UCSD

Lecture : MATH 20D Introduction to Differential Equations

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Contents

1 Introduction

1.1 Background

Definition 1.1. An equation that contains derivatives of unknown functions is called a differential equation.

Example 1.1.1. *A falling object. By Newton's second law:*

$$F = ma$$

where F denotes an external force, m denotes mass and a is the acceleration.

Suppose one object is h meters above the ground. Let v, a be, respectively, the object's velocity and acceleration. Then we have

$$v(t) = \frac{dh(t)}{dt} = h'(t), \quad a(t) = \frac{d^2h(t)}{dt^2} = h''(t).$$

Now we study the force on the object. There are two forces: gravity and the air resistance. By physics, gavity $= mg$ where g is the gravity constant, and air resistance=$\frac{1}{2}\rho Ac|v|^2$ where ρ, A, c are air density, cross sectional area and the drag coefficient. Note that the air resistance is proportional to velocity's square. By Newton's second law, we obtain

$$mh''(t) = -mg + \frac{1}{2}\rho Ac|h'(t)|^2$$

which is a differential equation.

Definition 1.2. A differential equation always involves the derivative of one variable with respect to another. The former is called a **dependent variable** and the latter an **independent variable**.

Definition 1.3. A differential equation involving only derivatives with respect to one independent variable is called an **ordinary differential equation (ODE)**. Otherwise it is called a **partial differential equation (PDE)**.

Definition 1.4. The **order** is the order of the highest derivatives present in the equation.

Definition 1.5. A **linear differential equation** is one in which the dependent variable and its derivatives appear in additive combinations of their first powers. More precisely, linear differential equation is of the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

Example 1.1.2. $\frac{d^{10}y}{dx^{10}} + y + x^{100} + \sin x = 0$ *is linear. While* $\sqrt{y'+1} + x = 1$, $y'' + y' + y^2 = 1$ *are nonlinear.*

1.2 Initial Value Problem

Definition 1.6. By an **initial value problem** for an nth-order differential equation, we mean

$$F(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}) = 0,$$
$$y(x_0) = y_0,\ \frac{dy}{dx}(x_0) = y_1,\ ...,\ \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}.$$

By an **explicit solution** to the above equation, we mean a function $y = y(x)$ such that the above n equalities hold.

The goal of this course is to solve several types of differential equations.

Example 1.2.1. *Consider $\frac{dy}{dx} = f(x)$ with initial data $y(0) = 1$. Then the anti-derivatives of f are solutions of the equation. We have*

$$y(x) = \int f(x)dx = F(x) + C.$$

Use the initial data to solve for C: $y(0) = F(0) + C = 1$ implies $y(x) = F(x) + 1 - F(0)$.

For example: $\frac{dy}{dx} = 2e^{-x}$ with initial data $y(0) = 1$. The solution $y(x) = -2e^{-x} + 3$.

2 First Order Equations

2.1 Separable Equations

From example 1.2.1 we know that equations of the form $\frac{dy}{dx} = f(x)$ can be solved. More generally, consider the equations of the following form.

Definition 2.1. Consider the equation $\frac{dy}{dx} = f(x, y)$. If $f(x, y) = g(x)p(y)$ for some functions g, p, then the differential equation is called **separable differential equation**.

Method of solving separable equations. Suppose

$$\frac{dy}{dx} = g(x)p(y).$$

Then the **implicitly defined** solution is

$$\int \frac{1}{p(y)}dy = \int g(x)dx.$$

Example 2.1.1. *Solve $\frac{dy}{dx} = \frac{x^2-1}{y^3+1}$. (Give implicit solutions.)*

Example 2.1.2. *Solve for the initial value problem*

$$\frac{dy}{dx} = \frac{y-1}{x^2+3}, \quad y(-1) = 0.$$

(Compare the problem with Example 2 on textbook page 43.)

Solution. From the equation

$$\frac{dy}{y-1} = \frac{dx}{x^2+3},$$

and then

$$\int \frac{dy}{y-1} = \int \frac{dx}{x^2+3}. \tag{1}$$

Recall $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$. By the chain rule

$$\frac{d}{dx} \arctan ax = \frac{1}{1+(ax)^2} \frac{d}{dx}(ax) = \frac{a}{1+a^2x^2}.$$

Therefore

$$\frac{d}{dx} a \arctan ax = \frac{a^2}{1+a^2x^2}.$$

Pick $a = 1/\sqrt{3}$. We get

$$\frac{d}{dx} \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} = \frac{1/3}{1+(1/3)x^2}$$

which implies

$$\int \frac{1}{3+x^2} dx = \int \frac{1/3}{1+(1/3)x^2} dx = \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C.$$

Now it follows from (1),

$$\ln|y-1| = \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C.$$

Then

$$|y-1| = \exp(\frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C) = C_1 \exp(\frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}})$$

where $C_1 := e^C$ and thus $C_1 > 0$. Then

$$y = 1 \pm C_1 \exp(\frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}}).$$

Because C_1 is positive, we can replace $\pm C_1$ by C_2 where C_2 represents an arbitrary nonzero constant. We have

$$y = 1 + C_2 \exp(\frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}}).$$

Since $y(-1) = 0$,

$$0 = 1 + C_2 \exp(\frac{1}{\sqrt{3}} \arctan \frac{-1}{\sqrt{3}}).$$

We get

$$C_2 = -\exp(\frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}}).$$

The solution is

$$y = 1 - \exp(\frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + \frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}}).$$

□

Remark 2.2. In the above problem, if without the initial condition, we have obtained that

$$y = 1 + C_2 \exp(\frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}})$$

are solutions for all $C_2 \neq 0$. Note $C_2 = \pm e^C$ and thus the only constrain is $C_2 \neq 0$. However this constrain can be removed. When $C_2 = 0$, we get $y = 1$, a constant function. It is not hard to verify that $y = 1$ is also a solution to $\frac{dy}{dx} = 0 = \frac{y-1}{x^2+3}$.

2.2 Linear Equations

In this section we are going to deal with the equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2}$$

The key idea: multiply $\mu(x)$ on both sides of the equation and hope that we can combine the terms $\mu(x)\frac{dy}{dx}$, $\mu(x)P(x)y$ by

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}(\mu(x)y).$$

Note the RHS

$$= \mu(x)\frac{dy}{dx} + \mu'(x)y.$$

Then the requirement becomes a equation of μ:

$$\mu'(x) = \mu(x)P(x)$$

which is a separable equation. We get

$$\mu(x) = \exp(\int P(x)dx).$$

With this choice of μ, the original equation becomes

$$\frac{dy}{dx}(\mu(x)y) = \mu(x)Q(x)$$

which has the solution

$$y(x) = \frac{1}{\mu(x)}(\int \mu(x)Q(x)dx).$$

This is often referred to as the **general solution** to (2). μ is called the **integrating factor** to (2).

Example 2.2.1. *Find the general solution to*

$$\frac{1}{x}\frac{dy}{dx} - \frac{3y}{x^2} = x^3 \cos x.$$

Solution. **Step one. Write the equation into the standard form.** Multiplying x on both sides, we get

$$\frac{dy}{dx} - \frac{3y}{x} = x^4 \cos x.$$

Step two. Calculating the integrating factor:

$$\exp(\int \frac{-3}{x} dx) = \exp(-3 \ln |x| + C).$$

Since we only need one integrating factor, let us select $C = 0$ and suppose for the moment $x > 0$. Then

$$\mu(x) := \exp(-3 \ln x) = \frac{1}{x^3}.$$

Step three. Multiply the integrating factor and do the computation. We have

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4} = x \cos x,$$

$$\frac{d}{dx}(\frac{1}{x^3} y) = x \cos x.$$

Integrate both sides:

$$\frac{y}{x^3} = \int x \cos x dx.$$

Apply integration by parts and then we get

$$\frac{y}{x^3} = \int x \, d \sin x = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

Thus the general solution is

$$y = x^4 \sin x + x^3 \cos x + C x^3.$$

□

Example 2.2.2. *For the initial value problem*

$$y' = \sqrt{1 + \cos^2 x} - y, \quad y(1) = 4,$$

find $y(2)$.

Solution. The standard form of the equation is

$$y' + y = \sqrt{1 + \cos^2 x}.$$

Then

$$\mu(x) = \exp(\int 1dx) = e^{x+C}.$$

Take $C = 0$. Multiplying $\mu(x)$ on both sides of the equation gives

$$e^x y' + e^x y = e^x \sqrt{1 + \cos^2 x}.$$

Then

$$\frac{d}{dx}(e^x y) = e^x \sqrt{1 + \cos^2 x}$$

and we have

$$e^x y = \int e^x \sqrt{1 + \cos^2 x} dx.$$

However this indefinite integral cannot be expressed in finite terms with elementary functions.

So instead of indefinite integral, let us do definite integral for x over $[1, 2]$. Then

$$(e^x y)\Big|_{x=1}^{x=2} = \int_1^2 e^x \sqrt{1 + \cos^2 x} dx.$$

By the initoal data

$$(e^x y)\Big|_{x=1}^{x=2} = e^2 y(2) - ey(1) = e^2 y(2) - 4e.$$

Hence

$$y(2) = e^{-2}(4e + \int_1^2 e^x \sqrt{1 + \cos^2 x} dx)$$

which is the answer. We can use computer or the Simpson's rule to approximate the value.

□

2.3 Exact Equations

Consider the following general first order equation:

$$\frac{dy}{dx} = f(x, y). \tag{3}$$

We can rewrite it into the following form:

$$f(x, y)dx - dy = 0.$$

Doing this we ignore the fact of viewing y as a dependent variable, and x as an independent variable. But we are more focused on removing the “differentiations" and to obtain an implicit defined solution.

For example

Example 2.3.1. *The solution to*

$$xdx + ydy = 0$$

is

$$x^2 + y^2 = C.$$

As for the equation

$$ydx - xdy = 0,$$

notice that it is the same as

$$\frac{ydx - xdy}{y^2} = 0.$$

The RHS is $d(\frac{x}{y})$. *Thus we get solutions*

$$\frac{x}{y} = C.$$

Definition 2.3. For any constant C, $F(x, y) = C$ is said to be an **implicit solution** of (3) if

$$f(x, y) = -\frac{\partial_x F}{\partial_y F} \tag{4}$$

where $\partial_x F = \frac{\partial F}{\partial x}$ and $\partial_y F = \frac{\partial F}{\partial y}$.

Remark 2.4. The derivation of (4). View y as a function of x, and then

$$\frac{d}{dx}F(x, y) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0,$$

which is the same as

$$\frac{dy}{dx} = -\frac{\partial_x F}{\partial_y F}.$$

Suppose

$$f(x, y) = -\frac{M(x, y)}{N(x, y)}.$$

For instance, we can pick $M(x, y) = -f(x, y)$ and $N(x, y) = 1$. Now let us rewrite (3) as

$$M(x, y)dx + N(x, y)dy = 0.$$

The advantage of this notation is that we don't really distinguish the role of dependent variable (x) and independent variable (y).

Definition 2.5. $M(x, y)dx + N(x, y)dy$ is called a **differential form**. The differential form is said to be **exact** if there is a function $F(x, y)$ such that

$$\partial_x F(x, y) = M(x, y) \text{ and } \partial_y F(x, y) = N(x, y).$$

In such a case, we can write

$$dF = M(x, y)dx + N(x, y)dy$$

which is called **the total differential** of F, and the equation

$$dF = M(x,y)dx + N(x,y)dy = 0$$

is called an **exact equation**.

Example 2.3.2. *Consider*

$$(2xy^2+1)dx + (2x^2y)dy = 0.$$

It is an exact equation. Because $F = x^2y^2 + x$ satisfies

$$dF = (2xy^2+1)dx + (2x^2y)dy = 0.$$

Then $F(x,y) = C$, which is

$$x^2y^2 + x = C$$

is the general implicit solution to the oringinal equation.

Theorem 2.6. *Test for Exactness. The differential form $M(x,y)dx + N(x,y)dy$ is exact if and only if*

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y).$$

Example 2.3.3. *Solve*

$$(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0.$$

Solution. **Step one. Check exactness**. Here $M = 2xy - \sec^2 x$ and $N = x^2 + 2y$. Because

$$\partial_y M = 2x = \partial_x N,$$

the equation is exact.

