UCSD

Lecture : MATH 20B Calculus for Science and Engineering

2020 WI

Lecturer: Yuming Paul Zhang *Scribe:*

Contents

1 Integration

1.1 Net Change as the Integral of a Rate of Change

Recall the fundamental theorem of calculus:

Theorem 1.1. *Suppose F is an antiderivative of f on $[a, b]$. Then*

$$\int_a^b f(x)dx = F(a) - F(b).$$

If we are given the antiderivative first, say $F = s(x)$, then f becomes $s'(x)$. The theorem becomes

Theorem 1.2.

$$\int_a^b s'(x)dx = s(a) - s(b).$$

Example 1.1.1. *Water leaks from a tank at a rate of $2 + 5t$ L/hour, where t is the number of hours after 9. How much water is lost between 9 and 11 pm?*

Solution. Let $s(t)$ be the quantity of water in the tank at time t. Since $2 + 5t$ represents the rate the water is leaving, the rate of the change of the water is $-(2 + 5t)$. Then

$$s(2) - s(0) = \int_0^2 s'(t) = \int_0^2 -(2 + 5t)dt = (2t + \frac{5}{2}t^2)\Big|_0^2 = -14.$$

Therefore there is a loss of 14L of water. □

Remark 1.3. We sometimes call $s(2) - s(0)$ the **net change** in $s(t)$ over the interval $[0, 2]$. Since s' is the rate of the change, we know the net change equals the integral of the rage of change.

Theorem 1.4. ***The integral of Velocity.*** *Let $v = v(t)$ be a function representing velocity. Then*

$$\text{displacement during } [t_1, t_2] = \int_{t_1}^{t_2} v(t)dt, \tag{1}$$

$$\text{distance traveled during } [t_1, t_2] = \int_{t_1}^{t_2} |v(t)|dt. \tag{2}$$

Example 1.1.2. *A particle has velocity $v(t) = t^2 + t - 2$. Compute the displacement and the total distance traveled over $[0, 4]$.*

Solution. Compute

$$\begin{aligned}\int_0^4 v(t)dt &= \int_0^4 t^2 + t - 2\,dt \\ &= (\frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t)\Big|_0^4 = \frac{64}{3}.\end{aligned}$$

So the displacement is $\frac{64}{3}$. Next,

$$\int_0^4 |v(t)|dt = \int_0^4 |t^2 + t - 2|dt.$$

Since

$$t^2 + t - 2 = (t+2)(t-1)$$

which is positive when $t > 1$ and otherwise when $t \in (0,1)$. So

$$\begin{aligned}\int_0^4 |v(t)|dt &= \int_0^1 -(t^2 + t - 2)dt - \int_4^1 (t^2 + t - 2)dt \\ &= -(\frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t)\Big|_0^1 + (\frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t)\Big|_1^4 \\ &= \frac{7}{6} + \frac{64}{3} + \frac{7}{6} = \frac{71}{3}.\end{aligned}$$

Thus the total distance traveled is $\frac{71}{3}$.

□

1.2 The Substitution Method

The method is about the change of variable. For example, if we change the variable from x to u, then we view $u = u(x)$ and so by chain rule,

$$du = u'(x)dx.$$

Theorem 1.5. *If $F'(x) = f(x)$, and u is a differentiable function, then*

$$\int f(u(x))u'(x)dx = \int f(u(x))du(x) = F(u(x)) + C.$$

Example 1.2.1. *Evaluate $\int x(x^2+9)^5 dx$.*

Solution. Let $u = x^2 + 9$. Then $du = 2xdx$ and so $xdx = \frac{1}{2}du$. We apply substitution

$$\begin{aligned}\int x(x^2+9)^5 dx &= \int (x^2+9)^5 xdx \\ &= \int u^5 \frac{1}{2}du = \frac{1}{12}u^6 + C.\end{aligned}$$

□

Example 1.2.2. *Evaluate $\int \cot\theta d\theta$.*

Solution. If letting $u = \sin\theta$, then $u = \cos\theta d\theta$. Since $\cot\theta = \frac{\cos\theta}{\sin\theta}$, then

$$\int cot\theta d\theta = \int \frac{\cos\theta}{\sin\theta} d\theta = \int \frac{du}{u} = \ln|u| + C.$$

The answer is $(\ln|\sin\theta| + C)$. □

Example 1.2.3. *Evaluate* $\int \frac{dx}{(1+\sqrt{x})^2}$.

Solution. Let $u = 1 + \sqrt{x}$. Then

$$du = d(1 + \sqrt{x}) = \frac{1}{2\sqrt{x}} dx.$$

Since $\sqrt{x} = u - 1$, we get $du = \frac{1}{2(u-1)} dx$ and so

$$dx = 2(u-1)du.$$

Then

$$\begin{aligned} \int \frac{dx}{(1+\sqrt{x})^2} &= \int \frac{2(u-1)du}{u^2} \\ &= \int \frac{2}{u} - \frac{2}{u^2} du = 2\ln|u| + \frac{2}{u} + C. \end{aligned}$$

□

The change of variables formula can be applied to definite integrals.

Theorem 1.6. *Suppose* $F'(x) = f(x)$.

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du = F(u(b)) - F(u(a)).$$

Example 1.2.4. *Calculate the area under the graph of* $y = \frac{x}{x^2+1}$ *over* $[1, 3]$.

Solution. The area equals $\int_1^3 \frac{x}{x^2+1} dx$. Let $u = x^2$ and then

$$du = 2x\,dx,\ u(1) = 1,\ u(3) = 9.$$

We have

$$\begin{aligned} \int_1^3 \frac{x}{x^2+1} dx &= \frac{1}{2} \int_1^9 \frac{du}{u+1} \\ &= \frac{1}{2} \ln|u+1|\Big|_1^9 = \frac{1}{2}\ln 5. \end{aligned}$$

□

Example 1.2.5. *A particle has velocity* $v(t) = \sin^2(t)\cos(t)$. *Compute the displacement and the total distance traveled over* $[0, \pi]$.

Solution. Compute

$$\begin{aligned}\int_0^\pi v(t)dt = \int_0^\pi \sin^2 t \cos t dt = \int_0^\pi \sin^2 t d\sin t \\ = (\frac{1}{3}\sin^3 t)\Big|_0^\pi = 0.\end{aligned}$$

So the displacement is 0. Next,

$$\begin{aligned}\int_0^\pi |v(t)|dt &= \int_0^\pi |\sin^2 t \cos t| dt \\ &= \int_0^{\pi/2} \sin^2 t \cos t dt - \int_{\pi/2}^{\pi} \sin^2 t \cos t dt.\end{aligned}$$

By symmetry: $\sin^2 t \cos t = -\sin^2(\pi - t)\cos(\pi - t)$. Therefore the above

$$\begin{aligned}&= 2\int_0^{\pi/2} \sin^2 t \cos t dt = 2\int_0^{\pi/2} \sin^2 t \, d\sin t \\ &= \frac{2}{3}\sin^3 t\Big|_0^{\pi/2} = \frac{2}{3}.\end{aligned}$$

Thus the total distance traveled is $\frac{2}{3}$.

□

2 Applications of the integral

2.1 Area between two curves

Theorem 2.1. *Let* $y = f(x)$ *and* $y = g(x)$ *be two graphs. The signed area from the graph of* $g(x)$ *to graph of* $f(x)$ *over interval* $[a, b]$ *is*

$$\int_a^b f(x) - g(x) dx.$$

The area between the graphs over interval $[a, b]$ *is*

$$\int_a^b |f(x) - g(x)| dx.$$

Example 2.1.1. *Find the area between the graphs of* $f(x) = x^2 - 5x - 7$ *and* $g(x) = x - 12$ *over* $[-2, 5]$.

Solution. **Step 1. Sketch the region.** (You can skip this step when you are familiar with this type of problems.)

Step 2. Find out the signs of $f(x) - g(x)$. Since

$$f(x) - g(x) = (x^2 - 5x - 7) - (x - 12) = x^2 - 6x + 5 = (x - 1)(x - 5),$$

$f(x) - g(x) > 0$ for $x \in (-2, 1)$ and < 0 for $x \in (1, 5)$.

Step 3. Compute the integral. We have

$$\begin{aligned}\int_{-2}^{5} |f(x) - g(x)| &= \int_{-2}^{1} f(x) - g(x)dx + \int_{1}^{5} g(x) - f(x)dx \\ &= \int_{-2}^{1} x^2 - 6x + 5dx - \int_{1}^{5} x^2 - 6x + 5dx \\ &= (\frac{1}{3}x^3 - 3x^2 + 5x)\Big|_{-2}^{1} - (\frac{1}{3}x^3 - 3x^2 + 5x)\Big|_{1}^{5} \\ &= (\frac{7}{3} - \frac{(-74)}{3}) - (\frac{-7}{3} - \frac{25}{3}) = \frac{113}{3}.\end{aligned}$$

□

Example 2.1.2. *Find the area of the region bounded by the graphs of $y = \frac{8}{x^2}$ (with positive x), $y = 8x$ and $y = x$.*

Solution. Let us find the intersection of graphs. Suppose $f_1 = \frac{8}{x^2}$, $f_2 = 8x$, $f_3 = x$. The intersection of f_1, f_2 is given by

$$f_1 = f_2 \quad \Longrightarrow \quad x = 1.$$

The intersection of f_1, f_3 is

$$f_1 = f_3 \quad \Longrightarrow \quad x = 2.$$

Finally

$$f_2 = f_3 \quad \Longrightarrow \quad x = 0.$$

When $x \in (0, 1)$, $f_1 \geq f_2 \geq f_3$ and when $x \in (1, 2)$, $f_2 \geq f_1 \geq f_3$. Therefore the area equals

$$\begin{aligned}&\int_{0}^{1} f_2 - f_3 dx + \int_{1}^{2} f_1 - f_3 \\ =& \int_{0}^{1} 8x - xdx + \int_{1}^{2} \frac{8}{x^2} - xdx \\ =& \int_{0}^{1} 7xdx + \int_{1}^{2} \frac{8}{x^2} - xdx \\ =& (\frac{7}{2}x^2)\Big|_{0}^{1} + (-\frac{8}{x} - \frac{x^2}{2})\Big|_{1}^{2} \\ =& \frac{7}{2} + \frac{5}{2} = 6.\end{aligned}$$

□

We can also do integration along y-axis. Then we need to rewrite the curves as functions of y.

Example 2.1.3. *Find the area of the region (in $\{x > 0, y > 0\}$) that is bounded by $y = x^2$, $y = (x-2)^2$ and $x = 0$.*

Solution. Let us rewrite the curves as functions of y in the region $y \le 1$ as follows

$$f_1(y) = \sqrt{y}, \quad f_2(y) = 2 - \sqrt{y}.$$

Set $f_1(y) = 0, f_2(y) = 0$ respectively and then we get $y = 0, 4$. We find $y \in (0, 4)$.

