1)$ such that $c \hat{g}$ dominates $\hat{g}$.

By the bias-variance relationship,

$$R_{c\hat{g}}(\theta) = \text{Var}(c \hat{g}) + \text{Bias}^2(c \hat{g})$$

$$= c^2 \text{Var}(\hat{g}) + (c - 1)^2 g^2(\theta)$$

$$= c^2 (R_{\hat{g}}(\theta) + q^2(\theta)) - 2g^2(\theta)c + g^2(\theta)$$

This is minimized by $c = \frac{g^2(\theta)}{R_{\hat{g}}(\theta) + q^2(\theta)}$

Since $c = 1$ is a root for equation $R_{c\hat{g}}(\theta) = R_{\hat{g}}(\theta)$ for all $\theta$, then for given $\theta$, any number in $(\frac{g^2(\theta)}{R_{\hat{g}}(\theta) + q^2(\theta)}, 1)$ will make $R_{c\hat{g}}(\theta) < R_{\hat{g}}(\theta)$.

To make such a choice independent of $\theta$, just take

$$c = \bigcap_{\theta \in \Theta} \left( \frac{g^2(\theta)}{R_{\hat{g}}(\theta) + q^2(\theta)}, 1 \right)$$

$$= \left( -\sup_{\theta} \frac{g^2(\theta)}{R_{\hat{g}}(\theta) + q^2(\theta)}, 1 \right).$$
5. $X \sim \text{POI}(\lambda)$, for which $(a, b)$ such that $aX + b$ admissible?

First: $R_{\hat{\theta}}(X) = \text{var}(\hat{\theta}) + \text{bias}^2(\hat{\theta}) = a^2 \lambda + (a-1)\lambda + b$.

We consider a prior $\text{Gamma}(\alpha, \beta)$ on $\lambda$ (shape-rate parameterization).

It's easy to see that:

$$p(\lambda | x) \propto p(\lambda) \cdot f(x | \lambda) \propto \lambda^{x+\alpha-1} e^{-(\beta+1)\lambda} \lambda^\beta$$

So the posterior is $\text{Gamma}(\alpha+x, \beta+1)$, with Bayes estimator

$$\frac{\alpha + x}{\beta + 1} = \frac{1}{\beta+1} \lambda + \frac{\lambda}{\beta+1}.$$ This is admissible, and by taking different $(\alpha, \beta)$, we see that $a < 1, \ b > 0$ makes $aX + b$ admissible.

Now we rule out other options. First, $(a, b) > 0$, otherwise it's dominated by $\bar{f} = \int 1(\bar{f} > 0)$. Since $\lambda > 0$.

When $a > 1$: $R_{\bar{f}}(\lambda) \geq a^2 \lambda > a \lambda = R_{X}(\lambda)$, so it's dominated by $X$.

The case $\bar{f} = X$ is not admissible by question (4), since $X$ is unbiased to $\lambda$. 