Step two. View y as a constant and solve for F as a function of x. By exactness, there is F such that

$$\partial_x F = M \text{ and } \partial_y F = N.$$

Then for some $g(y)$,

$$\begin{aligned} F(x,y) &= \int M(x,y)dx + g(y) = \int (2xy - \sec^2 x)dx + g(y) \\ &= x^2y - \tan x + g(y). \end{aligned}$$

Step three. Solve for g. Now view y as a variable and x as a constant. Since $\partial_y F = N$,

$$N(x,y) = x^2 + 2y = \partial_y(x^2y - \tan x + g(y)).$$

Then

$$x^2 + 2y = x^2 + g'(y)$$

which gives

$$g = y^2.$$

And we have

$$F(x,y) = x^2y - \tan x + y^2.$$

The general solutions are given implicitly by

$$x^2y - \tan x + y^2 = C.$$

$\square$

2.4 Integrating factors

Definition 2.7. If the equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{5}$$

is not exact, but for some function $\mu(x,y)$ the equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact, then μ is called an **integrating factor** of (5).

In general finding integrating factors is a hard problem. In view of Theorem 2.6, μ is an integrating factor is the same as μ satisfies

$$\frac{\partial}{\partial y}[\mu(x,y)M(x,y)] = \frac{\partial}{\partial x}[\mu(x,y)N(x,y)].$$

This reduces to the equation

$$M\partial_y\mu - N\partial_x\mu = (\partial_x N - \partial_y M)\mu. \tag{6}$$

Unfortunately solving this equation is as hard as solving the original equation. There are, however, two exceptions.

If

$$\frac{1}{N}(\partial_y M - \partial_x N) \tag{7}$$

is only a function of x, then we can assume that μ is also only a function of x. In such a case (6) is reduced to

$$\frac{d}{dx}\mu = \frac{\mu}{N}(\partial_y M - \partial_x N), \tag{8}$$

which is a separable differential equation and can be solved.

Theorem 2.8. *If* (7) *only depends on x (not y), then*

$$\mu = \mu(x) = \exp\left(\int \frac{1}{N}(\partial_y M - \partial_x N)dx\right)$$

is an integrating factor for (5)*. Similarly if*

$$\frac{1}{M}(\partial_x N - \partial_y M)$$

only depends on y, then

$$\mu = \mu(y) = \exp\left(\int \frac{1}{M}(\partial_x N - \partial_y M)\,dy\right)$$

is an integrating factor for (5)*.*

Example 2.4.1. *Solve*

$$(2x^2 + y)dx + (x^2 y - x)dy = 0.$$

Solution. A quick inspection shows that this equation is neither separable nor linear. Let us try the method given in the above theorem. Notice

$$\partial_y M = 1 \neq (2xy - 1) = \partial_x N.$$

Then the equation is not exact. We compute

$$\frac{1}{N}(\partial_y M - \partial_x N) = \frac{1 - (2xy - 1)}{x^2 y - x} = \frac{-2}{x}$$

which is a function of only x. So an integrating factor is given by

$$\mu(x) = \exp\left(\int \frac{-2}{x} dx\right) = x^{-2}.$$

After multiplying $\mu = x^{-2}$ on both sides of the equation, we get an exact equation

$$(2 + yx^{-2})dx + (y - x^{-1})dy = 0.$$

Suppose it is the total differential of F. Then $\partial_x F = 2 + yx^{-2}$ which says that

$$F(x, y) = 2x - yx^{-1} + g(y)$$

for some function $g(y)$. In view of $\partial_y F = y - x^{-1}$, we have

$$F(x, y) = 2x - yx^{-1} + \frac{y^2}{2}.$$

and

$$F(x, y) = 2x - yx^{-1} + \frac{y^2}{2} = C$$

are the solutions.

$\square$

Sometimes, we don't really need to aim at making the whole differential form as a total differential of a function F.

Example 2.4.2. *Solve*

$$\frac{dy}{dx} = \frac{y + x^2 \cos x}{x}.$$

Solution. We can rewrite the equation as

$$xdy - ydx = x^2 \cos x dx.$$

By the method we have just introduced, the integrating factor for the differential form $xdy - ydx$ equals

$$\mu = \mu(x) = x^{-2}.$$

Multiplying μ, we get

$$\frac{xdy - ydx}{x^2} = \cos x dx.$$

Since

$$\frac{d}{dx}\frac{y}{x} = \frac{xdy - ydx}{x^2}, \quad \cos x dx = d \sin x,$$

the equation becomes

$$\frac{d}{dx}\frac{y}{x} = d \sin x.$$

The solutions are

$$\frac{y}{x} + \sin x = C.$$

□

There are some useful total differential formulas:

$$\begin{aligned}
&ydx + xdy = d(xy), && \frac{ydx - xdy}{y^2} = d\frac{x}{y}, \\
&\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = d\sqrt{x^2 + y^2}, && \frac{ydx - xdy}{x^2 + y^2} = d \arctan \frac{x}{y}, \\
&\frac{xdx + ydy}{x^2 + y^2} = \frac{1}{2} d \ln(x^2 + y^2), && etc.
\end{aligned}$$

3 Linear Second-Order Equations

3.1 Homogeneous Linear Equations

In this section let us study the linear second-order constant-codfficient differential equation

$$ay'' + by' + cy = f(x). \tag{9}$$

First we study the **homogeneous** case when $f(x) = 0$.

Let us try a spatial solution of the form $e^{\lambda x}$. Substitute $y = e^{\lambda x}$ into the equation

$$ay'' + by' + cy = 0. \tag{10}$$

We get

$$e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0.$$

Thus if λ is a solution to

$$a\lambda^2 + b\lambda + c = 0, \tag{11}$$

which is called the **characteristic equation** (or **the auxiliary equation**), then $e^{\lambda x}$ is a solution to (10).

Example 3.1.1. *Find the solutions to*

$$y'' + 2y' - y = 0.$$

Find the solution to the equation with initial values $y(0) = 0, y'(0) = -1$.

Solution. First let us solve the characteristic equation

$$\lambda^2 + 2\lambda - 1 = 0.$$

We get the roots are

$$\lambda_1 = -1 + \sqrt{2}, \quad \lambda_2 = -1 - \sqrt{2}.$$

Therefore we find two special solutions

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}.$$

Since the equation is linear, any functions of the following form are solutions

$$y := C_1 y_1 + C_2 y_2 = C_1 e^{(-1+\sqrt{2})x} + C_2 e^{(-1-\sqrt{2})x}$$

where C_1, C_2 are any constants.

Now we solve for the initial value problem. By the condition

$$\begin{aligned} y(0) &= C_1 + C_2 = 0, \\ y'(0) &= C_1(-1+\sqrt{2}) + C_2(-1-\sqrt{2}) = -1. \end{aligned}$$

We obtain $C_1 = -C_2$ from the first equality. And then from the second, we have

$$-1 = C_1(-1+\sqrt{2}) - C_1(-1-\sqrt{2}) = C_1 2\sqrt{2}.$$

We have

$$C_1 = -\frac{\sqrt{2}}{4}, \quad C_2 = \frac{\sqrt{2}}{4}.$$

Thus

$$y = -\frac{\sqrt{2}}{4} e^{(-1+\sqrt{2})x} + \frac{\sqrt{2}}{4} e^{(-1-\sqrt{2})x}.$$

□

Now let us answer the question that how many solutions are there. We need the following definition.

Definition 3.1. A pair of functions $y_1(x)$ and $y_2(x)$ is said to be **linearly independent** on an interval I if NEITHER of them is a constant multiple of the other on I. We say that they are **linearly dependent** is one of them is a constant multiple of the other.

Lemma 3.2. *The **Wronskian** of $y_1(x), y_2(x)$ is defined to be*

$$W(y_1, y_2) := y_1 y_2' - y_2 y_1'.$$

y_1, y_2 are linearly dependent on an interval I if and only if $W(y_1, y_2)(x) = 0$ for all $x \in I$.

Example 3.1.2. *Say $y_1 = x + 1, y_2 = e^x$. It can be checked that $W(x + 1, e^x) = xe^x$. It does not matter that $xe^x = 0$ at a single point $x = 0$. We have y_1, y_2 are linearly independent on any interval in $\mathbb{R}$ e.g. $[-1, 1]$.*

Theorem 3.3. *If y_1, y_2 are two solutions to (10) and they are linearly independent on $\mathbb{R}$, then*

$$\{C_1 y_1 + C_2 y_2 \quad \text{with } C_1, C_2 \in \mathbb{R}\}$$

are all the solutions to (10). In particular, if the characteristic equation (11) has two different real roots λ_1, λ_2, then all the solutions to (10) are of the form

$$C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x},$$

where C_1, C_2 are constants.

The theorem is useful since it tells us that to find all solutions to (10), we only need to find two linearly independent particular solutions. And we have already shown the way to find two solutions if the characteristic function has two different real roots. Now we consider the case if the characteristic function has only one repeated real root.

Theorem 3.4. *It the characteristic function has only one repeated real root λ, then both $y_1(x) = e^{\lambda x}$ and $y_2(x) = xe^{\lambda x}$ are solutions to (10). In such a case*

$$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$$

with $C_1, C_2 \in \mathbb{R}$, are the general solutions.

Example 3.1.3. *Find a solution to the initial value problem*

$$y'' + 4y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 3.$$

Solution. The corresponding characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

which has a repeated root $\lambda = -2$. Hence the general solutions to the differential equation are

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x}.$$

Using the initial data, we can solve for the value of C_1, C_2. We get

$$y(x) = e^{-2x} + 5xe^{-2x}.$$

□

Actually the same idea applies to high order equations.

Example 3.1.4. *Find a general solution to*

$$y''' + y'' - 5y' + 3y = 0.$$

Solution. The corresponding characteristic equation is

$$0 = \lambda^3 + \lambda^2 - 5\lambda + 3 = (\lambda - 1)^2(\lambda + 3).$$

Then $\lambda = 1$ is a root with multiplicity 2 and $\lambda = -3$ is another root. It is not hard to check that $y_1 = e^x$, $y_2 = xe^x$ and $y_3 = e^{-3x}$ are solutions. The general solutions are then given by

$$y(x) = C_1 e^x + C_2 x e^x + C_3 e^{-3x}.$$

□

3.2 Complex roots

Consider the equation

$$ay'' + by' + cy = 0$$

and its characteristic function

$$a\lambda^2 + b\lambda + c = 0.$$

Suppose there are two complex roots to the characteristic function:

$$\lambda_1 = \alpha + \beta i, \quad \lambda_2 = \alpha - \beta i.$$

Theorem 3.3 implies that all the solutions to the equation are of the form

$$C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}. \tag{12}$$

Notice here we have exponential function valued at a complex number. We introduce the well-known **Euler's formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Hence we have

$$e^{(\alpha+i\beta)x} = e^{\alpha x}e^{(i\beta)x} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x),$$
$$e^{(\alpha-i\beta)x} = e^{\alpha x}e^{(-i\beta)x} = e^{\alpha x}\cos(\beta x) - ie^{\alpha x}\sin(\beta x).$$

Now we pick $C_1 = \frac{1}{2}, C_2 = \frac{1}{2}$ in (12), and then $C_1 = -\frac{i}{2}$ and $C_2 = \frac{i}{2}$. We get respectively that the real and complex parts of the above particular solutions, which are

$$e^{\alpha x}\cos(\beta x), \quad e^{\alpha x}\sin(\beta x),$$

are two linearly independent solutions to the original equations.

In view of Theorem 3.3, we have the following theorem.

Theorem 3.5. *If the characteristic equation has two complex conjugate roots $\alpha \pm i\beta$, then the general real solutions to the equations are*

$$y(x) = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

where c_1, c_2 are arbitrary real numbers.