The intersection of the two curves is given by

$$\sqrt{y} = 2 - \sqrt{y} \quad \Longrightarrow \quad y = 1.$$

For $y \in (0, 1)$, the region is given by those point in between the graphs of $x = 0, x = f_1$. For $y \in (1, 4)$, the region is given by $x = 0, x = f_2$. Thus the area equals

$$\begin{aligned}
&\int_0^1 f_1(y) - 0\, dy + \int_1^4 f_2(y) - 0\, dy \\
&= \int_0^1 \sqrt{y}\, dy + \int_1^4 2 - \sqrt{y}\, dy \\
&= (\frac{2}{3} y^{3/2})\Big|_0^1 + (2y - \frac{2}{3} y^{3/2})\Big|_1^4 \\
&= \frac{2}{3} + (8 - \frac{16}{3} - 2 + \frac{2}{3}) \\
&= 2.
\end{aligned}$$

□

2.2 Volume, Density

Let us compute the volume of a solid body in $\mathbb{R}^3$. Suppose that a solid body extends from height $y = a$ to $y = b$. Let us cut the solid body with a hyperplane $y = y_0$, then we get a **horizontal cross section** and we denote the area of the section as $A(y)$. Then we have the following formula:

Theorem 2.2.

$$\textit{The volume of the solid body (given above)} = \int_a^b A(y) dy.$$

Example 2.2.1. ***Volume of a Pyramid.*** *Calculate the volume V of a pyramid of height $12m$ whise base is a square of side $4m$.*

Solution. **Step 1. Find $A(y)$.** Consider a horizontal cross section at height y. It is a square denoted by S_y and suppose the length of its side is s. Apply the law of similar triangles to the triangle given by one side of S_y and the top point. See Figure 3 on page 367 of the textbook. We find

$$\text{the proportion of the base } = \frac{s}{4} = \frac{12-y}{y}.$$

This shows that $s = \frac{1}{3}(12-y)$ and therefore

$$A(y) = \text{ the area of } S_y = s^2 = \frac{1}{9}(12-y)^2.$$

Step 2. Compute V.

$$V = \int_0^{12} A(y)dy = \int_0^{12} \frac{1}{9}(12-y)dy = -\frac{1}{27}(12-y)^3\Big|_0^{12} = 64.$$

□

Remark 2.3. Suppose a pyramid of base area A and heigh h, the volume formula is

$$V = \frac{1}{3}Ah.$$

Example 2.2.2. ***Volume of a Sphere.*** *Compute the volume of a sphere of radius R.*

Solution. Place the sphere centered at the origin. Let $A(y)$ denotes the area of the horizontal cross section. See Figure 5 on page 367. Then y is from $-R$ to R. For each such y, the section is a circle with radius r such that

$$r^2 + y^2 = R^2 \quad \text{ and so } r = \sqrt{R^2 - y^2}.$$

Then

$$A(y) = \pi r^2 = \pi(R^2 - y^2).$$

Therefore the volume of the sphere

$$\int_{-R}^{R} \pi(R^2 - y^2)dy = \pi(R^2 y - \frac{y^3}{3})\Big|_{-R}^{R} = \frac{4}{3}\pi R^3.$$

□

Example 2.2.3. *The population of one city and its surrounding suburbs has radial density function $\rho(r) = 15(1+r^2)^{-1/2}$, where r is the distance from the city center in kilometers and ρ has units of thousands per square kilometer. How many people live in the ring between 10 and 30 km from the city center?*

Solution. Knowing the density of population, say ρ, then the population equals the integration of the density. Suppose A (in dimension 2) is the region and the the population equals

$$\int_A \rho dx dy.$$

We have the following formula where (r, θ) is the polar coordinates in dimension 2:

$$dxdy = rdrd\theta.$$

So the population is

$$\begin{aligned}
\int_0^{2\pi}(\int_{10}^{30} 15(1+r^2)^{-1/2}\, rdr)d\theta &= 30\pi\int_{10}^{30}\frac{rdr}{(1+r^2)^{1/2}} \\
&= 15\pi\int_{10}^{30}\frac{dr^2}{(1+r^2)^{1/2}} \\
&= 15\pi\int_{100}^{900}\frac{du}{(1+u)^{1/2}} \qquad (u=r^2) \\
&= 30\pi(1+u)^{1/2}\Big|_{100}^{900} \\
&= 30\pi(\sqrt{901}-\sqrt{101}).
\end{aligned}$$

□

2.3 Volumes of Revolution

A **solid of revolution** is a solid obtained by rotating a 2-dimensional region about an axis in 3-dimensional space.

Theorem 2.4. *If $f \geq 0$ on $[a, b]$, then the solid obtained by rotating the region under the graph about the x-axis has volume*

$$V = \pi\int_a^b f(x)^2dx.$$

Theorem 2.5. *If $f \geq g \geq 0$ on $[a, b]$, then the solid obtained by rotating the region between the graphs f, g about the x-axis has volume*

$$V = \pi\int_a^b f(x)^2 - g(x)^2dx.$$

Example 2.3.1. *(Region between two curves) Find the volume V obtained by revolving the region between $y = x^2 + 4$ and $y = 2$ about the x-axis for $1 \leq x \leq 3$.*

Solution. The volume of the solid given by rotating the region under $y = x^2 + 4$ is

$$V_1 = \pi\int_1^3 (x^2+4)^2dx.$$

The volume of the solid given by rotating the region under $y = 2$ is

$$V_2 = \pi\int_1^3 2^2dx.$$

The volume wanted is the difference of the above two volumes:

$$V = V_1 - V_2 = \pi \int_1^3 (x^2+4)^2 - 2^2 dx = \pi \int_1^3 (x^4 + 8x^2 + 12) dx = \frac{2126}{15}\pi.$$

□

In the next example, we calculate a volume of revolution about a vertical line that is parallel to the y-axis.

Example 2.3.2. *Find the volume of the solid obtained by rotating the region under the graph of $f(x) = 9 - x^2$ for $0 \le x \le 3$ about the vertical axis $x = -2$.*

Solution. The solid can be viewed as the difference rotating two graphs about $x = -2$. Let us write the inner and outer radii as R_1 and R_2 (note that the center is at $x = -2$). Since $y = f(x) = 9 - x^2$, we have $x = \sqrt{9-y}$. Because the center is at $x = -2$, then

$$R_1 = \sqrt{9-y} + 2.$$

Since the region accounts for $0 \le x \le 3$,

$$R_2 = 2.$$

When $0 \le x \le 3$, from the picture of the graphs, y is from 0 to 9. So the volume

$$V = \pi \int_0^9 R_1^2 - R_2^2 dy = \pi \int_0^9 (9 - y + 4\sqrt{9-y}) dy$$
$$= \pi (9y - \frac{1}{2}y^2 - \frac{8}{3}(9-y)^{3/2})\Big|_0^9 = \frac{225}{2}\pi.$$

□

3 Techniques of integration

3.1 Integration by parts

In this section, we give a formula that often allows us to convert an integral to another.

Theorem 3.1. *Let u, v be (differentiable) functions of x.*

$$\int u dv = uv - \int v du.$$

Proof. Notice by the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

So after integrating both sides,

$$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}.$$

The formula follows by moving the second integral on the right to the other side. □

Example 3.1.1. *Evaluate* $\int \ln x dx$.

Solution. View x as v and $\ln x$ as u. Then

$$\begin{aligned}\int \ln x dx &= (\ln x)x - \int x d \ln x \\ &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - x + C.\end{aligned}$$

□

Sometimes we need to do integration by parts more than once.

Example 3.1.2. *Evaluate* $\int x^2 \sin x \, dx$.

Solution. Since $\sin x \, dx = d(-\cos x)$, view x^2 as u and $-\cos x$ as v. Then

$$\begin{aligned}\int x^2 \sin x \, dx &= \int x^2 d(-\cos x) \\ &= -x^2 \cos x - \int (-\cos x) dx^2 \\ &= -x^2 \cos x + \int 2x \cos x \, dx.\end{aligned}$$

Let us do integration by parts once again for $\int 2x \cos x \, dx$.

$$\begin{aligned}\int 2x \cos x \, dx &= \int 2x \, d \sin x \\ &= 2x \sin x - \int \sin x d(2x) \\ &= 2x \sin x - 2 \int \sin x dx \\ &= 2x \sin x + 2 \cos x + C.\end{aligned}$$

In all we have

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

□

The definite integral version of the integration by parts:

Theorem 3.2.

$$\int_a^b u \, dv = uv \,\Big|_b^a - \int_a^b v \, du.$$

Example 3.1.3. *Evaluate* $\int_0^\pi e^x \cos x\, dx$.

Solution.

$$\begin{aligned}\int_0^\pi e^x \cos x\, dx &= \int_0^\pi e^x\, d\sin x \\ &= e^x \sin x\Big|_0^\pi - \int_0^\pi \sin x\, de^x \\ &= -\int_0^\pi e^x \sin x\, dx.\end{aligned}$$

If we do integration by parts once again, we get the above

$$\begin{aligned}&= \int_0^\pi e^x d\cos x \\ &= e^x \cos x\Big|_0^\pi - \int_0^\pi \cos x\, de^x \\ &= -e^\pi - 1 - \int_0^\pi e^x \sin x\, dx.\end{aligned}$$

In all we obtained

$$\int_0^\pi e^x \sin x\, dx = -e^\pi - 1 - \int_0^\pi e^x \sin x\, dx,$$

which implies

$$\int_0^\pi e^x \sin x\, dx = -\frac{1}{2}e^\pi - \frac{1}{2}.$$

□

3.2 Polar Coordinates

The rectangular coordinates describe the position of one point on the $x - y$ plane. For example if a point P has coordinate (x_1, y_1), it means that x_1 is the projection of P onto the x-axis and y_1 is the projection of P onto the y-axis.

There is another way to describe the position of P. In polar coordinates, we label it by coordinates (r, θ), meaning that the distance from P to the origin is $r(= \overline{\mathrm{OP}})$ and the angle between $\overline{\mathrm{OP}}$ and the x-axis is θ.

Theorem 3.3. *Suppose a point has (x, y) rectangular coordinate and (r, θ) polar coordinate. Then polar to rectangular coordinates:*

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and rectangular to polar

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Here we do not write $\theta = \arctan \frac{y}{x}$. Because by convention, the range of $\arctan$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$. We also remark that (r, θ) and $(r, \theta + 2\pi k)$ with $k \in \mathbb{Z}$ label the same point.

We commonly choose $r \geq 0$ and $\theta \in [0, 2\pi)$. When $r > 0$,

$$\begin{aligned}
&\theta = \arctan \frac{y}{x} && \text{if the point lies in the first quadrant,}\\
&\theta = \arctan \frac{y}{x} + \pi && \text{if the point lies in the second or third quadrant,}\\
&\theta = \arctan \frac{y}{x} + 2\pi && \text{if the point lies in the fourth quadrant,}\\
&\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} && \text{if } x = 0.
\end{aligned}$$

Example 3.2.1. *Find the rectangular coordinate of point Q which has polar coordinate $(3, \frac{5\pi}{6})$. Find polar coordinate for the point W which has a rectangular coordinate $(3, 2)$.*

Solution. Since the polar coordinate of Q is $(r, \theta) = (3, \frac{5\pi}{6})$, so the rectangular coordinate is

$$(x, y) = (r \cos \theta, r \sin \theta) = (-\frac{3\sqrt{3}}{2}, \frac{3}{2}).$$

As for W, since $(x, y) = (3, 2)$ lies in the first quadrant, we have

$$r = \sqrt{x^2 + y^2} = \sqrt{13} \approx 3.6,$$
$$\theta = \arctan(2/3) \approx 0.588.$$

So the polar coordinate for W is $(\sqrt{13}, \arctan(2/3)) \approx (3.6,\ 0.588)$. □

By convention, we allow negative radial coordinates (though not common!). By definition

$$(-r, \theta) \text{ is the reflection of } (r, \theta) \text{ through the origin.}$$

Hence $(-r, \theta)$ and $(r, \theta + \pi)$ represent the same point.