Example 3.2.1. *Find the general solutions to*

$$36y'' - 12y' + 37y = 0.$$

Solution. The corresponding characteristic equation is

$$36\lambda^2 - 12\lambda + 37 = 0.$$

The roots are

$$\lambda_{1,2} = \frac{1}{6} \pm i.$$

Therefore

$$e^{x/6}\cos x, \quad e^{x/6}\sin x$$

are two linearly independent solutions. By Theorem 3.5, the general real solutions are

$$y(x) = c_1 e^{x/6}\cos x + c_2 e^{x/6}\sin x$$

with $c_1, c_2 \in \mathbb{R}$. □

Example 3.2.2. *Find the general solution to*

$$y^{(4)} + 13y'' + 36y = 0.$$

Solution. The corresponding characteristic equation is

$$\lambda^4 + 13\lambda^2 + 36 = 0.$$

Since

$$\lambda^4 + 13\lambda^2 + 36 = (\lambda^2 + 4)(\lambda^2 + 9),$$

the roots are

$$\lambda_{1,2} = \pm 2i, \quad \lambda_{3,4} = \pm 3i.$$

Thus

$$\cos 2x, \quad \sin 2x, \quad \cos 3x, \quad \sin 3x$$

are four linearly independent solutions. It follows from Theorem 3.5 that the general solutions are

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x.$$

□

3.3 Undetermined Coefficients

In this section let us attack the nonhomogeneous equation with constant coefficients:

$$ay'' + by' + cy = f(x). \tag{13}$$

In the following example, we first find one solution to the equation. We usually call any one solution as a **particular solution**.

Example 3.3.1. *Find a particular solution to*

$$y'' + 3y' + 2y = 3x.$$

Solution. Let us try $y = ax + b$. Plug in $y = ax + b$ into the equation, we get

$$\text{the RHS } = 3a + 2ax + 2b.$$

In order to have the RHS the same as the LHS, we need

$$3a + 2b + 2ax = 3x$$

which gives $a = \frac{3}{2}$ and $b = -\frac{9}{4}$. Thus $y = \frac{3}{2}x - \frac{9}{4}$ is one solution. □

Remark 3.6. The **undetermined coefficient method**:

1. If $f = \Sigma_{i=0}^{n} a_i x^i$, then in the case that r is not a root to the char. eqn. of the corresponding homogeneous equation ($a\lambda^2 + b\lambda + c = 0$), we try

   $$y(x) = \Sigma_{i=0}^{n} b_i x^i$$

   and solve for $\{b_i\}$ (which are constants) to get one particular solution. Next

   $$y(x) = x(\Sigma_{i=0}^{n} b_i x^i), \quad \text{if 0 is a single root of the char. eqn.}$$
   $$y(x) = x^2(\Sigma_{i=0}^{n} b_i x^i), \quad \text{if 0 is a repeated root of the char. eqn.}$$

2. If $f = (a_0 + a_1 x)e^{rx}$, then in the case that r is not a root to the char. eqn. of the corresponding homogeneous equation, we try

$$y(x) = (b_0 + b_1 x)e^{rx}$$

and solve for b_0, b_1 to get one particular solution. We have

$$\begin{aligned} y(x) &= x(b_0 + b_1 x)e^{rx}, && \text{if } r \text{ is a single root of the char. eqn.} \\ y(x) &= x^2(b_0 + b_1 x)e^{rx} && \text{if } r \text{ is a repeated root of the char. eqn.} \end{aligned}$$

3. If $f = a_0 \cos(rx) + a_1 \sin(rx)$, then we try

$$y(x) = b_0 \cos(rx) + b_1 \sin(rx) \quad \text{if } \pm ir \text{ are not the solutions to the char. eqn.}$$

and solve for b_0, b_1 to get one particular solution. We have

$$b_0 x \cos(rx) + b_1 x \sin(rx) \quad \text{if } \pm ir \text{ are the solutions to the char. eqn. .}$$

Example 3.3.2. *Find a particular solution to*

$$y'' - 4y' + 4y = e^{2t} + 5e^{-3t}.$$

Solution. Let us consider the following two equations separately:

$$y'' - 4y' + 4y = e^{2t}, \tag{14}$$

$$y'' - 4y' + 4y = 5e^{-3t}. \tag{15}$$

For the first equation try

$$y_1(t) = ae^{2t}$$

and we find

$$y_1'' - 4y_1' + 4y_1 = 0$$

for all a. Also it is not possible to have $y_1(t) = ate^{2t}$ being a solution. Let us try $y_1(t) = at^2e^{2t}$. Since

$$y_1' = 2ate^{2t} + 2a^2e^{2t},$$

$$y_1'' = 2ae^{2t} + 8ate^{2t} + 4at^2e^{2t}.$$

We get

$$y_1'' - 4y_1' + 4y_1 = 2ae^{2t},$$

which implies $a = \frac{1}{2}$ and $y_1 = \frac{1}{2}t^2e^{2t}$ is one solution to (14).

Now we try $y_2 = be^{-3t}$ for the equation (15). We get

$$y_2'' - 4y_2' + 4y_2 = (9b + 12b - 4b)e^{-3t} = 17be^{-3t}.$$

Thus $y_2 = \frac{5}{17}e^{-3t}$ is a solution to (15).

We claim that

$$y = y_1 + y_2 = \frac{1}{2}t^2e^{2t} + \frac{5}{17}e^{-3t}$$

is a particular solution to the original equation. This is because

$$\begin{aligned}(y_1+y_2)'' - 4(y_1+y_2)' + 4(y_1+y_2) &= y_1'' - 4y_1' + 4y_1 + y_2'' - 4y_2' + 4y_2 \\ &= e^{2t} + 5e^{-3t}.\end{aligned}$$

□

Example 3.3.3. *Find the form for a particular solution to*

$$y'' - 2y' + 3y = 2te^t \sin t.$$

Solution. Try

$$y_p(t) = (a_0 + a_1 t)e^t \cos t + (b_0 + b_1 t)e^t \sin t.$$

□

3.4 General solutions to Nonhomogeneous equation

Theorem 3.7 (Superposition Principle)**.** *If y_1, y_2 are respectively solutions to*

$$ay'' + by' + cy = f_1(x), \quad ay'' + by' + cy = f_2(x),$$

then $k_1y_1 + k_2y_2$ is a solution to

$$ay'' + by' + cy = k_1f_1(x) + k_2f_2(x).$$

Theorem 3.8. *Suppose y_p is a particular solution to*

$$ay'' + by' + cy = f(x) \tag{16}$$

and y_1, y_2 are two linearly independent solutions to the homogeneous equation

$$ay'' + by' + cy = 0.$$

Then the general solutions to (16) are (all solutions are of the following form)

$$y_p + C_1y_1 + C_2y_2$$

with $C_1, C_2 \in \mathbb{R}$.

Example 3.4.1. *Given that $y_p(x) = x^2$ is a particular solution to*

$$y'' - y = 2 - x^2,$$

find a solution satisfying $y(0) = 1, y'(0) = 0$.

Solution. The corresponding homogeneous equation,

$$y'' - y = 0,$$

has the associated auxiliary equation $\lambda^2 - 1 = 0$. The equation has $\lambda = \pm 1$ two different real roots. So $y_1 = e^{\pm x}$ are two linearly independent solutions to the homogeneous equation. We find that the general solutions to the original nonhomogenous equation are

$$y(x) = x^2 + C_1 e^x + C_2 e^{-x}.$$

To meet the initial conditions, set

$$\begin{aligned} y(0) &= C_1 + C_2 = 1, \\ y'(0) &= C_1 - C_2 = 0, \end{aligned}$$

which yields $C_1 = C_2 = \frac{1}{2}$. The answer is

$$y(x) = x^2 + \frac{1}{2}(e^x + e^{-x}) = x^2 + \cosh x.$$

□

Example 3.4.2. *A mass-spring system is driven by a sinusoidal external force* $(5\sin t + 5\cos t)$. *The equation is*

$$my'' + by' + ky = F_{ext}(t).$$

The mass m *equals* 1, *the spring constant* $k = 2$ *and the damping coefficient* $b = 2$. *If the mass is initially located at* $y(0) = 1$ *with velocity* $y'(0) = 2$, *find its equation of motion.*

Solution. Since $F_{ext}(t) = 5\sin t + 5\cos t$, let us use the method of undetermined coefficients and try

$$y_p = A\sin t + B\cos t.$$

Plugging in y_p into the equation and solving for A, B gives

$$y_p = 3\sin t - \cos t.$$

The associated homogeneous equation is

$$y'' + 2y' + 2y = 0.$$

Solve for the corresponding characteristic equation and we get two complex roots: $-1 \pm i$. Therefore the general solutions to the homogeneous equation are

$$C_1 e^{-t}\cos t + C_2 e^{-t}\sin t.$$

The general solutions to the original nonhomogenous equation are

$$C_1 e^{-t}\cos t + C_2 e^{-t}\sin t + 3\sin t - \cos t.$$

Finally the initial data implies that $C_1 = 2, C_2 = 1$, and thus the solution is

$$y(t) = 2e^{-t}\cos t + e^{-t}\sin t + 3\sin t - \cos t.$$

□

3.5 Variation of Parameters

Variation of parameters is a more general method (comparing to the undetermined coefficients method) to find a particular solution to a linear second order equation.

Consider

$$ay'' + by' + cy = f(x) \tag{17}$$

and suppose $y_1(x), y_2(x)$ are two linearly independent solutions to

$$ay'' + by' + cy = 0.$$

Then we know that $C_1y_1(x)+C_2y_2(x)$ are all solutions to the homogeneous equation. The variation of parameters method is the strategy to replace the constants C_1, C_2 by functions i.e. we seek a solution of the form

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x).$$

Let us formally plug y_p into the RHS of the equation (17). Direct computation yields

$$y_p' = v_1'y_1 + v_2'y_2 + v_1y_1' + v_2y_2'.$$

Since v_1, v_2 give us two much freedom, it is hard to find the right ones. We impose the requirement

$$v_1'y_1 + v_2'y_2 = 0.$$

Such requirement is good, because it also simplifies the expression of y_p':

$$y_p' = v_1y_1' + v_2y_2'.$$

And then

$$y_p'' = v_1'y_1' + v_2'y_2' + v_1y_1'' + v_2y_2''.$$

Substituting y_p, y_p', y_p'' into (17), we find

$$\begin{aligned} f(x) &= ay_p'' + by_p' + cy_p \\ &= a(v_1'y_1' + v_2'y_2' + v_1y_1'' + v_2y_2'') + b(v_1y_1' + v_2y_2') + c(v_1(x)y_1(x) + v_2(x)y_2(x)) \\ &= a(v_1'y_1' + v_2'y_2') + v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) \\ &= a(v_1'y_1' + v_2'y_2'). \end{aligned}$$

To summarize, if we can find v_1, v_2 that satify

$$\begin{aligned} &a(v_1'y_1' + v_2'y_2') = f(x), \\ &v_1'y_1 + v_2'y_2 = 0, \end{aligned}$$

then y_p will be a particular solution.

Also notice in the above argument, we do not really used that a, b, c are constants. Indeed the method applies to non-constant coefficients equation!

Example 3.5.1. *Find a general solution on* $(-\pi/2, \pi/2)$ *to*

$$y'' + y = \tan t.$$

Solution. Consider y_p of the form

$$y_p = v_1 y_1 + v_2 y_2$$

where v_1, v_2 are two functions and y_1, y_2 solves the homogeneous equation. So $y_1 = \cos t$ and $y_2 = \sin t$.

By the method of variation of parameter, we need v_1, v_2 to satisfy

$$\begin{aligned} v_1' y_1' + v_2' y_2' &= \tan t, \\ v_1' y_1 + v_2' y_2 &= 0, \end{aligned}$$

this is

$$\begin{aligned} -v_1' \sin t + v_2' \cos t &= \tan t, \\ v_1' \cos t + v_2' \sin t &= 0. \end{aligned}$$

This is a linear system for v_1', v_2'. We get

$$v_1' = -\tan t \sin t, \quad v_2' = \sin t.$$

After integrating

$$\begin{aligned} v_1 &= -\int \tan t \sin t dt = -\int \frac{\sin^2 t}{\cos^2 t} d \sin t \\ &= \sin t - \ln|\sec t + \tan t| + C, \\ v_2 &= \int \sin t dt = -\cos t + C. \end{aligned}$$

We only need one particular solution, so pick $C = 0$ in the above. We obtain

$$y_p(t) = v_1 y_1 + v_2 y_2 = (\sin t - \ln|\sec t + \tan t|) \cos t - \cos t \sin t.$$

The general solutions are

$$y = C_1 \cos t + C_2 \sin t - (\cos t) \ln|\sec t + \tan t|).$$

□

Remark 3.9. The formula for v_1, v_2 is

$$v_1 = \int \frac{-f y_2}{aW(y_1, y_2)} dx, \quad v_1 = \int \frac{f y_1}{aW(y_1, y_2)} dx.$$

Example 3.5.2. *Find a particular solution of the variable coefficient linear equation*

$$t^2y'' - 4ty' + 6y = 4t^3$$

given $y_1 = t^2$ and $y_2 = t^3$ are solutions to the corresponding homogeneous equation.