Example 3.2.2. *Find two polar representations of $P = (-1, 1)$, one with $r > 0$ and one with $r < 0$.*

Solution. Let (r, θ) be one polar coordinate of P. Then

$$r^2 = 2 \quad \text{and} \quad \tan \theta = \frac{y}{x} = -1.$$

First consider $r > 0$, then $r = \sqrt{2}$. Since P is in the second quadrant, the correct angle is

$$\theta = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}.$$

If we wish to use the negative radial coordinate $r = -\sqrt{2}$, then the angle becomes $\frac{3\pi}{4} + \pi = \frac{7\pi}{4}$. Thus

$$P = (\sqrt{2}, \frac{3\pi}{4}) \quad \text{or} \quad (-\sqrt{2}, \frac{7\pi}{4}).$$

□

Example 3.2.3. *Convert the following to an equation in polar coordinates of the form $r = f(\theta)$:*

1. $xy = 1$;
2. *the line whose point closest to the origin is $P_0 = (d, \alpha)$ in polar coordinate.*

Solution. For 1, since $x = r\cos\theta$, $y = r\sin\theta$, we have

$$r^2 \cos\theta \sin\theta = 1.$$

Therefore

$$r = \sqrt{\frac{1}{\cos\theta\sin\theta}}.$$

We also need

$$\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$$

to be positive. Hence

$$\theta \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}).$$

Now consider 2. Consider $P = (r, \theta)$ to be any point on the line. Then ΔOPP_0 is a right triangle. Therefore

$$\frac{d}{r} = \cos(\theta - \alpha)$$

or

$$r = \frac{d}{\cos(\theta - \alpha)} = d\sec(\theta - \alpha).$$

From the picture, θ belongs to $(-\frac{\pi}{2} + \alpha, \frac{\pi}{2} + \alpha)$. □

Example 3.2.4. *Convert to rectangular coordinates and identify the curve with polar equation $r = 2a\cos\theta$ where a is a positive constant.*

Solution. Use the relation

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad x = r\cos\theta.$$

We get

$$\sqrt{x^2 + y^2} = 2ax/r$$

which gives

$$x^2 + y^2 = 2ax.$$

It can be rewritten into

$$(x - a)^2 + y^2 = a^2.$$

This is the equation of the circle of radius a and center $(a, 0)$. □

3.3 Area in Polar Coordinates

Theorem 3.4. *Let f be a continuous function. The area bounded by a curve in polar form $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ (with $\alpha < \beta$) is equal to*

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 d\theta.$$

Example 3.3.1. *Sketch $r = \sin 3\theta$ and compute the area of one "petal".*

Solution. To sketch the curve, we first graph $r = \sin 3\theta$ in r versus θ rectangular coordinates (see figure 5 on page 647 of the textbook). By periodicity, we only need to look at $\theta \in [0, 2\pi)$. Also since $r(\theta) = -r(\theta + \pi)$, we only need to look at $\theta \in [0, \pi)$.

We know that r varies from 0 to 1 and back to 0 as θ increases from 0 to $\frac{\pi}{3}$. This gives one petal. When θ increase from $\frac{\pi}{3}$ to $\frac{2\pi}{3}$, $r \leq 0$. We shift the angle by π if we use positive radius. This gives that r varies from 0 to 1 and back to 0 as θ increases from $\frac{4\pi}{3}$ to $\frac{5\pi}{3}$. Lastly when θ increases from $\frac{2\pi}{3}$ to π, r varies from 0 to 1 and back to 0.

The area is

$$\frac{1}{2}\int_{0}^{\pi/3} (\sin 3\theta)^2 d\theta = \frac{1}{2}\int_{0}^{\pi/3} \left(\frac{1-\cos 6\theta}{2}\right) d\theta = \left(\frac{1}{4}\theta - \frac{1}{24}\sin 6\theta\right)\Big|_0^{\pi/3} = \frac{\pi}{12}.$$

□

Example 3.3.2. *Find the area of the region inside the circle $r = 2\cos\theta$ but outside the circle $r = 1$.*

Solution. The two circles intersect at the points where $2\cos\theta = 1$, which gives $\cos\theta = \frac{1}{2}$ and so $\theta = \pm\frac{\pi}{3}$.

From the picture (see figure 7 on page 648 of the textbook) that the target region can be viewed as the difference of two "sectors".

$$\begin{aligned}
\text{the area } &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3} (1)^2 d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4(\cos\theta)^2 - 1\, d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos(2\theta) + 1\, d\theta \\
&= \frac{1}{2}(2\sin\theta + \theta)\Big|_{-\pi/3}^{\pi/3} \\
&= \frac{\sqrt{3}}{2} + \frac{\pi}{3}.
\end{aligned}$$

□

4 Review for Midterm 1

Velocity and Displacement.

Example 4.0.1. *Find the distance travelled and displacement from $t = 1$ to 4 for an object with velocity $8t - t^2$.*

Integration method.

Example 4.0.2.

$$\int \cos x \cos(\sin x) dx, \int x5^x dx, \int_0^2 \frac{dx}{\sqrt{2x+5}}.$$

Area and Volume.

Example 4.0.3. *Sketch the region enclosed by the curves and compute its area as an integral.*

$$y = x\sqrt{x-2},\ y = -x\sqrt{x-2},\ x = 4.$$

Solution. Note by the given expression, $x \geq 2$. So x is from 2 to 4.

$$\begin{aligned} \text{area } &= \int_2^4 x\sqrt{x-2} - (-x\sqrt{x-2}) dx \\ &= 2\int_2^4 x\sqrt{x-2}\, dx \end{aligned}$$

Set $t := \sqrt{x-2}$ and then $x = t^2 + 2$, $dx = 2tdt$. The integration

$$\begin{aligned} &= 2\int_0^{\sqrt{2}} (t^2+2)t \times 2t\, dt \\ &= 4\int_0^{\sqrt{2}} t^4 + 2t^2\, dt \\ &= 4\left(\frac{t^5}{5} + \frac{2t^3}{3}\right)\Big|_0^{\sqrt{2}} \\ &= \frac{128}{15}\sqrt{2}. \end{aligned}$$

□

Example 4.0.4. *Find the volume V of the solid whose base is the circle $x^2 + y^2 = 16$ and whose cross sections perpendicular to the x-axis are triangles whose height and base are equal.*

Solution. For each hyperplane location $x = a$, it intersects with the solid and the cross section is a triangle with bottom side of length $2y = 2\sqrt{16 - a^2}$. The height of the triangle is the same with the bottom side.

Therefore, the volume of the solid is

$$\begin{aligned}\int_{-4}^{4}\frac{1}{2}(2y)^2dx &= \int_{-4}^{4}\frac{1}{2}2(16-x^2)dx\\ &= 2\int_{-4}^{4}(16-x^2)dx\\ &= 512/3.\end{aligned}$$

□

Example 4.0.5. *A plane inclined at an angle of* 45 *degree passes through a diameter of the base of a cylinder of radius r. Find the volume of the region with the cylinder and below the plane (see Figure 23 on page 374 of the textbook).*

Solution. Let h be the height. When $h = 0$, the cross section is a half disc. When $h > 0$, the cross section is the smaller part of the intersection of a disc and a straight line which is h away from its center. The area of the cross section is the difference of sector of angle $2\arccos\frac{h}{r}$ and a triangle:

$$\pi r^2\frac{2\arccos(h/r)}{2\pi} - \frac{1}{2}h(2\sqrt{r^2-h^2}).$$

Therefore the volume

$$\begin{aligned}&\int_0^r r^2\arccos(h/r) - h\sqrt{r^2-h^2}\,dh\\ &= r^3\int_0^r \arccos(h/r)d(h/r) - \frac{1}{2}\int_0^r\sqrt{r^2-h^2}\,dh^2\\ &= r^3\int_0^1 \arccos x dx - \frac{1}{2}\int_0^{r^2}\sqrt{r^2-y}\,dy\\ &= r^3(x\arccos x - \sqrt{1-x^2})\Big|_0^1 + \frac{1}{3}(r^2-y)^{3/2}\Big|_0^{r^2}\\ &= r^3 - \frac{1}{3}r^3\\ &= \frac{2}{3}r^3.\end{aligned}$$

□

Example 4.0.6. *The solid* S *(see Figure 24 on page 375 of the textbook) is the intersection of two cylinders of radius* r *whose axes are perpendicular.*

(a) The horizontal cross section of each cylinder at distance y *from the central axis is a rectangular strip. Find the strip's width.*

(b) Find the area of the horizontal cross section of S *at distance* y*.*

(c) Find the volume of S as a function of r.

Solution. The strip's side is given by the intersection of a disc of radius r (the base of the cylinder) and a straight line which is y away from the disc's center. Thus the width is $2\sqrt{r^2-y^2}$.

The horizontal cross section of a cylinder (placed in a way that its bottom is perpendicular to the horizontal plane) is a rectangle. And therefore the horizontal cross section of S is a square of side length $2\sqrt{r^2-y^2}$. Thus the area is $4(r^2-y^2)$.

With the above information, the volume is

$$\int_{-r}^{r} 4(r^2-y^2)dy = 8r^3 - \frac{4}{3}y^3\Big|_{-r}^{r} = \frac{16}{3}r^3.$$

□

Example 4.0.7. *Find the volume of the solid obtained by rotating the region enclosed by the graphs about the given axis.*

$$y = 2\sqrt{x},\ y = x,\ \textit{about } x = -20.$$

Solution. Let us find the intersection of the two curves by setting $2\sqrt{x} = x$. We get the two curves intersect at $x = 0$ and $x = 4$. Rotating the region enclosed by $y = 2\sqrt{x}$ and $y = x$ about $x = -20$ produces a solid whose cross sections are washers with outer radius $R = y - (-20) = y + 20$ and inner radius $r = \frac{1}{4}y^2 - (-20) = \frac{1}{4}y^2 + 20$. The volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^4 \left((y+20)^2 - (\frac{1}{4}y^2+20)^2 \right) dy \\ &= \pi \int_0^4 \left(0 + 40y - 9y^2 - \frac{1}{16}y^4 \right) dy \\ &= \pi \left(20y^2 - 3y3 - \frac{1}{80}y^5 \right) \Big|_0^4 \\ &= \frac{576}{5}\pi. \end{aligned}$$

□

Example 4.0.8. *The torus (see Figure 15 on page 384 of the textbook) is obtained by rotating the circle $(x-a)^2 + y^2 = b^2$ around y-axis (assume $a > b > 0$). Show that it has volume $2\pi^2ab^2$.*

Solution. In order to find out the outer and inner radius, we find the distance from the circle to the y-axis. For each y, the inner radius equals $a - \frac{1}{2}s$ where s is the length of the intersection of the circle and a straight line y away from its center. While the outer radius is $a + \frac{1}{2}s$. Thus the volume

$$V = \pi \int_{-b}^{b} (a + \frac{1}{2}s)^2 - (a - \frac{1}{2}s)^2 dy = 2a\pi \int_{-b}^{b} s dy.$$

Notice $\int_{-b}^{b} s dy$ is exactly the area of a circle of radius b. Therefore

$$V = 2a\pi \times \pi b^2 = 2\pi^2 ab^2.$$

Alternatively, we can compute $s = 2\sqrt{b^2 - y^2}$ and do the integration by considering the substitution $y = b \sin\theta$. $\square$

Example 4.0.9 (exercise 20 on textbook pate 650)**.** *Find the area between* $r = 2 + \sin 2\theta$ *and* $r = \sin 2\theta$.

Solution. [Sketch] Think about why the two curves attach at two points in the picture?

$\theta \in (0, \pi/2)$:

$$S_1 = \frac{1}{2}\int_0^{\pi/2} (2 + \sin 2\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/2} (\sin 2\theta)^2 d\theta.$$

$\theta \in (\pi/2, \pi)$:

$$S_2 = \frac{1}{2}\int_{\pi/2}^{\pi} (2 + \sin 2\theta)^2 d\theta - \frac{1}{2}\int_{\pi/2}^{\pi} (\sin 2(\theta + \pi))^2 d\theta.$$

The area $= 2S_1 + 2S_2$. $\square$

Example 4.0.10. *Set up, but do not evaluate, the area of the region between the inner and outer loop given by* $r = 2\cos\theta - 1$. *See Figure 19 on textbook page 650.*

Solution. What is the graph of $r(\theta)$? Which parts translates to the which parts of the graph? What happens when $r < 0$? This answer is:

$$\int_{\pi/3}^{\pi} (2\cos\theta - 1)^2 d\theta - \int_0^{\pi/3} (2\cos\theta - 1)^2 d\theta.$$

Think about why? $\square$

5 Supplement

5.1 Complex Numbers

Definition 5.1. A complex number is a number of the form $a + bi$, where a and b are real numbers and $i^2 = -1$. Here a is called the real part and b is called the imaginary part.

Remark 5.2. A real number is also a complex number (with 0 imaginary part).

Here are some properties: if $\alpha = a + bi$ and $\beta = c + di$ are complex numbers, then

$$\begin{aligned} \alpha + \beta &= (a + c) + (b + d)i, \\ \alpha - \beta &= (a - c) + (b - d)i, \\ \alpha\beta &= (ac - bd) + (ad + bc)i. \end{aligned}$$

The last property is true, because

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 \quad \text{and} \quad i^2 = -1.$$

Example 5.1.1. *Let α, β be as the above. What is the real part and the imaginary part of $\frac{\alpha}{\beta}$?*

Solution. It follows from the direct computation

$$\begin{aligned}\frac{a+bi}{c+di} &= \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right) \\ &= \frac{ac+adi+bci+bdi^2}{c^2+d^2i^2} \\ &= \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{-ad+bc}{c^2+d^2}\right)i.\end{aligned}$$

Thus the real part is $\frac{ac+bd}{c^2+d^2}$ and the imaginary part is $\frac{-ad+bc}{c^2+d^2}$. □

Definition 5.3. Let $\alpha = a + bi$ be a complex number.

(a) The complex conjugate of α is the complex number $\bar{\alpha} = a - bi$.

(b) The magnitude of α, written $|\alpha|$, is given by $|\alpha| = \sqrt{\alpha\bar{\alpha}} = \sqrt{a^2 + b^2}$.

(c) $|\alpha|$ is also called the modulus, length or absolute value of α.

Theorem 5.4. *Let α and β be complex variables. Then,*

$$\overline{(\alpha \pm \beta)} = \bar{\alpha} \pm \bar{\beta}, \quad \overline{(\alpha\beta)} = \bar{\alpha}\bar{\beta}, \quad \overline{\left(\frac{\alpha}{\beta}\right)} = \frac{\bar{\alpha}}{\bar{\beta}}.$$

A complex variable $z = x + iy$ is represented geometrically by a point (x, y) in the plane. Any point in the plane can be represented in polar coordinates (r, θ). Thus, we can write z as

$$z = r[\cos(\theta) + i\sin(\theta)].$$

This is called the polar form of z. Note that $|z| = r$. The angle θ is called the argument of z, written $\theta = arg(z)$. It is important to realize that $\theta = arg(z)$ is not uniquely determined.

Theorem 5.5. *Suppose for $j = 1, 2$,*

$$z_j = r_j[\cos(\theta_j) + i\sin(\theta_j)].$$

Then

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

Example 5.1.2. *Let $\alpha = 1 + i$. By what angle will multiplication by $\beta = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ rotate α? Compute $\alpha\beta$.*

Solution. Since
$$\beta = -\frac{\sqrt{3}}{2} + \frac{1}{2}i = \cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6}),$$
thus multiplication by β produces a rotation by $\frac{5\pi}{6}$. Also since
$$\alpha = \sqrt{2}[\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})],$$
we obtain
$$\alpha\beta = |\alpha||\beta|[\cos(\frac{\pi}{4} + \frac{5\pi}{6}) + i\sin(\frac{\pi}{4} + \frac{5\pi}{6})] = \sqrt{2}[\cos(\frac{13\pi}{12}) + i\sin(\frac{13\pi}{12})].$$
□

Theorem 5.6 (de Moivre's Theorem)**.** *Let $z = r[\cos(\theta) + i\sin(\theta)]$ be a complex number in polar form and let n be an integer. Then,*
$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Example 5.1.3. *Compute $(1+\sqrt{3}i)^8$ and write the result in standard $(a+bi)$ form.*

Solution. Notice
$$1 + \sqrt{3}i = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 2[\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})].$$
Therefore
$$\begin{aligned}(1+\sqrt{3}i)^8 &= 2^8[\cos(\frac{8\pi}{3}) + i\sin(\frac{8\pi}{3})]\\ &= 256(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)\\ &= -128 + 128\sqrt{3}i.\end{aligned}$$
□

Example 5.1.4. *Find the sixth root of -64.*

Solution. Since
$$-64 = 2^6[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)],$$
the 6^{th} roots are given by
$$w_k = 2\left[\cos\left(\frac{\pi + 2\pi k}{6}\right) + i\sin\left(\frac{\pi + 2\pi k}{6}\right)\right], \quad k = 0, 1, 2, 3, 4, 5.$$
□

5.2 Complex Exponential

Definition 5.7. Complex Exponential.

$$e^{a+bi} = e^a[\cos b + i\sin b].$$

Why this definition? We are going to provide one answer after learning Taylor series. In particular, we have the well-known Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{i\pi} = 1.$$

Theorem 5.8. *Let z_1, z_2 be complex numbers. Then*

$$e^{z_1}e^{z_2} = e^{z_1+z_2}.$$

Example 5.2.1. *Show*

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}.$$

Proof. Notice

$$e^{ix} = \cos x + i\sin x, \quad e^{-ix} = \cos x - i\sin x.$$

Adding and dividing by 2 gives us $\cos(x)$ whereas subtracting and dividing by $2i$ gives us $\sin(x)$. □

This expression can be used to define $\cos z$, $\sin z$ for complex z.

Example 5.2.2. *Use Theorem 5.8 to show*

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

Proof. Consider $e^{i\alpha}, e^{i\beta}$. Then

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta},$$

which implies

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos(\alpha)+i\sin(\alpha)) \times (\cos(\beta)+i\sin(\beta)).$$

The RHS of the above equals

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).$$

To have the RHS = the LHS, we need the corresponding real parts equal and imaginary parts equal. Thus

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

□

Remark 5.9. The proof of example 5.2.2 also implies

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

If we take $\alpha = \beta$, we get

$$\sin 2\alpha = 2\sin\alpha\sin\beta,$$
$$\cos 2\alpha = (\cos\alpha)^2 - (\sin\alpha)^2 = 2\cos^2\alpha - 1.$$

5.3 Trig integration

Complex exponential method.

Example 5.3.1. *Let us integrate*

$$8\cos(3x)\sin(x).$$

Solution. We have

$$\begin{aligned} 8\cos(3x)\sin(x) &= 8\left(\frac{e^{3ix}+e^{-3ix}}{2}\right)\left(\frac{e^{ix}-e^{-ix}}{2i}\right) \\ &= \frac{2}{i}\left(e^{4ix}+e^{-2ix}-e^{2ix}-e^{-4ix}\right). \end{aligned}$$

After integration,

$$\begin{aligned} \int 8\cos(3x)\sin(x)dx &= \int \frac{2}{i}\left(e^{4ix}+e^{-2ix}-e^{2ix}-e^{-4ix}\right)dx \\ &= \frac{2}{i}\left(\frac{e^{4ix}}{4i}-\frac{e^{-2ix}}{2i}-\frac{e^{2ix}}{2i}+\frac{e^{-4ix}}{4i}\right)+C \\ &= -\left(\frac{e^{4ix}}{2}+\frac{e^{-4ix}}{2}\right)+e^{-2ix}+e^{2ix}+C \\ &= -\cos(4x)+2\cos(2x)+C. \end{aligned}$$

□

Example 5.3.2. *Let us integrate* $e^{2x}\sin(x)$.

Solution.

$$\begin{aligned} \int e^{2x}\sin(x)dx &= \frac{1}{2i}\int e^{2x}(e^{ix}-e^{-ix})dx \\ &= \frac{1}{2i}\int e^{(2+i)x}-e^{(2-i)x})dx \\ &= \frac{1}{2i}\left(\frac{e^{(2+i)x}}{2+i}-\frac{e^{(2-i)x})}{2-i}\right)+C \\ &= -\frac{e^{2x}}{2}\left(\frac{e^{ix}}{1-2i}+\frac{e^{-ix}}{1+2i}\right)+C. \end{aligned}$$

Since the two expressions are conjugate to each other, the above equals,

$$\begin{aligned} &= -e^{2x}\,\mathrm{Re}\left(\frac{e^{ix}}{1-2i}\right)+C \\ &= -e^{2x}\,\mathrm{Re}\left(\frac{(1+2i)e^{ix}}{5}\right)+C \\ &= -\frac{e^{2x}}{5}\left(\cos(x)-2\sin(x)\right)+C. \end{aligned}$$

□

Substitution.
Sometimes doing substitution:

$$u = \sin x, \quad u = \cos x, \quad u = \tan x \quad etc.$$

helps solving an integration problem.

Example 5.3.3. *Evaluate*

$$\int \tan^3 x \sec^5 x \, dx.$$

Solution. Notice

$$\tan^3 x \sec^5 x = \frac{\sin^3 x}{\cos^8 x} = \frac{1 - \cos^2 x}{\cos^8 x} \sin x.$$

Let $u = \cos x$ and then $du = -\sin x dx$. Therefore

$$\begin{aligned}
\int \tan^3 x \sec^5 x dx &= \int \frac{1 - \cos^2 x}{\cos^8 x} \sin x \, dx \\
&= -\int \frac{1 - u^2}{u^8} du \\
&= -\int u^{-8} - u^{-6} du \\
&= \frac{u^{-7}}{7} - \frac{u^{-5}}{5} + C \\
&= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{6} + C.
\end{aligned}$$

□

Example 5.3.4. *Evaluate*

$$\int \tan^2 x \sec^4 x \, dx.$$

Solution. Notice

$$\tan^2 x \sec^4 x = \frac{\sin^2 x}{\cos^6 x}$$

and doing substitution for as in the previous example won't work. Instead, let us consider

$$u = \tan x \quad \text{and then } du = \frac{1}{\cos^2 x} dx.$$

Hence

$$\begin{aligned}\int \tan^2 x \sec^4 x\, dx &= \int \frac{\sin^2 x}{\cos^4 x} d\tan x\\ &= \int \frac{\sin^2 x(\sin^2 x + \cos^2 x)}{\cos^4 x} d\tan x\\ &= \int u^4 + u^2\, du\\ &= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C.\end{aligned}$$

□

Example 5.3.5. *Evaluate*

$$(1). \int \tan^3 x\, dx, \quad (2). \int \tan^4 x\, dx.$$

Solution. For (1). consider $u = \sin x$. Then

$$\begin{aligned}\int \tan^3 x\, dx &= \int \frac{\sin^3 x}{\cos^3 x}\frac{1}{\cos x} d\sin x\\ &= \int \frac{\sin^3 x}{(1-\sin^2 x)^2} d\sin x\\ &= \int \frac{u^3}{(1-u^2)^2} du.\end{aligned}$$

Then we apply the Partial fraction method (in Section 5.4) to get

$$\frac{u^3}{(1-u^2)^2} = \frac{u^3}{(u-1)^2(u+1)^2} = \frac{1}{2\,(u+1)} - \frac{1}{4\,(u+1)^2} + \frac{1}{2\,(u-1)} + \frac{1}{4\,(u-1)^2}.$$

After integration, we get

$$\begin{aligned}&\frac{1}{2}\ln|(u-1)(u+1)| + \frac{1}{4}\frac{1}{u+1} - \frac{1}{4}\frac{1}{u-1} + C\\ =& \frac{1}{2}\ln|1-u^2| + \frac{1}{2}\frac{1}{1-u^2}\\ =& -\ln|\sec x| + \frac{1}{2}\sec^2 x + C.\end{aligned}$$

As for (2), we let $v = \tan x$. Then

$$\begin{aligned}\int \tan^4 x\,dx &= \int \frac{\sin^4 x}{\cos^2 x}\,d\tan x\\ &= \int \frac{\sin^4 x}{\cos^2 x(\sin^2 x+\cos^2 x)}d\tan x\\ &= \int \frac{v^4}{v^2+1}dv = \int v^2 - 1 + \frac{1}{v^2+1}dv\\ &= \frac{v^3}{3} - v + \arctan v + C.\end{aligned}$$

The integral becomes:

$$\frac{1}{3}\tan^3(x) + x - \tan(x) + C.$$

□

Remark 5.10. We can also apply the following formula to Example 5.3.5:

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx.$$

Please do it as an exercise.