Solution. Let us use the variation of parameter method. Write $y_p = v_1y_1 + v_2y_2$. Given y_1, y_2, we have

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = t^4.$$

Then

$$\begin{aligned} v_1 &= \int \frac{-fy_2}{aW(y_1,y_2)}dt = \int \frac{-4t^3t^3}{t^2(t^4)}dt \\ &= \int 4dt = -4t + C, \\ v_2 &= \int \frac{fy_1}{aW(y_1,y_2)}dt = \int \frac{4t^3t^2}{t^6} \\ &= \int \frac{4}{t}dt = 4\ln|t| + C. \end{aligned}$$

Take $C = 0$. The particular solution is

$$y_p = v_1y_1 + v_2y_2 = 4t^3(-1 + \ln|t|).$$

□

3.6 Variable-Coefficient Equations

In this section let us consider first- and second- order linear differential equation. Let us start with the following two theorems concerning existence and uniqueness of solutions (in a local region) to the initial value problems.

Theorem 3.10. *Consider*

$$y' + p(x)y = f(x).$$

If p, f are continuous on an interval (a, b) that contains x_0, then for any choice of the initial values Y_0, there exits a unique solution $y(x)$ to the equation satisfying the initial data $y(x_0) = Y_0$.

Theorem 3.11. *Consider*

$$y'' + p(x)y' + q(x)y = f(x).$$

If p, q, f are continuous on an interval (a, b) that contains x_0, then for any choice of the initial values Y_0, Y_1, there exits a unique solution $y(x)$ to the equation satisfying the initial data

$$y(x_0) = Y_0, \quad y'(x_0) = Y_1.$$

The following example is one problem in the first midterm.

Example 3.6.1 (Loss of uniqueness)**.** *Consider the initial value problem*

$$\frac{dx}{dt} = \frac{(t+1)^3}{xt}, \quad x(1) = 0.$$

Using the separable equation method, we get

$$\frac{1}{2}x^2 = \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 + t - \frac{15}{4}.$$

So there are actually two solutions:

$$x = \pm\sqrt{\frac{1}{2}t^4 + 2t^3 + 3t^2 + 2t - \frac{15}{2}}.$$

Explain why this does not contradicts with theorem.

Definition 3.12. A linear second order equation that can be expressed in the form

$$ax^2y''(x) + bxy'(x) + cy(x) = f(x) \tag{18}$$

where a, b, c are constants, is called a **Cauchy-Euler**, or **equidimensinal equation**.

The idea is to look for solutions of the form $y = x^r$. Let us explain the idea through the following example.

Example 3.6.2. *Find two linearly independent solutions to*

$$3t^2y'' + 11ty' - 3y = 0, \quad t > 0.$$

Solution. Let $y = t^r$. Direct computation yields

$$y' = rt^{r-1}, \quad y'' = r(r-1)t^{r-2}.$$

Inserting these into the equation, we get

$$\begin{aligned} RHS &= 3t^2r(r-1)t^{r-2} + 11trt^{r-1} - 3t^r \\ &= 3(r^2 - r)t^r + 11rt^r - 3t^r \\ &= (3r^2 + 8r - 3)t^r. \end{aligned}$$

To have $RHS = 0$, we only need r to be one root of $3r^2 + 8r - 3 = 0$. Hence $r = \frac{1}{3}$ and -3. We obtain two linearly independent solutions

$$y_1 = t^{1/3}, \quad y_2 = t^{-3}.$$

□

Method for (18): try $y = x^r$. Then we get the associated characteristic equation:

$$ar^2 + (b-a)r + c = 0.$$

1. the char. eqn. has two real roots r_1, r_2. x^{r_1}, x^{r_2} are two solutions.
2. the char. eqn. has one repeated real root r. $x^r, x^r \ln x$ are two solutions.
3. the char. eqn. has two complex roots $\alpha \pm \beta i$. $x^\alpha \cos(\beta \ln x)$, $x^\alpha \sin(\beta \ln x)$ are two solutions.

Now let me check that in the case 2, $y = x^r \ln x$ is indeed the solution. We have

$$y' = rx^{r-1} \ln x + x^{r-1}, \quad y'' = r(r-1)x^{r-2} \ln x + (2r-1)x^{r-2}.$$

Plug them into the equation we get

$$\begin{aligned} ax^2y'' + bxy' + cy &= ar(r-1)x^r \ln x + a(2r-1)x^r + brx^r \ln x + bx^r + cx^r \ln x \\ &= (ar^2 + (b-a)r + c)x^r \ln x + (2ar + b - a)x^r \\ &= 0. \end{aligned}$$

Here we used that $r = \frac{b-a}{2a}$ is the repeated root.

Example 3.6.3. *Find two linearly independent solutions to*

$$(1). \quad x^2y'' + 5xy' + 5y = 0, \quad (2). \quad x^2y'' + xy' = 0.$$

Solution. For (1), the associated characteristic equation is

$$r^2 + 4r + 5 = 0,$$

which gives two complex roots $r = -2 \pm i$. So the two solutions are

$$x^{-2} \cos(\ln x), \quad x^{-2} \sin(\ln x).$$

For (2), the associated characteristic equation: $r^2 = 0$, has a repeated root $r = 0$. So the two solutions are

$$x^0 = 1, \quad \ln x.$$

□

4 Laplace Transforms

4.1 Definition of Laplace Transform

Definition 4.1. Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by

$$F(s) := \int_0^\infty e^{-st} f(t) dt.$$

We usually use the notation $F(s) = \mathcal{L}\{f\}(s)$.

Example 4.1.1. *Determine the Laplace transform of*

$$(1).\, f(t) = e^{at},\ t \geq 0. \quad (2).\, g(t) = 2\chi_{[0,5]}.$$

Here $\chi_{[0,5]}$ *denotes the function that takes value* 1 *when* $x \in [0,5]$ *and value* 0 *otherwise.*

Solution.

$$\begin{aligned}\mathcal{L}\{f\} &= \int_0^\infty e^{-st} e^{at}\, dt = \lim_{N\to\infty} \int_0^N e^{(a-s)t}\, dt \\ &= \lim_{N\to\infty} \frac{e^{(a-s)t}}{a-s}\Big|_0^N = \lim_{N\to\infty} \left(\frac{e^{(a-s)N}}{a-s} - \frac{1}{a-s} \right).\end{aligned}$$

When $s \leq a$, the limit does not exist. When $s > a$, the limit equals $\frac{1}{s-a}$. So

$$\mathcal{L}\{f\}(s) = \frac{1}{s-a}$$

with the domain $s > a$.

Next

$$\begin{aligned}\mathcal{L}\{g\} &= \int_0^\infty e^{-st} 2\chi_{[0,5]}\, dt = \int_0^5 2e^{-st}\, dt \\ &= \frac{2}{s} - \frac{2e^{-5s}}{s}\end{aligned}$$

when $s \neq 0$. And when $s = 0$, $\mathcal{L}\{g\}(0) = 10$.

□

Theorem 4.2 (Linearity of the transform)**.** *Let* f_1, f_2 *be functions whose Laplace transform exist for* $s \in I$ *and let* c_1, c_2 *be constants. Then for* $s \in I$*,*

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\}(s) = c_1 \mathcal{L}\{f_1\}(s) + c_2 \mathcal{L}\{f_1\}(s).$$

Existence of Laplace Transforms

Definition 4.3. A function $f(t)$ is said to be **piecewise continuous on** $[a,b]$ if $f(t)$ is continuous at every point in $[a,b]$, except possibly for a finite number of points at which f has a jump discontinuity.

A function $f(t)$ is said to be **piecewise continuous on** $[0,\infty)$ if f is piecewise continuous on $[0,N]$ for all $N > 0$.

Definition 4.4. A function $f(t)$ is said to be of **exponential order** α if there exist T, M such that

$$|f(t)| \leq Me^{\alpha t}, \quad \text{for all } t \geq T.$$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$	Region of convergence
c, c a constant	$\frac{c}{s}$	$\mathrm{Re}(s) > 0$
t	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
e^{kt}, k a constant	$\frac{1}{s-k}$	$\mathrm{Re}(s) > \mathrm{Re}(k)$
$\sin at$, a a real constant	$\frac{a}{s^2+a^2}$	$\mathrm{Re}(s) > 0$
$\cos at$, a a real constant	$\frac{s}{s^2+a^2}$	$\mathrm{Re}(s) > 0$
$e^{-kt}\sin at$, k and a real constants	$\frac{a}{(s+k)^2+a^2}$	$\mathrm{Re}(s) > -k$
$e^{-kt}\cos at$, k and a real constants	$\frac{s+k}{(s+k)^2+a^2}$	$\mathrm{Re}(s) > -k$

Figure 1: Table of Laplace transforms

Example 4.1.2.

$$f(t) = \begin{cases} t, & 0 < t \leq 1, \\ 2, & 1 < t \leq 2, \\ e^{5t}\sin 2t, & t > 2, \end{cases}$$

is a piecewise continuous function in $[0, \infty)$ of exponential order α for all $\alpha \geq 5$.

However,

$$f(t) = e^{t^2}, \quad f(t) = e^{e^t}$$

are functions of NO exponential order. They grow too fast at ∞.

Theorem 4.5. *If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.*

Proof. We need to show that

$$\int_0^\infty e^{-st} f(t)dt$$

exists (converges) for $s > \alpha$.

Since f is of exponential order α, there are T, M such that

$$|f(t)| \leq Me^{\alpha t}, \quad \text{for } t > T.$$

Hence write

$$\int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt.$$

The first integral is finite because $e^{-st}f(t)$ is a bounded function and the domain is finite.

As for the second integral, we apply the comparison test. In view of

$$|f(t)| \leq Me^{\alpha t},$$

and

$$\int_T^\infty e^{-st}Me^{\alpha t}dt = M\int_T^\infty e^{-(s-\alpha)t}dt = \frac{Me^{-(s-\alpha)T}}{s-\alpha} < \infty,$$

the second integral converges for all $s > \alpha$. $\square$

4.2 Properties of the Laplace Transform

Theorem 4.6 (Translation). *If $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$, then*

$$\mathcal{L}\{e^{\beta t}f(t)\}(s) = \mathcal{L}\{f\}(s-\beta)$$

for $s > \alpha + \beta$.

Theorem 4.7 (Derivative). *Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then for $s > \alpha$,*

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

Corollary 4.8 (Integral). *The following equality holds whenever all the terms inside are well-defined*

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s)$$

Theorem 4.9 (Higher-Order Derivatives). *Let $f(t), f'(t), ..., f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then for $s > \alpha$,*

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0).$$

Theorem 4.10. *[**Multiple of a polynomial**] Let $F(s) = \mathcal{L}\{f\}(s)$ for $s > \alpha$. Then, for $s > \alpha$,*

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n\frac{d^nF}{ds^n}(s).$$

Example 4.2.1. *Determine $\mathcal{L}\{e^{at}\sin bt\}$.*

Solution. Recall

$$\mathcal{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2+b^2}.$$

Thus, by the translation property,

$$\mathcal{L}\{e^{at}\sin bt\}(s) = F(s-a) = \frac{b}{(s-a)^2+b^2}.$$

$\square$

Example 4.2.2. *Determine* $\mathcal{L}\{t \sin bt\}$.