Example 5.3.6 (Integral of Secant)**.** *Derive the formula*

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C.$$

Solution. Let $u = \tan\frac{x}{2}$. Then

$$du = \frac{1}{2\cos^2(x/2)}dx = \frac{1}{2}(u^2+1)dx,$$

$$\sec x = \frac{1}{\cos^2(x/2) - \sin^2(x/2)} = \frac{u^2+1}{u^2-1}.$$

The integral becomes

$$\begin{aligned}\int \sec x\,dx &= \int \frac{u^2+1}{u^2-1}\frac{2}{u^2+1}du\\ &= \int \frac{1}{1-u} + \frac{1}{1+u}du\\ &= \ln\left|\frac{1+u}{1-u}\right| + C.\end{aligned}$$

After transferring from $\tan(x/2)$ to u, we can show the formula. I will leave it to you. :) □

5.4 Partial fractions

Theorem 5.11. ***[Fundamental Theorem of algebra]*** *A polynomial P of degree n is a function of the form*

$$P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x_n$$

where a_i are complex numbers and $a_n \neq 0$. Then $P(x)$ has n complex roots (counting multiplicity) and we can factor $P(x)$ as

$$P(x) = a_n (x - z_1)^{n_1} (x - z_2)^{n_2} ... (x - z_k)^{n_k},$$

where z_k are k distinct complex numbers and $\sum_i n_i = n$.

Now we consider a polynomial differentiate another polynomial and we study partial fractions.

Theorem 5.12. *Suppose that the n numbers $\alpha_1, ..., \alpha_n$ are pairwise distinct and that $P(x)$ is a polynomial with degree less than n. Then, there are constants $C_1, ..., C_n$ such that*

$$\frac{P(x)}{(x - \alpha_1)...(x - \alpha_n)} = \frac{C_1}{x - \alpha_1} + ... + \frac{C_n}{x - \alpha_n}. \tag{3}$$

To determine the constants $C_1, ..., C_n$, we carry out the following steps:

- Multiply both sides of (3) by $x - \alpha_j$ and then set $x = \alpha_j$. The left side will evaluate to a number Z_j.
- The right side evaluates to C_j, since the other terms have a factor of $x - \alpha_j$ which is 0 when $x = \alpha_j$. We conclude that $Z_j = C_j$.

Now for some illustrations, we consider the following example.

Example 5.4.1. *Lets expand $f(x) := \frac{x^2+2}{(x-1)(x+2)(x+3)}$ by partial fractions.*

Solution. By the theorem,

$$f(x) = \frac{x^2 + 2}{(x - 1)(x + 2)(x + 3)} = \frac{C_1}{x - 1} + \frac{C_2}{x + 2} + \frac{C_3}{x + 3}.$$

Multiply by $x - 1$ to eliminate the pole at $x = 1$ and get

$$\frac{x^2 + 2}{(x + 2)(x + 3)} = C_1 + \frac{C_2(x - 1)}{x + 2} + \frac{C_3(x - 1)}{x + 3}.$$

Set $x = 1$ and obtain

$$\frac{1 + 2}{(1 + 2)(1 + 3)} = C_1.$$

Similarly,

$$C_2 = \frac{x^2 + 2}{(x - 1)(x + 3)}|_{x=-2} = -2.$$

and

$$C_3 = \frac{x^2+2}{(x-1)(x+2)}|_{x=-3} = \frac{11}{4}.$$

We conclude that

$$f(x) = \frac{1}{4(x-1)} - \frac{2}{(x+2)} + \frac{11}{4(x+3)}.$$

□

A cultural aside is that the numbers C_1, C_2, C_3 are often called the **residues** of the poles at $1, -2, -3$, many of you will see them later in your career under that name. If we wish to find the antiderivatives of f from this, we immediately get

$$\int f(x)dx = \frac{1}{4}\ln|x-1| + 2\ln|x+2| + \frac{11}{4}\ln|x+3| + C.$$

Repeated roots.

When we have repeated root, each factor $(x-a)^n$ contributes the following sum of terms to the partial fraction decomposition

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + ... + \frac{A_n}{(x-a)^n}.$$

Let us apply the method to

$$f(x) = \frac{1}{(x-1)^2(x-3)}.$$

Solution. The partial fraction expansion is of the form

$$f(x) = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{x-3}.$$

We can find C quickly from

$$C = (x-3)f(x)|_{x=3} = \frac{1}{(3-1)^2} = \frac{1}{4}$$

and A from

$$A = (x-1)^2 f(x)|_{x=1} = \frac{1}{1-3} = -\frac{1}{2}.$$

We get

$$f(x) = -\frac{1}{2(x-1)^2} + \frac{B}{(x-1)} + \frac{1}{4(x-3)}.$$

Let us plug in a convenient value of x, say $x=0$ and obtain

$$f(0) = \frac{1}{(-1)^2(-3)} = -\frac{1}{2} - B - (\frac{1}{3})(\frac{1}{4})$$

We get

$$B = -\frac{1}{4}.$$

□

Quadratic factor.

Irreducible quadratic factors $(x^2 + ax + b)^N$ contributes the following sum of terms to the partial fraction decomposition

$$\frac{A_1x + B_1}{(x^2 + ax + b)} + \frac{A_2x + B_2}{(x^2 + ax + b)^2} + ... + \frac{A_Nx + B_N}{(x^2 + ax + b)^N}.$$

Example 5.4.2. *Evaluate*

$$\int \frac{4 - x}{x(x^2 + 2)^2} dx.$$

Solution. The partial fraction decomposition has the form

$$\frac{4 - x}{x(x^2 + 2)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2}.$$

Multiplying both side by x and then set $x = 0$, we get $A = 1$.

Then multiplying both side by $x(x^2 + 2)^2$, we get

$$\begin{aligned} 4 - x &= (x^2 + 2)^2 + (Bx + C)x(x^2 + 2) + (Dx + E)x \\ &= (1 + B)x^4 + Cx^3 + (4 + 2B + D)x^2 + (2C + E)x. \end{aligned}$$

Now equate the coefficients on the two sides gives

$$B = -1, \quad C = 0, \quad D = -2, \quad E = -1.$$

Thus

$$\begin{aligned} \int \frac{4 - x}{x(x^2 + 2)^2} dx &= \int \frac{dx}{x} - \int \frac{xdx}{x^2 + 2} - \int \frac{2x + 1}{(x^2 + 2)^2} dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2 + 2) - \int \frac{2x + 1}{(x^2 + 2)^2} dx. \end{aligned}$$

Seeing from example 5.5.2, we have

$$\int \frac{4 - x}{x(x^2 + 2)^2} dx = \ln|x| - \frac{1}{2}\ln(x^2 + 2) + \frac{1}{4}\frac{4 - x}{x^2 + 2} - \frac{1}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C.$$

□

5.5 Trig Substitution

In this section, let us use substitution with trigonometric function to integrate functions.

Example 5.5.1. *Evaluate*

$$\int \frac{1}{\sqrt{1 - x^2}} dx.$$

Solution. Set $x = \sin\theta$ and then

$$dx = \cos\theta d\theta, \quad \sqrt{1-x^2} = \cos\theta.$$

So

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{1}{\cos\theta}\cos\theta d\theta = \theta + C = \sin^{-1} x + C.$$

□

Example 5.5.2. *Solve*

$$\int \frac{2x+1}{(x^2+2)^2}dx.$$

Solution. Note

$$\int \frac{2x+1}{(x^2+2)^2}dx = \int \frac{2x}{(x^2+2)^2}dx + \int \frac{1}{(x^2+2)^2}dx =: A + B.$$

For A, we have

$$A = \int \frac{d(x^2+2)}{(x^2+2)^2} = -\frac{1}{(x^2+2)}.$$

As for B, we use the trigonometric substitution

$$x = \sqrt{2}\tan\theta.$$

And then

$$dx = \frac{\sqrt{2}}{\cos^2\theta}d\theta, \quad x^2 + 2 = 2\tan^2\theta + 2 = \frac{2}{\cos^2\theta}.$$

$$\begin{aligned} B &= \int \frac{\cos^4\theta}{4}\frac{\sqrt{2}}{\cos^2\theta}d\theta = \frac{\sqrt{2}}{4}\int \cos^2\theta\, d\theta \\ &= \frac{\sqrt{2}}{8}\int (1 + 2\cos 2\theta)\, d\theta \\ &= \frac{\sqrt{2}}{8}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{1}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{4}\frac{x}{x^2+2}) + C. \end{aligned}$$

Therefore

$$\int \frac{4-x}{x(x^2+2)^2}\, dx = \frac{1}{4}\frac{x-4}{x^2+2} + \frac{1}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C.$$

□

Summary of trig substitution.

$$
\begin{array}{lll}
\sqrt{a^2-x^2}, & \text{try } x = a\sin\theta & \text{and then } dx = a\cos\theta d\theta, \quad \sqrt{a^2-x^2} = a\cos\theta; \\
\sqrt{a^2+x^2}, & \text{try } x = a\tan\theta & \text{and then } dx = a\cos^{-2}\theta d\theta, \quad \sqrt{a^2+x^2} = a\cos^{-1}\theta; \\
\sqrt{x^2-a^2}, & \text{try } x = a\cos^{-1}\theta & \text{and then } dx = a\sin\theta\cos^{-2}\theta d\theta, \quad \sqrt{a^2-x^2} = a\tan\theta.
\end{array}
$$

Sometimes you might need to do the substitution for more than once.

Example 5.5.3. *Evaluate*

$$\int \frac{dx}{(x^2+2x+3)^{3/2}}.$$

Solution. Since

$$x^2+2x+3 = (x+1)^2+2,$$

let $u = x+1$ and we get

$$\int \frac{dx}{(x^2+2x+3)^{3/2}} = \int \frac{du}{(u^2+2)^{3/2}}.$$

Now we set

$$u = \sqrt{2}\tan\theta,$$

and then

$$du = \sqrt{2}\cos^{-2}\theta d\theta, \quad (x^2+2x+3)^{3/2} = (2\cos^{-2}\theta)^{3/2}.$$

The integration becomes

$$\begin{aligned}
\int \frac{du}{(u^2+2)^{3/2}} &= \int \frac{\sqrt{2}\cos^{-2}\theta}{(2\cos^{-2}\theta)^{3/2}} = \frac{1}{2}\int \cos\theta\, d\theta \\
&= \frac{1}{2}\sin\theta + C.
\end{aligned}$$

Use the fact that

$$\sin\theta = \sqrt{\frac{1}{1+1/\tan^2\theta}}.$$

We get the above

$$= \frac{u}{2\sqrt{u^2+2}} + C.$$

Convert to the original x variable, we obtain

$$\frac{x+1}{2\sqrt{x^2+2x+3}} + C.$$

□

5.6 Improper Integrals

For the first part of section 5.6, I refer students to the lecture notes of Prof. Caroline Moosmüller (see 2-14 note).

We want to consider integations of a function in unbounded domain, which is often referred to as **improper integrals**.

Definition 5.13. Let a be a real number. The improper integral of f over $[a, \infty)$ is defined as the following limit (if it exists):

$$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx.$$

We say that the improper integral **converges** if the limit exists and it **diverges** if the limit does not exist.

Similarly we can define

$$\int_{-\infty}^a f(x)dx = \lim_{R\to-\infty} \int_R^a f(x)dx.$$

Example 5.6.1. *Show that $\int_2^\infty \frac{dx}{x^{1.1}}$ converges and compute its value. Show that $\int_{100}^\infty \frac{dx}{x}$ diverges.*

In the following theorem, we consider the integration with integrand $= x^{-p}$ in a unbounded region away from 0.

Theorem 5.14. *For $a > 0$,*

$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{p-1} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

Example 5.6.2. *Determine if $\int_{-\infty}^\infty \frac{1}{1+x^2}dx$ converges and, if so, compute its value.*

Example 5.6.3. *Use L'Hopital's rule to calculate $\int_0^\infty xe^{-x}dx$.*

Example 5.6.4. ***Escape Velocity.*** *The earth exerts a gravitational force of magnitude $F(r) = GM_e m/r^2$ on an object of mass m at distance r from the center of the earth.*

(a) Find the work required to move the object infinitely far from the earth.

(b) Calculate the escape velocity v_{esc} on the earth's surface.

5.6.1 Unbounded Functions

An integral over a finite interval for a unbounded integrand is also **improper**.

Definition 5.15. If f is continuous on $[a, b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$, we define

$$\int_a^b f(x)dx = \lim_{R\to b^-}\int_a^R f(x)dx.$$

We say that the improper integral **converges** if the limit exists and that it **diverges** otherwise.

Example 5.6.5. *Calculate*

$$(a). \int_0^9 \frac{dx}{\sqrt{x}}, \quad (b). \int_0^{1/2} \frac{dx}{x}.$$

Example 5.6.6. *Calculate* $\int_0^9 \frac{dx}{(x-1)^{2/3}}$.

In the following theorem, we study the integration with integrand $= x^{-p}$ in a region containing 0.

Theorem 5.16. *For* $a > 0$,

$$\int_0^a \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{1-p} & \textit{if } p < 1, \\ \textit{diverges} & \textit{if } p \geq 1. \end{cases}$$

5.6.2 Comparing Integrals

Sometimes we need to determining whether an improper integral converges or not, without finding its exact value. Then we can do comparison.

Theorem 5.17. *Assume for* $x > a$, $f(x) \geq g(x) \geq 0$. *We have*

if $\int_a^\infty f(x)dx$ *converges, then* $\int_a^\infty g(x)dx$ *also converges;*

if $\int_a^\infty g(x)dx$ *diverges, then* $\int_a^\infty g(x)dx$ *also diverges.*

The comparison test is also valid for improper integrals of unbounded functions.

Example 5.6.7. *Show that* $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ *converges.*

Proof. Let us use the comparison test. To show convergence, we need to construct a simpler and larger function. Looking at the denominator, x^3 is the main ingredient comparing to 1 when x is large. So consider

$$\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}} = x^{-3/2}.$$

Notice

$$\int_1^\infty x^{-3/2}dx \quad \text{converges}$$

and therefore $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converges by the comparison test. □

Example 5.6.8. *Determine whether the following integral converges or not:*

$$\int_1^\infty \frac{dx}{\sqrt{x}+e^{3x}}, \quad \int_0^2 \frac{dx}{x^8+x^2}.$$

Solution. When x is large $\frac{1}{\sqrt{x}+e^{3x}}$ behaves like $\frac{1}{e^{3x}}$ and the integration from 1 to ∞ for the latter function converges:

$$\int_1^\infty \frac{dx}{e^{3x}} = \frac{1}{3}e^{-3}.$$

Since

$$\frac{1}{\sqrt{x}+e^{3x}} \leq \frac{1}{e^3 x},$$

comparison test yields the convergence of the first integral.

For the second one, the integral is improper near $x=0$. When $x=0$,

$$x^8+x^2 \quad \text{is comparable to} \quad x^2.$$

Notice

$$\int_0^1 \frac{dx}{x^2} \quad \text{diverges.}$$

To do comparison, we observe that when $x \leq 1$

$$x^8+x^2 \leq 2x^2.$$

Since $\int_0^1 1/(2x^2)dx$ diverges,

$$\int_0^2 \frac{dx}{x^8+x^2} \geq \int_0^1 \frac{dx}{x^8+x^2} \geq \int_0^1 \frac{dx}{2x^2} \quad \text{diverges.}$$

□

6 Infinite Series

6.1 Sequences

By sequence we just mean a sequence of numbers, denoted as $\{a_n\} = \{a_1, a_2, ...\}$.

First let me introduce to you the well-known **Fibonacci Sequence**:

- We define the sequence by taking $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.

Given the first two terms, we can easily find out the first 10 terms one by one:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

The Fibonacci sequence appears in a surprisingly wide variety of situations, particularly in nature. For instance, the number of spiral arms in a sunflower almost always turns out to be a number from the Fibonacci sequence.

Recursive Sequence. Compute the two terms a_2, a_3 for the sequence defined recursively by

$$a_1 = 1, a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right).$$

Solution.

$$a_2 = \frac{1}{2}(1 + 2/1) = \frac{3}{2},$$
$$a_3 = \frac{1}{2}(\frac{3}{2} + \frac{2}{3/2}) = \frac{17}{12}.$$

□

Our main goal is to study convergence of sequences. A sequence $\{a_n\}$ converges to a limit L, if $|a_n - L|$ becomes arbitrarily small when n is sufficiently large.

Definition 6.1. (not required) We say $\{a_n\}$ converges to a limit L and write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L$$

if, for every $\epsilon > 0$, there is a number M such that $|a_n - L| < \epsilon$ for all $n > M$.

- If no limit exists, we say that $\{a_n\}$ diverges.
- If the terms increase without bound, we say that $\{a_n\}$ diverges to infinity.

Example 6.1.1. *The Fibonacci sequence increases without a bound, and so the sequence diverges. The sequence $a_n := \cos(\frac{n\pi}{2})$ has no limit because*

$$\{a_n\} = \{1, 0, -1, 0, 1, ...\}.$$

6.1.1 Prove convergence

Example 6.1.2. *Let $a_n = \frac{n+4}{n+3}$. Prove that $\lim_{n\to\infty} a_n = 1$.*

Proof. By definition, we need to find, for every $\epsilon > 0$, a number M (which depends on ϵ) such that

$$|a_n - 1| \le \epsilon \quad \text{for } n \ge M.$$

We have

$$a_n - 1 = \frac{1}{n+3}$$

which can be arbitrarily small when n is large. Indeed for every $\epsilon > 0$, when $n \ge \frac{1}{\epsilon}$,

$$|a_n - 1| = \frac{1}{n+3} \le \epsilon.$$

This proves the convergence. □

For some sequence $\{a_n\}$, we can understand it as a sequence defined by a function:

$$a_n = f(n).$$

Theorem 6.2. *If* $\lim_{x\to\infty} f(x)$ *exists and equals* L*. Then the sequence* $a_n = f(n)$ *converges to* L*.*

Can you apply this theorem to Example 6.1.2 with $f(x) = \frac{x+4}{x+3}$? Do you have $\lim_{x\to\infty} f(x) = 1$?

Example 6.1.3. *Calculate the limit of the sequence* $a_n = \frac{n+\ln n}{n^2}$*.*

Solution. Consider the following function

$$f(x) = \frac{x + \ln x}{x^2}.$$

By L'Hopital's rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x + \ln x}{x^2} = \lim_{x\to\infty} \frac{1 + 1/x}{2x} = 0.$$

Therefore the sequence $a_n \to 0$ as $n \to \infty$.

□

6.1.2 Geometric sequence

A **geometric sequence** is a sequence of the form

$$a_n = cr^n.$$

The number r is called the **common ratio**.

Example 6.1.4. *Prove that for* $r \geq 0, c > 0$*,*

$$\lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if } r \in [0,1), \\ c & \text{if } r = 1, \\ \infty & \text{if } r \in (1,\infty). \end{cases}$$

The proof follows by considering the function $f(x) = cr^x$.

Theorem 6.3. ***[Squeeze Theorem].*** *Let* $\{a_n\}, \{b_n\}, \{c_n\}$ *be sequences such that for some M,*

$$a_n \leq b_n \leq c_n \quad \text{for } n > M$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then

$$\lim_{n\to\infty} b_n = L.$$

Example 6.1.5. *Suppose $c \neq 0$. Prove*

$$\lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if } r \in (-1,0), \\ \text{diverges} & \text{if } r \leq -1. \end{cases}$$

Proof. Notice

$$-c|r|^n \leq cr^n \leq c|r|^n.$$

It follows from example 6.1.4, when $r \in (-1, 0)$, which implies $|r| < 1$, we have both $\{-c|r|^n\}$ and $\{c|r|^n\}$ converge to 0. Hence $cr^n \to 0$.

When $r = -1$, the sequence is an alternating sequence with values $\{c, -c, c, -c...\}$ which is diverging.

When $r < -1$, the sequence also diverges since $cr^{2n} \to \infty$ and $cr^{2n+1} \to -\infty$. □

Example 6.1.6. *Prove that $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for any R.*

Proof. Assume $R > 0$ and let M be an integer such that

$$M \leq R < M + 1.$$

We have

$$\frac{R^n}{n!} = \left(\frac{R}{1}\frac{R}{2}\cdots\frac{R}{M}\right)\left(\frac{R}{M+1}\frac{R}{M+2}\cdots\frac{R}{n}\right).$$

Let us call the value in the first bracket C. It is a constant which is independent of n. Then we obtain that

$$0 \leq \frac{R^n}{n!} \leq C\frac{R}{n}$$

and the latter $\to 0$ as $n \to \infty$. By Squeeze theorem, $\frac{R^n}{n!} \to 0$. □

Given a converging sequence $\{a_n\}$ and a function f, we can form the new sequence $\{f(a_n)\}$. When f is continuous, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n).$$

Example 6.1.7. *For example given a sequence $a_n = \frac{3n}{n+e^{-n}}$ which converges to 3. Then for any continuous function f, we have $f(a_n) \to f(3)$. For example, if $f = x^2$, then $f(a_n) = (\frac{3n}{n+e^{-n}})^2$ converges to 9; if $f = \ln x$, then $f(a_n) \to \ln 3$.*

6.1.3 Bounded sequence and Monotonic sequence (not required)

Next, we define the concepts of a bounded sequence and a monotonic sequence, concepts of great importance for understanding convergence.

Definition 6.4. A sequence $\{a_n\}$ is

- Bounded from above if there is a number M such that $a_n \leq M$ for all n. The number M is called an upper bound.

- Bounded from below if there is a number m such that $a_n \geq m$ for all n. The number m is called a lower bound.

The sequence $\{a_n\}$ is called bounded if it is bounded from above and below. A sequence that is not bounded is called an unbounded sequence.

Theorem 6.5. *Convergent sequences are bounded sequences.*

Divergent sequences can be both bounded or unbounded. Can you give both the examples?

Definition 6.6. $\{a_n\}$ is monotonic if either $a_n \leq a_{n+1}$ for all n or $a_n \geq a_{n+1}$. We call

- $\{a_n\}$ is (strictly) increasing if $a_n < a_{n+1}$ for all n,
- $\{a_n\}$ is (strictly) decreasing if $a_n > a_{n+1}$ for all n,
- $\{a_n\}$ is non-decreasing if $a_n \leq a_{n+1}$ for all n,
- $\{a_n\}$ is non-increasing if $a_n \geq a_{n+1}$ for all n.

Theorem 6.7. *Bounded Monotonic Sequences Converge.*

If $\{a_n\}$ is increasing and $a_n \leq M$, then $\{a_n\}$ converges and the limit $\leq M$.

If $\{a_n\}$ is decreasing and $a_n \geq m$, then $\{a_n\}$ converges and the limit $\geq m$.

Example 6.1.8. *Verify that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded from below. What is the limit of a_n?*

Solution. Let us consider the function $f(x) = \sqrt{x+1} - \sqrt{x}$. The function is decreasing because $f'(x) < 0$ for all $x > 0$. So $a_n = f(n)$ is a decreasing sequence. It is not hard to see that the sequence is bounded below by 0 i.e. $a_n \geq 0$ for all n. Therefore by the theorem, the sequence has a limit.

Notice

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Therefore $a_n \to 0$. □

Example 6.1.9. *Show that the following sequence is bounded and increasing:*

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

Prove the limit $\lim_{n\to\infty} a_n$ exits and find out the value.

Solution. **Step 1. Bounded from the above.** We certainly have that $a_1 < 2$. Suppose $a_k < 2$. Notice

$$a_{k+1} = \sqrt{2\sqrt{a_k}},$$

then

$$a_{k=1} < \sqrt{2\sqrt{2}} \leq 2.$$

Thus the sequence is bounded from the above by 2.

Step 2. Increasing.

Since a_n is positive and $a_n \leq 2$, then

$$a_{n+1} = \sqrt{2a_n} < \sqrt{a_n \times a_n} = a_n$$

which implies that the sequence is increasing. From the monotone convergence theorem, a_n converges and let us suppose the limit equals L.