Solution. We know

$$\mathcal{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2 + b^2}.$$

It follows from Theorem 4.10,

$$\mathcal{L}\{t \sin bt\} = -\frac{dF}{ds}(s) = \frac{2bs}{(s^2 + b^2)^2}.$$

□

Example 4.2.3. *Determine* $\mathcal{L}\{\sin^2 t + e^{3t}t^2\}$.

Solution. Since $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$,

$$\mathcal{L}\{\sin^2 t\} = \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2t\} = \frac{1}{2s} - \frac{s}{2(s^2 + 4)}.$$

Next,

$$\mathcal{L}\{e^{3t}t^2\} = \frac{d^2}{ds^2}\mathcal{L}\{e^{3t}\}(s) = \frac{d^2}{ds^2}\frac{1}{s-3} = \frac{2}{(s-3)^3}.$$

So

$$\mathcal{L}\{\sin^2 t + e^{3t}t^2\} = \frac{1}{2s} - \frac{s}{2(s^2+4)} + \frac{2}{(s-3)^3}.$$

□

Example 4.2.4. *Prove Theorem 4.9 for* $n = 3$.

Solution. Apply Theorem 4.7 for f and f':

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0),$$
$$\mathcal{L}\{f''\}(s) = s\mathcal{L}\{f'\}(s) - f'(0).$$

Applying Theorem 4.7 for f' and f'', we get

$$\mathcal{L}\{f^{(3)}\}(s) = s\mathcal{L}\{f''\}(s) - f''(0).$$

Use the above three:

$$\begin{aligned}
\mathcal{L}\{f^{(3)}\}(s) &= s\mathcal{L}\{f''\}(s) - f''(0) \\
&= s(s\mathcal{L}\{f'\}(s) - f'(0)) - f''(0) \\
&= s^2\mathcal{L}\{f'\}(s) - sf'(0) - f''(0) \\
&= s^3\mathcal{L}\{f\}(s) - s^2 f(0) - sf'(0) - f''(0).
\end{aligned}$$

□

4.3 Inverse Laplace Transform

In this section we consider the problem of finding the inverse of Laplace transform. For example

Example 4.3.1. *Given* $F(s) = \frac{10}{s^3}$, *determine* $\mathcal{L}^{-1}\{F\}$.

Solution. To compute the inverse Laplace transform, we refer to the Laplace transform table 1.

$$\mathcal{L}^{-1}(F)(t) = 5\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = 5t^2.$$

□

Theorem 4.11. ***[Linearity]*** $c_1, c_2 \in \mathbb{R}$.

$$\mathcal{L}^{-1}\{c_1F_1 + c_2F_2\} = c_1\mathcal{L}^{-1}\{F_1\} + c_2\mathcal{L}^{-1}\{F_2\}.$$

Example 4.3.2. *Determine the inverse Laplace Transform of* $F(s) = \frac{10}{s^3} + \frac{s-1}{s^2-2s+5}$.

Solution.

$$\begin{aligned}
\mathcal{L}^{-1}(F)(t) &= 5\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\} \\
&= 5t^2 + \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} \\
&= 5t^2 + e^t\cos 2t.
\end{aligned}$$

□

4.3.1 Partial Fractions.

Sometimes in order to find the inverse Laplace transform, we need partial fractions.

Distinct roots. Suppose that the n numbers $\alpha_1, ..., \alpha_n$ are pairwise distinct and that $P(x)$ is a polynomial with degree less than n. Then, there are constants $C_1, ..., C_n$ such that

$$\frac{P(x)}{(x-\alpha_1)...(x-\alpha_n)} = \frac{C_1}{x-\alpha_1} + ... + \frac{C_n}{x-\alpha_n}.$$

Repeated roots. When we have repeated root, each factor $(x-a)^n$ contributes the following sum of terms to the partial fraction decomposition

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + ... + \frac{A_n}{(x-a)^n}.$$

Quadratic factor. Irreducible quadratic factors $(x^2+ax+b)^N$ contributes the following sum of terms to the partial fraction decomposition

$$\frac{A_1x+B_1}{(x^2+ax+b)} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + ... + \frac{A_Nx+B_N}{(x^2+ax+b)^N}.$$

Example 4.3.3. *Determine*
$$\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}.$$

Solution. First let find the partial fractions. Known for some A, B, C,
$$\frac{s^2+9s+2}{(s-1)^2(s+3)}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+3}.$$
After multiplying both sides by $(s-1)^2(s+3)$ and evaluating at $s=1$, $s=-3$, we obtain
$$B=3,\quad C=-1.$$
To find the value of A, let $s=0$ in the above equality. We can get $A=2$.

So
$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}(t)&=\mathcal{L}^{-1}\left\{\frac{2}{s-1}+\frac{3}{(s-1)^2}+\frac{-1}{s+3}\right\}\\&=2e^t+3te^t-e^{-3t}.\end{aligned}$$
Here we used, by Theorem 4.10,
$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}=\mathcal{L}^{-1}\left\{-\frac{d}{ds}\frac{1}{(s-1)}\right\}=t\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}=te^t.$$

□

Example 4.3.4. *Determine*
$$\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}.$$

Solution. We know that the partial fraction expansion has the form
$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)}=\frac{As+B}{(s^2-2s+5)}+\frac{C}{s+1}.$$
After solving for A, B, C, we get
$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)}=\frac{3s+5}{(s^2-2s+5)}+\frac{-1}{s+1}=\frac{3(s-1)+8}{(s^2-2s+5)}-\frac{1}{s+1}.$$
Hence
$$\begin{aligned}&\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)^2(s+1)}\right\}\\&=\mathcal{L}^{-1}\left\{\frac{3(s-1)+8}{(s^2-2s+5)}-\frac{1}{s+1}\right\}\\&=3\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+4}\right\}+4\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\\&=3e^t\cos 2t+4e^t\sin 2t-e^{-t}.\end{aligned}$$

□

4.4 Solving Initial Value Problems

Given a equation $y'' + by' + cy = f(x)$, we can apply Laplace transform on both sides. Let us try this in the following example (and it turns out that we can use this idea to solve initial value problems).

Example 4.4.1. *Solve the initial value problem*

$$y'' - 2y' + 5y = -8e^{-t}; \quad y(0) = 2, \quad y'(0) = 12.$$

Solution. Let us apply the Laplace transform on both sides of the equation. Let us write $\mathcal{L}\{y\}(s) = Y(s)$. Notice

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY - 2,$$
$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y - 2s - 12.$$

Therefore the LHS becomes

$$\mathcal{L}\{y'' - 2y' + 5y\} = s^2Y - 2s - 12 - 2(sY - 2) + 5Y.$$

While the RHS

$$\mathcal{L}\{-8e^{-t}\} = \frac{-8}{s+1}.$$

Using RHS = LHS, after the simplification, we get

$$Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)}.$$

Recall Example 4.3.4, we have

$$y = \mathcal{L}^{-1}\{Y\} = 3e^t \cos 2t + 4e^t \sin 2t - e^{-t}.$$

□

Remark 4.12. One way for us to verify our solution is to check whether or not the obtained solution satisfies the initial condition.

Before doing the following example, let us discuss one property of the Laplace transform. If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\lim_{s\to\infty} \mathcal{L}\{f\}(s) = \lim_{s\to\infty} \int_0^\infty e^{-st} f(t)dt = 0.$$

Example 4.4.2. *Solve the initial value problem*

$$y'' + 2ty' - 4y = 1, \quad y(0) = y'(0) = 0.$$

Solution. Let us write $\mathcal{L}\{y\}(s) = Y(s)$. Taking Laplace transform on both sides of the equation gives

$$\mathcal{L}\{y''\}(s) + 2\mathcal{L}\{ty'\}(s) - 4\mathcal{L}\{y\} = \frac{1}{s}.$$

Using the initial conditions, we find

$$\mathcal{L}\{y''\}(s) = s^2 Y - sy(0) - y'(0) = s^2 Y,$$

$$\mathcal{L}\{ty'\}(s) = -\frac{d}{ds}\mathcal{L}\{y'\}(s) = -\frac{d}{ds}(sY - y(0)) = -sY' - Y.$$

The equation becomes

$$Y' + \left(\frac{3}{s} - \frac{s}{2}\right) Y = \frac{-1}{2s^2}.$$

This is a linear first order equation and we can apply the integrating factor method to solve it.

$$\mu(s) = \exp\left(\int \frac{3}{s} - \frac{s}{2} ds\right) = s^3 e^{-s^2/4}$$

(Here, as before, we chose one integrating factor). Multiplying the equation of Y by μ, we obtain

$$\frac{d}{ds}(\mu Y) = \frac{d}{ds}(s^3 e^{-s^2/4} Y) = -\frac{s}{2} e^{-s^2/4}.$$

Then

$$s^3 e^{-s^2/4} Y = -\int \frac{s}{2} e^{-s^2/4} ds = e^{-s^2/4} + C.$$

We get

$$Y(s) = \frac{1}{s^3} + C\frac{e^{s^2/4}}{s^3}.$$

Since $Y(s) \to 0$ as $s \to \infty$. Then C has to be 0. Hence $Y = \frac{1}{s^3}$, which implies that

$$y(t) = \frac{t^2}{2}.$$

□

4.5 Transforms of Discontinuous Functions.

Let us start with the following definition. This is a typical example of a discontinuous function with a jump singularity. (think about what other singularities can we have?)

Definition 4.13. The **unit step function** $H(t)$ is defined by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

Question. Can you draw the graph of $H(t)$?

Definition 4.14. The **rectangular window function** $\Pi_{a,b}(t)$ with $b > a$ is defined by

$$\Pi_{a,b}(t) = H(t-a) - H(t-b) = \begin{cases} 0, & t < a, \\ 1, & t \in [a,b), \\ 0, & t \geq b. \end{cases}$$

Question. Can you draw the graph of $\Pi_{a,b}(t)$?

Example 4.5.1. *Write the function*

$$f(t) \begin{cases} 3, & t < 2, \\ 1, & t \in [2,5), \\ t, & t \in [5,8), \\ t^2/10, & t \geq 8. \end{cases}$$

Solution. (Sketch the graph of $f(t)$.) From the figure we want to window the function in the intervals $[0,2)$, $[2,5)$, $[5,8)$, and to introduce a step for $t \in [8,\infty)$. We get

$$f(t) = 3\Pi_{0,2}(t) + \Pi_{2,5}(t) + t\Pi_{5,8}(t) + (t^2/10)H(t-8).$$

□

Remark 4.15. When we do integration of a function, the value is unaffected if the integrand's value at a single point is changed by a finite amount. Therefore sometimes, people do not specify a value for $\Pi_{a,t}(t)$ at $t = a, b$. As a consequence, it is OK to write

$$f(t) = g(t)$$

where $f(t)$ is given as the above and

$$g(t) \begin{cases} 3, & t < 2, \\ 1, & t \in (2,5), \\ t, & t \in (5,8), \\ t^2/10, & t > 8 \end{cases}$$

(the value of g is not specified at points $2, 5, 8$).

Lemma 4.16. *The Laplace transform of $H(t-a)$ with $a \geq 0$ is*

$$\mathcal{L}\{H(t-a)\}(s) = \frac{e^{-as}}{s}$$

for $s > 0$.

Proof.

$$\begin{aligned}\mathcal{L}\{H(t-a)\}(s) &= \int_0^\infty e^{-st}H(t-a)dt \\ &= \int_a^\infty e^{-st}dt \\ &= \lim_{N\to\infty} \frac{-e^{-st}}{s}\Big|_a^N = \frac{e^{-as}}{s}.\end{aligned}$$

□

Remark 4.17. Conversely, we have

$$\mathcal{L}^{-1}\{\frac{e^{-as}}{s}\}(t) = H(t-a).$$

For the rectangular window function, we have

$$\mathcal{L}\{\Pi_{a,b}(t)\}(s) = \mathcal{L}\{H(t-a) - H(t-b)\}(s) = \frac{e^{-sa} - e^{-sb}}{s}.$$

Theorem 4.18. *Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If $c > 0$, then*

$$\mathcal{L}\{f(t-c)H(t-c)\}(s) = e^{-cs}F(s), \tag{19}$$

and, conversely, an inverse Laplace transform of $e^{-cs}F(s)$ is given by

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\}(t) = f(t-c)H(t-c).$$

We skip the proof.