Step 3. Find L.

Since $a_{n+1} = \sqrt{2a_n}$, by passing $n \to \infty$, we get

$$\lim_{n\to\infty} a_{n+1} = \sqrt{2 \lim_{n\to\infty} a_n}$$

which implies that

$$L = \sqrt{2L}.$$

Thus L can only be 0 or 2. We eliminate $L = 0$, because the terms a_n are positive and they all $\geq a_1 = \sqrt{2} > 0$. We must have $L = 2$. □

Example 6.1.10. *(Very hard!) The Fibonacci sequence $\{F_n\}$ diverges since it is unbounded. Please show the sequence defined by the ratios $a_n = \frac{F_{n+1}}{F_n}$ converges. The limit is known as the golden ratio.*

Proof. (Hint) Since $F_{n+2} = F_{n+1} + F_n$, dividing by F_{n+1}, we get

$$a_{n+1} = 1 + \frac{1}{a_n}. \tag{4}$$

Known

$$a_2 = F_3/F_2 = 2 \geq \frac{1+\sqrt{5}}{2} =: c.$$

It can be shown that for all $n \geq 2$, $a_n \geq c$. Why?

Next it can be shown that a_n is decreasing. Why? Therefore a_n converges. Suppose the limit is L.

Passing $n \to \infty$ on both sides of (4), we get

$$L = 1 + \frac{1}{L}$$

this tells that $L = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$. We pick $L = \frac{1+\sqrt{5}}{2} = c$, because $a_n \geq c$ for all n. □

6.2 Infinite Series

Given a sequence $\{a_n\}$, in this section we study the sum of a_n:

$$a_1 + a_2 + a_3 + ... + a_n + ..$$

This infinite sums is also called **infinite series**.

We define the **partial sums**:

$$\begin{aligned} &S_1 = a_1, \\ &S_2 = a_1 + a_2, \quad ... \\ &S_n = a_1 + a_2 + ... + a_n, \quad ... \end{aligned}$$

We use the following notation:

$$\Sigma_{n=1}^{N} a_n = a_1 + a_2 + ... + a_N, \quad \Sigma_{n=1}^{\infty} a_n = a_1 + a_2 + ...$$

Definition 6.8. An infinite series $\Sigma_{n=1}^{\infty} a_n$ converges to S if the sequence of its partial sums $\{S_n\}$ converges to S.

If the limit does not exist or it is ∞, we say that the infinite series diverges.

Example 6.2.1. *Compute*

$$\Sigma_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Solution. Notice

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore

$$\begin{aligned} S_N &= \Sigma_{n=1}^{N} \frac{1}{n(n+1)} \\ &= \Sigma_{n=1}^{N} \frac{1}{n} - \frac{1}{(n+1)} \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

which converges to 1 as $N \to \infty$. So the series equals 1. □

Theorem 6.9. ***[Geometric series]*** *For the geometric series $\Sigma_{n=0}^{\infty} cr^n$ with $r \neq 1$,*

$$S_N = c + cr + ... + cr^n = \frac{c(1 - r^{N+1})}{1 - r}.$$

If $|r| < 1$, then

$$\Sigma_{n=0}^{\infty} cr^n = \frac{c}{1 - r}.$$

Example 6.2.2. *Evaluate* $\Sigma_{n=0}^{\infty}\frac{2+3^n}{5^n}$.

Solution.

$$\begin{aligned}\Sigma_{n=0}^{\infty}\frac{2+3^n}{5^n} &= \Sigma_{n=0}^{\infty}\frac{2}{5^n}+\Sigma_{n=0}^{\infty}\frac{3^n}{5^n}\\ &= 2\frac{1}{1-(1/5)}+\frac{1}{1-(3/5)}=5.\end{aligned}$$

□

Theorem 6.10. ***[nth Term divergence test]*** *If* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\Sigma_{n=k}^{\infty}a_n$ *diverges for any* k*.*

While if $\lim_{n\to\infty} a_n = 0$*, the series* $\Sigma_{n=k}^{\infty}a_n$ *can be both divergent or convergent.*

Example 6.2.3. *Prove the divergent of* $\Sigma_{n=1}^{\infty}(-1)^n\frac{n}{4n+1}$.

Solution. Notice that $a_n = (-1)^n\frac{n}{4n+1}$ does not approach a limit. The series diverges by the test. □

Example 6.2.4. *Prove the divergence of* $\Sigma_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

Solution. Notice the partial sums

$$S_N = \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{N}}.$$

We have N terms in S_N and each one $\geq \frac{1}{\sqrt{N}}$. Therefore

$$S_N \geq N\times\frac{1}{\sqrt{N}}=\sqrt{N}$$

which $\to\infty$ as $N\to\infty$. The series diverges.

□

6.3 Convergence of Series

In this section we consider positive series $\Sigma_{n=1}^{\infty}a_n$ where each $a_n \geq 0$.

Theorem 6.11. *Given a positive series* $\Sigma_{n=1}^{\infty}a_n$*, then either*

(i) The partial sums S_N *are bounded from the above and then* $\Sigma_{n=1}^{\infty}a_n$ *converges. Or*

(ii) The partial sums S_N *are unbounded and then* $\Sigma_{n=1}^{\infty}a_n$ *diverges.*

Theorem 6.12. ***[Integral Test]*** *Suppose* $a_n = f(n) \geq 0$ *for some non-increasing, continuous function* $f(x), x\geq 1$.

(i) If $\int_1^\infty f(x)dx$ converges, then $\Sigma_{n=1}^\infty a_n$ converges.

(ii) If $\int_1^\infty f(x)dx$ diverges, then $\Sigma_{n=1}^\infty a_n$ diverges.

Example 6.3.1. *Show that $\Sigma_{n=1}^\infty \frac{1}{n}$ diverges.*

Solution. Consider $f(x) = \frac{1}{x}$ and then $a_n = f(n)$. Since the integral of $\int_1^\infty f(x)dx$ diverges, the series $\Sigma_{n=1}^\infty \frac{1}{n}$ also diverges. □

Example 6.3.2. *The infinite series $\Sigma_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.*

Example 6.3.3. *Does $\Sigma_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converge?*

Solution. Let us use the integral test for $f(x) = \frac{1}{x(\ln x)^2}$. The substitution $u = \ln x$ yields that

$$\begin{aligned}\int_2^\infty \frac{1}{x(\ln x)^2}dx &= \int_{\ln 2}^\infty \frac{du}{u} \quad \text{(here we do the integration from 2 to } \infty \text{)}\\ &= \lim_{R\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{R}\right)\\ &= \frac{1}{\ln 2} < \infty.\end{aligned}$$

The integral test shows that $\Sigma_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converges. □

Theorem 6.13 (Comparison test). *Assume for some $N > 0$, $b_n \geq a_n \geq 0$ holds for all $n \geq N$. Then*

(i) If $\Sigma_{n=1}^\infty b_n$ converges, then $\Sigma_{n=1}^\infty a_n$ converges.

(ii) If $\Sigma_{n=1}^\infty a_n$ diverges, then $\Sigma_{n=1}^\infty b_n$ diverges.

Theorem 6.14 (Limit Comparison test). *Let a_n, b_n be non-negative sequences. Assume that there exist m, M such that*

$$m \leq \liminf_{n\to\infty} \frac{a_n}{b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n} \leq M.$$

(i) If $\infty > M > m > 0$, then $\Sigma_{n=1}^\infty a_n$ converges, if and only if, $\Sigma_{n=1}^\infty b_n$ converges.

(ii) If $m = \infty$ and $\Sigma_{n=1}^\infty a_n$ converges, then $\Sigma_{n=1}^\infty b_n$ converges.

(iii) If $M = 0$ and $\Sigma_{n=1}^\infty b_n$ converges, then $\Sigma_{n=1}^\infty a_n$ converges.

If the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then in the above we can select

$$m = M = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Example 6.3.4. *Show that $\Sigma_{n=2}^\infty \frac{n^2}{n^4-n-1}$ converges.*

Solution. Let us set

$$a_n = \frac{n^2}{n^4 - n - 1} \quad \text{and } b_n = \frac{1}{n^2}.$$

Notice

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4}{n^4 - n - 1} = 1.$$

Since $\Sigma_{n=2}^{\infty} b_n = \Sigma_{n=2}^{\infty} \frac{1}{n^2}$ converges, therefore by the Limit comparison test, $\Sigma_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ also converges. □

Example 6.3.5. *Does* $\Sigma_{n=2}^{\infty} \frac{1}{(n^2-3)^{1/3}}$ *converge?*

Solution. Notice

$$\lim_{n\to\infty} \left(\frac{1}{(n^2-3)^{1/3}}\right) / \left(\frac{1}{n^{2/3}}\right) = \lim_{n\to\infty} \left(\frac{1}{(1-3n^{-2})^{1/3}}\right) = 1.$$

By the limit comparison test, the divergence of $\Sigma \frac{1}{n^{2/3}}$ yields the divergence of $\Sigma_{n=2}^{\infty} \frac{1}{(n^2-3)^{1/3}}$. □

7 Review for Midterm 2

Complex numbers and trig integration

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$$e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2}(1+i), \quad e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i...$$

We can use this formula to compute trig integration: Example 5.3.1, Example 5.3.2. The idea is to use

$$\cos(cx) = \frac{e^{cix} + e^{-cix}}{2}, \quad \sin(cx) = \frac{e^{cix} - e^{-cix}}{2i}.$$

Substitution is another frequently used method. Recall Example 5.3.5: Evaluate $\int \tan^3 x\, dx$. We are going to use both substitution method and partial fractions.

Solution. For (1). consider $u = \sin x$. Then

$$\begin{aligned}\int \tan^3 x\, dx &= \int \frac{\sin^3 x}{\cos^3 x}\frac{1}{\cos x} d\sin x \\ &= \int \frac{\sin^3 x}{(1-\sin^2 x)^2} d\sin x \\ &= \int \frac{u^3}{(1-u^2)^2} du.\end{aligned}$$

Then we apply the Partial fraction method (in Section 5.4) to get

$$\frac{u^3}{(1-u^2)^2} = \frac{u^3}{(u-1)^2(u+1)^2} = \frac{1}{2\,(u+1)} - \frac{1}{4\,(u+1)^2} + \frac{1}{2\,(u-1)} + \frac{1}{4\,(u-1)^2}.$$

After integration, we get

$$\begin{aligned}
&\frac{1}{2}\ln|(u-1)(u+1)| + \frac{1}{4}\frac{1}{u+1} - \frac{1}{4}\frac{1}{u-1} + C \\
=& \frac{1}{2}\ln|1-u^2| + \frac{1}{2}\frac{1}{1-u^2} + C \\
=& -\ln|\sec x| + \frac{1}{2}\sec^2 x + C.
\end{aligned}$$

□

The last technique in this part is the trig substitution: Recall

$\sqrt{a^2-x^2}$, try $x = a\sin\theta$ and then $dx = a\cos\theta d\theta$, $\sqrt{a^2-x^2} = a\cos\theta$;
$\sqrt{a^2+x^2}$, try $x = a\tan\theta$ and then $dx = a\cos^{-2}\theta d\theta$, $\sqrt{a^2+x^2} = a\cos^{-1}\theta$;
$\sqrt{x^2-a^2}$, try $x = a\cos^{-1}\theta$ and then $dx = a\sin\theta\cos^{-2}\theta d\theta$, $\sqrt{a^2-x^2} = a\tan\theta$.

Sometimes the integrand is not exactly of the above forms, we might need to do substitution.

Example 7.0.1. *Evaluate*

$$\int \frac{x^2dx}{x^2+2x+3}.$$

Solution. Since

$$x^2+2x+3 = (x+1)^2+2,$$

let $u = x+1$ and we get

$$\begin{aligned}
\int \frac{x^2dx}{x^2+2x+3} &= \int \frac{(u-1)^2}{u^2+2}du \\
&= \int \frac{u^2-2u+1}{u^2+2}du \\
&= \int \frac{u^2+2}{u^2+2}du - \int \frac{2u}{u^2+2}du - \int \frac{1}{u^2+2}du \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

Then $A_1 = u + C$. For A_2, notice $2udu = du^2$ and thus we use substitution $w := u^2$. For A_3, we use trig substitution

$$u = \sqrt{2}\tan\theta.$$

Please finish the computations. :) □

Improper Integration

Example 7.0.2. *Compute* $\int_1^\infty \frac{1}{2+x^2}dx$.