Example 4.5.2. *Determine $\mathcal{L}\{\cos tH(t-\pi)\}$.*

Solution. Let $f(t) = \cos(t+\pi)$ and then $f(t-\pi) = \cos t$. Also notice that $f(t) = -\cos t$. It follows from (19) that

$$\begin{aligned}\mathcal{L}\{\cos tH(t-\pi)\} &= \mathcal{L}\{f(t-\pi)H(t-\pi)\} \\ &= e^{-\pi s}\mathcal{L}\{f(t)\} \\ &= e^{-\pi s}\mathcal{L}\{-\cos t\} \\ &= -e^{-\pi s}\frac{s}{s^2+1}.\end{aligned}$$

Here we used the formula $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$. □

Example 4.5.3. *Determine $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$.*

Solution. Let us write $F(s) = 1/s^2$ and then

$$f(t) := \mathcal{L}^{-1}\{F\}(t) = t.$$

In view of (19),

$$\mathcal{L}^{-1}\left\{e^{-2s}F(s)\right\} = f(t-2)H(t-2) = (t-2)H(t-2).$$

□

Example 4.5.4. *Determine*

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\}.$$

Solution. Let $F(s) = \frac{1}{s(s^2+4)}$. By partial fractions, we get

$$\frac{1}{s(s^2+4)} = \frac{1}{4s} - \frac{1}{4}\frac{s}{s^2+4}.$$

Then

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{4s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{s}{s^2+4}\right\} \\ &= \frac{1}{4} - \frac{1}{4}\cos 2t. \end{aligned}$$

Then as before (by (19))

$$\mathcal{L}^{-1}\{e^{-s}F(s)\} = f(t-1)H(t-1) = \left(\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right)H(t-1).$$

□

4.6 Convolutions

When using Laplace transform to solve differential equations, it would be common that we need to find the Laplace inverse of the product of two functions. The goal of this section is to introduce the following formula

$$\mathcal{L}^{-1}\{F(s)G(s)\}(t) = \mathcal{L}^{-1}\{F(s)\}(t) * \mathcal{L}^{-1}\{G(s)\}(t).$$

To do this we introduce convolutions.

Definition 4.19. Let $f(t), g(t)$ be piecewise continuous functions on $[0,\infty)$. The **convolution** of $f(t)$ and $g(t)$, denoted as $(f*g)(t)$ is defined by

$$(f*g)(t) := \int_0^t f(t-s)g(s)ds.$$

Theorem 4.20 (Properties of Convolution)**.** *Let $f(t), g(t), h(t)$ be piecewise continuous functions on $[0, \infty)$ and k_1, k_2 be two constants. Then*

$$f * g = g * f; \qquad f * (k_1 g + k_2 h) = k_1(f * g) + k_2(f * h);$$
$$(f * g) * h = f * (g * h).$$

Theorem 4.21 (Main)**.** *Let $f(t), g(t)$ be piecewise continuous functions on $[0, \infty)$ with exponential order α and set $F(s) = \mathcal{L}\{f\}(s)$, $G(s) = \mathcal{L}\{g\}(s)$. Then*

$$\mathcal{L}\{f * g\}(s) = F(s)G(s),$$

or equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t).$$

Proof. By the definition of convolution

$$\begin{aligned}
\mathcal{L}\{f * g\}(s) &= \int_0^\infty e^{-st} \int_0^t f(t-v)g(v)dvdt \\
&= \int_0^\infty e^{-st} \int_0^\infty H(t-v)f(t-v)g(v)dvdt \\
&= \int_0^\infty g(v) \int_0^\infty e^{-st}H(t-v)f(t-v)dtdv \quad \text{(reverse the order of integration)} \\
&= \int_0^\infty g(v) \int_v^\infty e^{-st}f(t-v)dtdv \\
&= \int_0^\infty g(v)e^{-sv}F(s)dv \\
&= F(s)G(s).
\end{aligned}$$

□

Example 4.6.1. *Find $\mathcal{L}^{-1}\{1/(s^2+1)^2\}$.*

Solution. Write

$$\frac{1}{(s^2+1)^2} = \frac{1}{(s^2+1)}\frac{1}{(s^2+1)}.$$

Since $\mathcal{L}^{-1}\{1/(s^2+1)\} = \sin t$, it follows from the convolution theorem that

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} &= \sin t * \sin t \\
&= \int_0^t \sin(t-v)\sin v\, dv \\
&= \frac{1}{2}\int_0^t [\cos(2v-t) - \cos t]dv \\
&= \frac{1}{2}\left[\frac{\sin(2v-t)}{2}\right]\Big|_0^t - \frac{1}{2}t\cos t \\
&= \frac{\sin t - t\cos t}{2}.
\end{aligned}$$

$\square$

We can also use this technique to find the inverse Laplace transform of $\frac{s}{(s^2+a^2)^2}$. Think about how?

In the following, we show that Laplace transform can be used to solve **integro-differential** equations.

Example 4.6.2. *Solve*

$$y'(t) = 1 - \int_0^t y(t-v)e^{-2v}dv, \quad y(0) = 1.$$

Solution. The equation can be rewritten as

$$y'(t) = 1 - y(t) * e^{-2t}.$$

Write $Y = \mathcal{L}\{y\}$. Apply the transform on both sides of the equation, we get

$$sY - 1 = \frac{1}{s} - Y(\frac{1}{s+2})$$

which simplifies to

$$Y(s) = \frac{2}{s} - \frac{1}{s+1}.$$

Hence $y = 2 - e^{-t}$. $\square$

Example 4.6.3. *Use the function g to represent the solution y to*

$$y'' - y = g(t); \quad y(0) = 1, \quad y'(0) = 1.$$

Solution. Write Y, G as the Laplace transform of y, g. Then

$$s^2Y - s - 1 - Y = G.$$

We get

$$Y = \frac{1}{s-1} + \frac{G}{s^2-1}.$$

Hence

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{\frac{1}{s-1}\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}G(s)\right\} \\ &= e^t + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} * \mathcal{L}^{-1}\left\{G(s)\right\} \\ &= e^t + (\sinh t) * g(t). \end{aligned}$$

$\square$

4.7 Dirac Delta Function

The Dirac delta is used to model a tall narrow spike function (an impulse), and other similar abstractions such as a point charge, point mass or electron point. For example, to calculate the dynamics of a billiard ball being struck, one can approximate the force of the impact by a delta function.

Definition 4.22. The **Dirac delta function** $\delta(t)$ is characterized by the following two properties

$$\delta(t) = \begin{cases} 0, & t \neq 0, \\ +\infty & t = 0, \end{cases}$$

and

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \tag{20}$$

for any function $f(t)$ that is continuous on an open interval containing $t = 0$.

Approximation of δ function. Typically a nascent delta function η_ϵ can be constructed in the following manner. Let η be an absolutely integrable function on $\mathbb{R}$ of total integral 1, and define

$$\eta_\epsilon(x) = \epsilon^{-1}\eta(\frac{x}{\epsilon}).$$

Then $\eta_\epsilon(x) \to \delta(x)$ (weakly) as $\epsilon \to 0$. This can be seen in terms of the formulation of (20).

Since

$$\int_{-\infty}^{t} \delta(x)dx = \begin{cases} 0, & t < 0, \\ 1, & t > 0, \end{cases}$$

which equals the unit step function $H(x)$. So formally we have

$$\delta(x) = H'(x).$$

Laplace transform. By shifting the argument of $\delta(t)$, we have for $\delta(t-a)$ satisfying

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

for any function $f(t)$ that is continuous on an open interval containing $t = a$.

The Laplace Transform of δ:

$$\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}.$$

We give the following example:

Example 4.7.1. *A mass attached to a spring is released from rest* 1 *m below the equilibrium position for the mass-spring and begins to vibrate. After* π *seconds, the mass is struck by a hammer exerting an impulse on the mass. The system is governed by the symbolic initial value problem*

$$x'' + 9x = 3\delta(t-\pi); \quad x(0) = 1, \quad x'(0) = 0,$$

where x *denotes the displacement from equilibrium at time* t. *Find* $x(t)$.

Let $X = \mathcal{L}\{x\}$. Since

$$\mathcal{L}\{x''\} = s^2 X - s \quad \text{and} \quad \mathcal{L}\{\delta(t-\pi)\}(s) = e^{-\pi s},$$

the equation can be transferred into

$$X(s) = \frac{s}{s^2+9} + e^{-\pi s}\frac{3}{s^2+9}.$$

Applying Theorem 4.18, we find

$$\begin{aligned} x(t) &= \cos(3t) + \sin 3(t-\pi)H(t-\pi) \\ &= \begin{cases} \cos 3t, & t < \pi, \\ \cos 3t - \sin 3t, & t > \pi, \end{cases} \\ &= \begin{cases} \cos 3t, & t < \pi, \\ \sqrt{2}\cos(3t + \frac{\pi}{4}), & t > \pi. \end{cases} \end{aligned}$$

5 Series Solutions of Differential Equation

In general, functions that can be explicitly represented by simple functions (like powers, log, and trig functions etc.) are just a very small amount of functions among all (smooth) functions (If all functions is a pool, functions that can be explicitly represented is like a water molecule which is not even visible by human).

However series can represent a larger class of smooth functions (still not all). In this section let us study series solutions of differential equations.

5.1 Power Series

Definition 5.1. A **power series** about a point x_0 is an expression of the form

$$\Sigma_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ...,$$

where x is a variable and a_n are constants. We say the series **converges** at $x = c$ if $\Sigma_{n=0}^{\infty} a_n(c-x_0)^n$ converges. If the limit does not exist, we say the series **diverges** at $x = c$.

Moreover if

$$\Sigma_{n=0}^{\infty} |a_n(c-x_0)^n|$$

converges, we say the series **converges absolutely** at point $x = c$.

Theorem 5.2. ***[Radius of convergence]*** *The radius of convergence r is a nonnegative real number or ∞ such that the series converges if $|x - x_0| < r$, and diverges if $|x - x_0| > r$.*

r can be derived through the following formulas:

Root test. $r = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{a_n}}$,

Ratio test. *when the following limit exists, it satisfies,* $r = \lim\limits_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right|$.

Example 5.1.1. *Determine the converge set of*

$$\Sigma_{n=0}^{\infty} \frac{(-2)^n}{n+1} (x-3)^n.$$

Theorem 5.3 (Vanishing Series)**.** *If $\Sigma_{n=0}^{\infty} a_n (x - x_0)^n = 0$ for all x in some open interval, then $a_n = 0$ for all n.*

Sum of two Power Series.

Given two power series:

$$f(x) = \Sigma_{n=0}^{\infty} a_n x^n, \quad g(x) = \Sigma_{n=0}^{\infty} b_n x^n.$$

Then

$$f(x) + g(x) = \Sigma_{n=0}^{\infty} (a_n + b_n) x^n.$$

Product of two Power Series.

$$\begin{aligned} f(x)g(x) &= (\Sigma_{n=0}^{\infty} a_n x^n) \times (\Sigma_{n=0}^{\infty} b_n x^n) \\ &= (a_0 b_0) + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + ... \end{aligned}$$

The general formula is

$$f(x)g(x) = \Sigma_{n=0}^{\infty} c_n x^n \quad \text{with } c_n := \Sigma_{k=0}^{n} a_k b_{n-k}. \tag{21}$$

This is called the **Cauchy Product**.

Theorem 5.4 (Differentiation and Integration)**.** *If*

$$f(x) = \Sigma_{n=0}^{\infty} a_n x^n$$

has a positive radius of convergence r, then f is differentiable in the interval $|x| < r$:

$$f'(x) = \Sigma_{n=1}^{\infty} n a_n x^{n-1}.$$

Also f has antiderivatives in $|x| < r$:

$$\int f(x)dx = \Sigma_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C.$$

Remark 5.5. We can replace x in the above theorem by $(x - x_0)$.

If we inductively apply the first part of the theorem, we know that f is nth differentiable for all $n \geq 1$.

Example 5.1.2. *Find the power series for* $\frac{1}{1-x}$.

Solution.

$$\frac{1}{1-x} = 1 + x + x^2 + ... = \Sigma_0^{\infty} x^n. \tag{22}$$

The radius of convergence is 1. □

Example 5.1.3. *Find a power series for each of the following functions:*

$$(a) \quad \frac{1}{1+x^2}, \quad (b) \quad \frac{1}{(x-1)^2}, \quad (c) \quad \arctan x.$$

Solution. Replacing x by $-x^2$ in (22), we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - ... = \Sigma_0^{\infty} (-1)^n x^{2n}. \tag{23}$$

For (b), since $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$, by differentiating (22) twice, we get

$$(\frac{1}{1-x})' = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... = \Sigma_1^n n x^{n-1}.$$

For (c), notice

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt.$$

Therefore we can integrate the series (23) to get

$$\begin{aligned} \arctan x &= \int_0^x \frac{1}{1+t^2} dt \\ &= \Sigma_0^{\infty} \int_0^x (-1)^n t^{2n} dt \\ &= \Sigma_0^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}. \end{aligned}$$

□

Shifting the Summation index.

Example 5.1.4. *Express the series*

$$\Sigma_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

as a series where the generic term is x^k.

Solution. Set $k = n - 2$. Then

$$\Sigma_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \Sigma_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k.$$

(This is like doing substitution in the summation index.) □

Example 5.1.5. *Show that the identity*

$$\Sigma_{n=1}^{\infty} n a_{n-1} x^{n-1} + \Sigma_{n=2}^{\infty} b_n x^{n+1} = 1$$

implies that $a_0 = 1, a_1 = a_2 = 0$ *and* $a_n = -\frac{b_{n-1}}{n+1}$ *for* $n \geq 3$.

Solution. The identity can be rewritten into

$$a_0 + 2a_1 x + 3a_2 x^2 + \Sigma_{n=4}^{\infty} n a_{n-1} x^{n-1} + \Sigma_{k=3}^{\infty} b_{k-1} x^k = 1,$$

and then

$$(a_0 - 1) + 2a_1 x + 3a_2 x^2 + \Sigma_{k=3}^{\infty}(k+1)a_k x^k + \Sigma_{k=3}^{\infty} b_{k-1} x^k = 0.$$

Thus we have

$$a_0 = 1, \quad a_1 = a_2 = 0, \quad (k+1)a_k + b_{k-1} = 0 \text{ for } k \geq 3.$$

□

5.1.1 Analytic Functions

Definition 5.6. A function f is said to be analytic at x_0 if, in an open interval about x_0, this function is the sum of a power series $\Sigma_{n=0}^{\infty} a_n (x - x_0)^n$ that has a positive radius of convergence.

Property: If $f = \Sigma_{n=0}^{\infty} a_n (x - x_0)^n$ and the power series has radius of convergence $r > 0$, then $f(x)$ is analytic in $|x - x_0| < r$.

Example 5.1.6.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \\ \ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \end{aligned}$$

Can you derive the second and the third equality from the first one? Can you derive the fourth equality from Example 5.1.2?

How to compute the coefficients a_n?

Suppose around point x_0,

$$f(x_0) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ...$$

Then

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_2 = \frac{f''(x_0)}{2!}, \quad a_3 = \frac{f^{(3)}(x_0)}{3!} ...$$

In general

$$a_n = f^{(n)}(x_0)/n!.$$

The equality

$$f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is often referred to as the Taylor expansion of f at point $x = x_0$.

5.2 Power Series Solutions

We begin with the differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

Definition 5.7. Let us call a point an ordinary point if p, q are analytic at x_0. Otherwise it is called a singular point.

Example 5.2.1. *Determine the singular point of*

$$xy'' + \frac{x}{1-x}y' + (\sin x)y = 0.$$

Solution. Dividing the equation by x, we get

$$p(x) = \frac{1}{1-x}, \quad q(x) = \frac{\sin x}{x}.$$

When $x \neq 0, 1$, then $p(x), q(x)$ are ratios of non-zero analytic functions. Therefore $p(x), q(x)$ are analytic.

Let us consider for $x = 0, 1$. As for $p(x)$, it is not defined at $x = 1$, hence $x = 1$ is singular. We consider $q(x)$. Notice

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{2!} + ...}{x} = 1 - \frac{x^2}{3!} + ...$$

Therefore q is analytic everywhere. □

If a equation has no singular point in an interval I, then we expect that it has power series solutions in that interval.

Example 5.2.2. *Find a power series solution about $x = 0$ to*

$$y' + 2xy = 0.$$

Solution. The coefficient of the equation is analytic in $\mathbb{R}$. We expect a solution is of the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + ... = \Sigma_{n=0}^{\infty} a_n x^n.$$

The goal is to find a_n.

By direct computations

$$y' = a_1 + 2a_2x + 3a_3x^2 + ... = \Sigma_{n=1}^{\infty} n a_n x^{n-1}.$$

From the equation,

$$\Sigma_{n=1}^{\infty} n a_n x^{n-1} + 2x\Sigma_{n=0}^{\infty} a_n x^n = 0$$

which simplifies to

$$\Sigma_{n=1}^{\infty} n a_n x^{n-1} + \Sigma_{n=0}^{\infty} 2a_n x^{n+1} = 0.$$

Use the shifting property, the above is equivalent to

$$\Sigma_{n=0}^{\infty} (n+1)a_{n+1}x^n + \Sigma_{n=1}^{\infty} 2a_{n-1}x^n = 0.$$

We obtain

$$a_1 + \Sigma_{n=1}^{\infty} \left((n+1)a_{n+1}x^n + 2a_{n-1}x^n\right) = 0.$$

By setting the coefficients to be zero, we get

$$a_1 = 0, \quad (n+1)a_{n+1}x^n + 2a_{n-1}x^n \quad \text{for all } n \geq 1.$$

This provides a **recurrence relation**:

$$a_{n+1} = -\frac{2}{n+1}a_{n-1}.$$

Let us start with $n = 1$:

$$a_2 = -a_0.$$

If we keep using the recurrence formula, we get

$$a_4 = \frac{1}{2}a_0, \quad a_6 = -\frac{1}{3!}a_0, \quad ... \quad a_{2n} = \frac{(-1)^n}{n!}a_0, \quad ...$$

If we start with $n = 2$, we get

$$a_3 = -\frac{2}{3}a_1 = 0, \quad a_5 = -\frac{2}{5}a_3 = 0, \quad ... \quad a_{2n+1} = 0, \quad ...$$

Submitting the values of a_n into the series, we obtain a series solution y which is

$$y = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{n!} a_0 x^{2n}.$$

□

Example 5.2.3. *Find the first four terms in the power series expansion about $x = 0$ for a general solution to*

$$(1 + x^2)y'' - y' + y = 0.$$

What you need to do if the question is for the series expansion about point $x = 2$?

Solution. Since $\frac{1}{1+x^2}$ is analytic in $\mathbb{R}$, then $x = 0$ is an ordinary point for the equation. Let us express the general solution in the form

$$y(x) = \Sigma_{n=0}^{\infty} a_n x^n.$$

Substituting this expansion into the equation yields

$$\begin{aligned}
0 &= (1 + x^2)\Sigma_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \Sigma_{n=1}^{\infty} n a_n x^{n-1} + \Sigma_{n=0}^{\infty} a_n x^n \\
&= \Sigma_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \Sigma_{n=2}^{\infty} n(n-1)a_n x^n - \Sigma_{n=1}^{\infty} n a_n x^{n-1} + \Sigma_{n=0}^{\infty} a_n x^n \\
&= \Sigma_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \Sigma_{k=2}^{\infty} k(k-1)a_k x^k - \Sigma_{k=0}^{\infty}(k+1)a_{k+1}x^k + \Sigma_{k=0}^{\infty} a_k x^k.
\end{aligned}$$

This implies

(zero order terms) $2a_2 - a_1 + a_0 = 0$,

(first order terms) $6a_3 - 2a_2 + a_1 = 0$,

(kth order terms with $k \geq 2$)

$$(k+2)(k+1)a_{k+2} - (k+1)a_{k+1} + (k^2 - k + 1)a_k = 0.$$

Let us view a_0, a_1 as known constants. Then use the first equation we get a_2. Next use the second one we get a_3. Finally apply the last one iteratively for $k = 2, 3, ...$ we are able to find the values for a_k with $k \geq 4$.

In the end we have

$$y(x) = a_0 \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + ...\right)$$

$$+a_1 \left(x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + ...\right).$$

$\square$

6 Linear Systems

In this section, we study systems of differential equations. By systems, we mean there are more than one differential equations and there are more than one unknown variable. For example

Example 6.0.1.

$$\begin{aligned}
x_1' &= 2x_1 + t^2 x_2 + (4t + e^t)x_4,\\
x_2' &= (\sin t)x_2 + (\cos t)x_3,\\
x_3' &= x_1 + x_2 + x_3 + x_4,\\
x_4' &= x_1.
\end{aligned}$$

Linear systems or equations can be represented using matrices and vectors. And it turns out that such representation is helpful even in solving the system. Therefore, let us discuss matrices and vectors in the next subsection.

6.1 Matrices and vectors

A **matrix** is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. An $m \times n$ matrix is with m rows and n columns:

$$A := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

We can simply write $[a_{ij}] \in \mathbb{R}^{m\times n}$ to denote the matrix.

Square matrices: $m = n$.

Diagonal matrices: $a_{ij} = 0$ for all $i \neq j$.

(Column) Vectors: $m \times 1$ matrices. **(Row) Vectors**: $1 \times m$ matrices.

Zero matrix: $a_{ij} = 0$ for all i, j, denoted as **0**.

Can you write the system in Example 6.0.1 into a matrix form where the matrix only depend on the free variable?

6.1.1 Algebra of Matrices

Scalar Multiplication. Let $r \in \mathbb{R}$ and $A = [a_{ij}]$ be a matrix. Then

$$rA = [ra_{ij}].$$

We write

$$-A := (-1)A = [-a_{ij}]$$

Matrix Addition. We can add up two $m \times n$ matrices. Suppose $A = [a_{ij}]$, $B = [b_{ij}]$ are two $m \times n$ matrices, then

$$A + B = [a_{ij} + b_{ij}], \quad A - B = [a_{ij} - b_{ij}].$$

If two matrices have different numbers of rows or columns, we can not add the matrices up.

Matrix Multiplication. Let $A = [a_{ij}]$ be a $m \times n$ matrix and let

$$B = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}$$

be a n-dimensional column vector. Then

$$AB = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + ...a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + ...a_{2n}b_n \\ ... \\ a_{m1}b_1 + a_{m2}b_2 + ...a_{mn}b_n \end{bmatrix}.$$

For example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+2+6 \\ 0+5+12 \\ 0+8+18 \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \\ 26 \end{bmatrix}.$$

In general, we are able to define AB if A is an $m \times n$ matrix and B is an $n \times p$ matrix:

$$AB = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & ... & b_{1p} \\ b_{21} & b_{22} & ... & b_{2p} \\ b_{n1} & b_{n2} & ... & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + ...a_{1n}b_{n1} & ... & a_{11}b_{1p} + a_{12}b_{2p} + ...a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}b_{21} + ...a_{2n}b_{n1} & ... & a_{21}b_{1p} + a_{22}b_{2p} + ...a_{2n}b_{np} \\ ... & & \\ a_{m1}b_{11} + a_{m2}b_{21} + ...a_{mn}b_{n1} & ... & a_{m1}b_{1p} + a_{m2}b_{2p} + ...a_{mn}b_{np} \end{bmatrix}.$$

If we denote $C := AB$ then $C = [c_{ij}]$ is a $m \times p$ matrix and for $i = 1, ..., m, j = 1, ..., p$

$$c_{ij} = \Sigma_{k=1}^{n} a_{ik}b_{kj}.$$

Theorem 6.1. *Suppose A, B, C are matrices and r is a number. The following holds as long as they are well-defined:*

$$A + B = B + A, \quad r(A + B) = rA + rB,$$

$$(AB)C = A(BC), \quad A(B + C) = AB + AC.$$

Example 6.1.1. *Let A, B be two $n \times n$ matrices. Remove the bracket of $(A + B)^2$.*

Solution.

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

Note that this is not the same as $A^2 + 2AB + B^2$. In matrices multiplication, $AB \neq BA$. $\square$

Example 6.1.2.

$$\begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

People often call the square matrix on the RHS of Example 6.1.2 as the **Identity matrix** denoted as I or I_3. Let us denote the two square matrices on LHS of Example 6.1.2 as A, B. Then

$$AB = I.$$

Then A is called the **inverse matrix** of B. Also we call B as the inverse matrix of A.

Theorem 6.2. *If $AB = I$, then $BA = I$.*

Example 6.1.3. *(not required) Find the inverse of*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution. Augment with a identity matrix:

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right].$$

Reduce the matrix to row echelon form: $\left[\begin{array}{ccc:ccc} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \end{array}\right].$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

Reduce the matrix to row echelon form: $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{array}\right].$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right].$$

The inverse matrix is
$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ -2 & 1 & 2 \end{bmatrix}.$$
□

Determinants. A square matrix is invertible if and only if its determinant is not zero.

- The determinant of a 2×2 matrix:
$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

- The determinant of a 3×3 matrix:
$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

- Similarly we define the determinant for $n \times n$ square matrix.

Example 6.1.4. *Can you compute the determinate of the matrix in Example 6.1.3.*

6.2 Linear Systems in Normal Form

Example 6.2.1. *Write the following linear system in matrix notation:*
$$x_t = -4x + 2y,$$
$$y_t = 4x - 4y.$$

Solution.
$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$
□

Example 6.2.2. *Write the following coupled mass-spring system in matrix notation.*
$$2x'' + 6x - 2y = 0,$$
$$y'' + 2y - 2x = 0.$$

Solution. We introduce
$$x_1 := x, \quad x_2 := x', \quad x_3 := y, \quad x_4 := y'.$$

Then

$$x_1' = x_2, \qquad\qquad x_2' = -3x_1 + x_3,$$
$$x_3' = x_4, \qquad\qquad x_4' = 2x_1 - 2x_3.$$

We get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

□

In general we say that a system of n linear differential equations is in a **normal form** if it is expressed as

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$$

where for each t, $\mathbf{x}(t), \mathbf{f}(t)$ are $n \times 1$ vectors and $\mathbf{A}(t)$ is an $n \times n$ matrix.

Theorems Part.

Theorem 6.3. *If $\mathbf{A}(t), \mathbf{f}(t)$ are continuous in an open interval I which contains point t_0, then for any choice of initial vector $\mathbf{x}_0$, there exists a unique solution $\mathbf{x}(t)$ to the initial value problem:*

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Definition 6.4. m vector functions $\mathbf{x}_1, ..., \mathbf{x}_m$ are said to be **linearly dependent on interval** I if there exist constants $c_1, ..., c_m$, not all zeros, such that

$$c_1\mathbf{x}_1 + ... + c_m\mathbf{x}_m = \mathbf{0}$$

for all $t \in I$. They are said to be **linearly independent on** I if they are not linearly dependent on I.

Theorem 6.5. *Let $\mathbf{x}_1, ..., \mathbf{x}_n$ be n $n \times 1$ vector functions on I. Suppose each $\mathbf{x}_i$ is a solution to the same linear system $\mathbf{x}' = A(t)\mathbf{x}$ on I. Then they are linearly independent if and only if*

$$\begin{vmatrix} x_{11}(t) & x_{12}(t) & \dots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \dots & x_{nn}(t) \end{vmatrix} \neq 0$$

for one single t in I.

Example 6.2.3. *Show that the vector functions*

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ -e^{2t} \end{bmatrix}, \quad \mathbf{x}_3(t) = \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ e^{2t} \end{bmatrix}$$

are linearly independent on $(-\infty, \infty)$.

Solution. Notice

$$\mathbf{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

We only need to compute the following determinate and show it is not zero:

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 5 - (-2) - 1 \neq 0.$$

□

Theorem 6.6. *If $\mathbf{x}_p$ is a particular solution to the system*

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{24}$$

on the interval I and $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ is a ***fundamental solution set*** *($\mathbf{x}_i$ is a solution; the n solutions are linearly independent) to the homogeneous system, then every solution to (24) is of the form*

$$\mathbf{x} = \mathbf{x}_p + c_1\mathbf{x}_1 + ... + c_n\mathbf{x}_n$$

where $c_1, ..., c_n$ are constants.

6.3 Homogeneous Linear System

Consider the homogeneous constant coefficients system

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t).$$

Recall that for homogeneous equation $x' + ax = 0$ or $x'' + ax' + bx = 0$, the most basic solutions are of the form $x = ce^{\lambda t}$. So similarly for system, let us suppose that one solution is of the form $e^{\lambda t}\mathbf{v}$ where $\mathbf{v}$ is a constant vector.

Plug $\mathbf{x} = e^{\lambda t}\mathbf{v}$ into the homogeneous system, we get

$$\mathbf{x}'(t) = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v}.$$

This equality is equivalent to

$$\lambda\mathbf{I}\mathbf{v} = \mathbf{A}\mathbf{v}.$$

We get

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = 0. \tag{25}$$

Now let us study (25). For a constant matrix $\mathbf{A}$, if (25) holds we call λ is one **eigenvalue** of $\mathbf{A}$ and $\mathbf{v}$ is one **eigenvector** of $\mathbf{A}$ that is associated with λ.

Theorem 6.7. *λ is one eigenvalue of $\mathbf{A}$ if and only if the determinate of $(\lambda\mathbf{I} - \mathbf{A})$ equals 0.*

Example 6.3.1. *Find the eigenvalues of the matrix*

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

Solution. Let us compute the determinate of $(\lambda\mathbf{I} - \mathbf{A})$:

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} \lambda - 2 & 3 \\ -1 & \lambda + 2 \end{vmatrix} \\ &= (\lambda - 2)(\lambda + 2) + 3 = \lambda^2 - 1. \end{aligned}$$

Set this to be zero, we get two eignvalues: $\lambda = \pm 1$. □

The determinate of $(\lambda\mathbf{I} - \mathbf{A})$ is a polynomial of λ. And if $\mathbf{A}$ is an $n \times n$ matrix, the polynomial is of order n. There are exactly n eigenvalues counting multiplicity (fundamental theorem of calculus). The polynomial is often called the **characteristic polynomial** of $\mathbf{A}$.

Example 6.3.2. *Find one eigenvector of $\mathbf{A}$ that is associated to the eigenvalue $\lambda = 1$.*

Solution. We need to find $\mathbf{v} = (v_1, v_2)^T$ (here $(v_1, v_2)^T$ is the 2×1 vector) such that

$$[\lambda\mathbf{I} - \mathbf{A}]\,\mathbf{v} = \begin{bmatrix} \lambda - 2 & 3 \\ -1 & \lambda + 2 \end{bmatrix} \mathbf{v} = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

Thus

$$\mathbf{v} = (v_1, v_2)^T = (3, 1)^T$$

is one eigenvector.

□

Example 6.3.3. *Find the eigenvalues and eigenvectors of the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

Solution. By direct computations, the characteristic equation is

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0.$$

For $\lambda_1 = 1$, we can find an eigenvector: $\mathbf{v}_1 = (-1, 1, 2)^T$. For $\lambda_2 = 2$, we can find an eigenvector: $\mathbf{v}_1 = (-2, 1, 4)^T$. For $\lambda_3 = 3$, we can find an eigenvector: $\mathbf{v}_1 = (-1, 1, 4)^T$. □

Example 6.3.4. *Find three linearly independent solutions to*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \quad \textit{where } \mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

Give the general solution.

Solution. known the eigenvalues and eigenvectors, we have three solutions:

$$\mathbf{x}_1 = e^t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$$

It can be checked that they are linearly independent. So according to Theorem 6.6, the general solutions are

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1 e^t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$$

□

For the linear system, in general we need n, where n is the dimension, independent solutions to get the general solutions.

For the equation with constant coefficients, in the case there are n distinct eigenvalues, we are able to find n independent solutions. Sometimes we cannot find n distinct real eigenvalues, but if we are able to find n independent eigenvectors, then the corresponding solutions are still independent.

Example 6.3.5. *Find a general solution of*

$$\mathbf{x}' = \mathbf{A}\boldsymbol{x}, \quad \textit{where} \quad \mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Solution. The characteristic equation for $\mathbf{A}$ is

$$(\lambda - 3)^2(\lambda + 3) = 0.$$

For $\lambda = 3$, $(3\mathbf{I} - \mathbf{A})\mathbf{u} = 0$ gives

$$\begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0.$$

There are two linearly independent vectors satisfies the above equation:

$$\mathbf{u} = [1, 0, 1]^T, \quad \mathbf{u} = [0, 1, 1]^T.$$

We have two linearly independent solutions:

$$\mathbf{x}_1 = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = e^{3t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -3$, we get one solution

$$\mathbf{x}_3 = e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The general solutions are

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3.$$

□

6.4 Nonhomogeneous Linear System

Undertermined coefficients method

Example 6.4.1. *Find the general solutions to*

$$\mathbf{x}' = \mathbf{A}\boldsymbol{x} + t\mathbf{g} \quad \textit{where} \quad \mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

Solution. From example (6.4.1), the general solution to the corresponding homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}_h = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Now let us find a particular solution of the form

$$\mathbf{x}_p = t\mathbf{a} + \mathbf{b} = t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Let us plug in $\mathbf{x}_p$ into the nonhomogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x} + t\mathbf{g}$. We get

$$\mathbf{x}_p' = \mathbf{a} = \mathbf{A}\mathbf{x}_p + t\mathbf{g} = \mathbf{A}(t\mathbf{a} + \mathbf{b}) + t\mathbf{g}.$$

We need

$$\mathbf{a} = \mathbf{A}\mathbf{b}, \quad \mathbf{A}\mathbf{a} = -\mathbf{g}.$$

First we consider $\mathbf{A}\mathbf{a} = -\mathbf{g}$. This yields one solution

$$\mathbf{a} = [-1, 0, 0]^T.$$

The first equality gives:

$$b_1 - 2b_2 + 2b_3 = -1, \quad -2b_1 + b_2 + 2b_3 = 0, \quad 2b_1 + 2b_2 + b_3 = 0. \tag{26}$$

From the first equality it follows that $b_1 = -1 + 2b_2 - 2b_3$. And then the second one, we get

$$b_2 = 2b_1 - 2b_3 = 2(-1 + 2b_2 - 2b_3) - 2b_3 = 2 + 4b_2 - 6b_3.$$

Thus

$$b_2 = -\frac{1}{3}(-2 - 6b_3) = \frac{2}{3} + 2b_3.$$

Then

$$b_1 = -1 + 2(\frac{2}{3} + 2b_3) - 2b_3 = \frac{1}{3} + 2b_3.$$

Use the above two and the third equality in (26), we obtain

$$b_3 = -\frac{2}{9}, \quad \text{and then } b_1 = -\frac{1}{9}, b_2 = \frac{2}{9}.$$

We get

$$\mathbf{x}_p = t\mathbf{a} + \mathbf{b} = t\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix}.$$

The general solutions are

$$\mathbf{x}_h + \mathbf{x}_p = c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix}.$$

□

Example 6.4.2. *Find the solution to*

$$\mathbf{x}' = \mathbf{A}\boldsymbol{x} + t\mathbf{g}, \mathbf{x}(0) = \mathbf{x}_0 \quad \textit{where} \quad \mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution. From the previous example, we know that the general solutions to the equations are

$$c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-3t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix}$$

where c_1, c_2, c_3 are constants.

Now let us use the initial data and solve for the constants. Plugging $t = 0$ gives

$$c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

We get

$$\begin{bmatrix} c_1 & 0 & -c_3 \\ 0 & c_2 & -c_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 10/9 \\ 7/9 \\ 2/9 \end{bmatrix}.$$

Let the first line minus the third line. We obtain

$$-c_2 - 2c_3 = \frac{8}{9},$$

and using the second line we find $c_3 = -\frac{5}{9}$. Then

$$c_1 = \frac{5}{9}, \quad c_2 = \frac{2}{9}.$$

Thus the solution is

$$\frac{5}{9}e^{3t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{2}{9}e^{3t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{9}e^{-3t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix}.$$

$\square$