Solution. Since

$$\frac{1}{2+x^2} \leq \frac{1}{x^2}$$

and

$$\int_1^\infty \frac{1}{x^2} < \infty,$$

So the original improper integral converges.

Actually we can use trig substitution to find its value: let $x = \sqrt{2}\tan\theta$ and we have

$$\begin{aligned}\int_1^\infty \frac{1}{2+x^2}dx &= \int_{\arctan(1/\sqrt{2})}^{\pi/2} \frac{\cos^2\theta\, d\tan\theta}{2} \\ &= \int_{\arctan(1/\sqrt{2})}^{\pi/2} \frac{d\theta}{2} \\ &= \frac{\pi}{4} - \frac{1}{2}\arctan(1/\sqrt{2})\end{aligned}$$

□

Let us solve Example 5.6.3: calculate $\int_0^\infty xe^{-x}dx$.

Solution. Use integration by parts we get

$$\int_0^R xe^{-x}dx = (-xe^{-x})|_0^R - \int_0^R e^{-x}dx = 1 - \frac{R+1}{e^R}.$$

Take $R \to \infty$ and apply the L'Hopital's rule,

$$\int_0^\infty xe^{-x}dx = 1 - \lim_{R\to\infty} \frac{R+1}{e^R} = 1.$$

□

Sequence and series

Example 7.0.3. *Determine the limit*

$$a_n = \ln\left(\frac{12n+2}{-9+4n}\right).$$

The answer is $\ln 3$.

Recall Series: $\Sigma_1^{\infty} a_n$.
Power series: for $|r| < 1$,

$$\Sigma_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + ... = \frac{c}{1-r}$$

We can use functions to determine whether a positive series is convergent. f needs to be positive, decreasing and continuous. See Example 6.3.1- Example 6.3.3

Another method is the comparison test and limit comparison test.

Example 7.0.4. *Does $\Sigma_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ converge?*

Solution. For $n \geq 1$,

$$\frac{1}{\sqrt{n}3^n} \leq \frac{1}{3^n}.$$

The larger series $\Sigma_{n=1}^{\infty} \frac{1}{3^n}$ converges and therefore the original one converges. □

8 Infinite Series (Continued)

8.1 Absolute and Conditional Convergence

In the previous section, we mainly considered positive series. Now let us study the general series.

Definition 8.1. [Absolute Convergence] The series $\Sigma_{n=1}^{\infty} a_n$ **converges absolutely** if $\Sigma_{n=1}^{\infty} |a_n|$ converges.

Theorem 8.2. ***[Absolute Convergence Implies Convergence]*** *If $\Sigma_{n=1}^{\infty} |a_n|$ converges, $\Sigma_{n=1}^{\infty} a_n$ also converges.*

Example 8.1.1. *Verify that $\Sigma_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges.*

Solution. Since

$$\Sigma_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \Sigma_{n=1}^{\infty} \frac{1}{n^2}$$

which is a convergent series, so $\Sigma_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent. By the theorem, $\Sigma_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is convergent. □

Definition 8.3. [Conditional Convergence] An infinite series $\Sigma_{n=1}^{\infty} a_n$ **converges conditionally** if $\Sigma_{n=1}^{\infty} a_n$ converges but $\Sigma_{n=1}^{\infty} |a_n|$ diverges.

Example 8.1.2. *Show $\Sigma_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.*

Proof. It is not hard to see that

$$\Sigma_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \Sigma_{n=1}^{\infty}\frac{1}{n}$$

diverges. To prove the claim, we only need to show the partial sum of $\Sigma_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges. Note

$$\begin{aligned} S_{2N} &= \Sigma_{n=1}^{2N}\frac{(-1)^n}{n} \\ &= -\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-...-\frac{1}{2N-1}+\frac{1}{2N} \\ &= -(\frac{1}{1}-\frac{1}{2})-(\frac{1}{3}-\frac{1}{4})-...-(\frac{1}{2N-1}-\frac{1}{2N}) \\ &= -\Sigma_{i=1}^{N}(\frac{1}{2i-1}-\frac{1}{2i}). \end{aligned}$$

So $\{S_{2N}\}$ (as a sequence) is a decreasing sequence. Since

$$\left|\frac{1}{2i-1}-\frac{1}{2i}\right| = \left|\frac{1}{(2i-1)(2i)}\right| \leq \frac{1}{(2i-1)^2},$$

triangle inequality implies that

$$|S_{2N}| \leq \Sigma_{i=1}^{N}\frac{1}{(2i-1)^2} \leq \frac{\pi^2}{6}.$$

It follows from Theorem 6.7 that $\{S_{2N}\}$ has a limit. Finally Since the difference of S_{2N+1} and S_{2N} converges to 0, S_n has the same limit as $\{S_{2N}\}$.

The convergence of the partial sum implies that $\Sigma_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges. □

The previous example gives a series which is similar to the **alternating series**:

$$\Sigma_{n=1}^{\infty}(-1)^{n-1}b_n$$

where b_n are positive and decrease to zero.

Theorem 8.4. ***[Alternating Series Test]*** *Assume that $\{b_n\}$ is a positive sequence that is decreasing and converges to 0. Then*

$$\Sigma_{n=1}^{\infty}(-1)^{n-1}b_n \quad \textit{converges.}$$

Example 8.1.3. *Show that*

$$\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$$

converges conditionally. Furthermore if S is the sum of the series, then $0 < S < 1$.

Proof. It is direct to see that $\Sigma_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{\sqrt{n}}\right|$ diverges.

Next the terms $b_n = \frac{(-1)^{n-1}}{\sqrt{n}}$ are positive and decreasing, and $\lim_{n\to\infty} b_n = 0$. By the Alternating series test, the series converges conditionally. Since

$$S = \Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}} = b_1 - \Sigma_{n=1}^{\infty}(b_{2n} - b_{2n+1}) < b_1,$$

we get $S < 1$. Also

$$S = \Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}} = \Sigma_{n=1}^{\infty}(b_{2n-1} - b_{2n}) > 0.$$

□

Remark 8.5. In general for alternating series converging to S with partial sums S_N, we have

$$S_p < S < S_q$$

for any even p and odd q.

Also if $b_n > 0$ for all n, we have

$$|S - S_N| < b_{N+1}.$$

(Please prove the two claims in this remark.)

Example 8.1.4. *Consider a conditionally convergent sequence* $\Sigma_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ *(why?). If* S *represent the sum, show that*

$$|S - S_6| < \frac{1}{7}.$$

Next find an N *such that* S_N *approximates* S *with error less than* 10^{-2}.

Solution. By the remark

$$|S - S_6| < b_7 = \frac{1}{7}.$$

Also we have

$$|S - S_N| < b_{N+1} = \frac{1}{N+1}.$$

To have error smaller than $\frac{1}{100}$, we need $N \geq 99$. □

8.2 Ratio and Root Tests

Let us present two tests concerning the problem whether a series converges or diverges.

Theorem 8.6. ***[Ratio Test]*** *Assume that the following limit exists:*

$$\rho := \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

If $\rho < 1$, *the the series* $\Sigma_{k=1}^{\infty}a_k$ *converges absolutely. If* $\rho > 1$, *then* $\Sigma_{k=1}^{\infty}a_k$ *diverges. If* $\rho = 1$, *the test is inconclusive.*

Remark 8.7. Let us compare the test with the geometric series:

$$a + ar + ar^2 + ar^3 + ...$$

which converges if $|r| < 1$ and diverges if $|r| > 1$.

Example 8.2.1. *Prove that* $\Sigma_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}$ *converges.*

Example 8.2.2. *Prove that* $\Sigma_{n=1}^{\infty} \frac{10^n}{n!}$ *converges.*

Example 8.2.3. *Show* $\Sigma_{n=1}^{\infty}(-1)^n \frac{n!}{10^n}$ *diverges.*

Theorem 8.8. ***[Root Test]*** *Assume that the following limit exists:* $L := \lim_{n\to\infty} \sqrt[n]{|a_n|}$. *If* $L < 1$, *then* $\Sigma_{n=1}^{\infty} a_n$ *converges absolutely. If* $L > 1$, *then* $\Sigma_{n=1}^{\infty} a_n$ *diverges.*

Example 8.2.4. *Show that* $\Sigma_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ *converges.*

Example 8.2.5. *Show* $\Sigma_{n=1}^{\infty} \frac{10^n}{n^{\sqrt{n}}}$ *diverges.*

Solution. Let us use the root test:

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{10^n}{n^{\sqrt{n}}}\right)^{1/n} = \lim_{n\to\infty} \frac{10}{n^{\frac{1}{\sqrt{n}}}}.$$

Notice by L'Hopital's rule

$$\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}} = e^{\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}} = e^{\lim_{n\to\infty} \frac{1/n}{1/(2\sqrt{n})}} = e^{\lim_{n\to\infty} \frac{2}{\sqrt{n}}} = 1.$$

Therefore

$$\lim_{n\to\infty} |a_n|^{1/n} = 10$$

which implies that the series diverges. □

8.3 Power Series

By power series we mean

$$a_0 + a_1 r + a_2 r^2 + ...$$

If we vary the "r", we get a polynomial of infinite degree:

$$f(x) = \Sigma_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ...$$

More generally, we consider a power series near point x_0:

$$f(x) = \Sigma_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ...$$

These are functions of x.

Shifting the Summation index.

Example 8.3.1. *Express the series*

$$\Sigma_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

as a series where the generic term is x^k.

Solution. Set $k = n - 2$. Then

$$\Sigma_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \Sigma_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k.$$

(This is like doing substitution in the summation index.) □

We say the series

$$\Sigma_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ...$$

converges at $x = c$ if $\Sigma_{n=0}^{\infty} a_n(c-x_0)^n$ converges. If the limit does not exist, we say the series **diverges** at $x = c$. Moreover if

$$\Sigma_{n=0}^{\infty}|a_n(c-x_0)^n|$$

converges, we say the series **converges absolutely** at point $x = c$.

There is a surprisingly simple way to describe the set of values x at which $f(x)$ converges.

Theorem 8.9. ***[Radius of convergence]*** *The radius of convergence r is a nonnegative real number or ∞ such that the series converges if $|x - x_0| < r$, and diverges if $|x - x_0| > r$.*

r can be derived through the following formulas:

Root test. $r = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{a_n}}$,

Ratio test. *when the following limit exists, it satisfies,* $r = \lim_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right|$.

Example 8.3.2. *Determine the converge set of*

$$\Sigma_{n=0}^{\infty} \frac{(-2)^n}{n+1} x^n.$$

Solution. By the ratio test, $r = \frac{1}{2}$. Let us check the endpoints. When $x = \frac{1}{2}$, we get an alternating series, which converges. When $x = -\frac{1}{2}$, the series diverges. Thus the series converges in $(-\frac{1}{2}, \frac{1}{2}]$. □

Sum of two Power Series.

Given two power series:

$$f(x) = \Sigma_{n=0}^{\infty} a_n x^n, \quad g(x) = \Sigma_{n=0}^{\infty} b_n x^n.$$

Then

$$f(x) + g(x) = \Sigma_{n=0}^{\infty}(a_n + b_n)x^n.$$

Product of two Power Series.

$$f(x)g(x) = (\Sigma_{n=0}^{\infty} a_n x^n) \times (\Sigma_{n=0}^{\infty} b_n x^n)$$
$$= (a_0b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + ...$$

The general formula is

$$f(x)g(x) = \Sigma_{n=0}^{\infty} c_n x^n \quad \text{with } c_n := \Sigma_{k=0}^{n} a_k b_{n-k}. \tag{5}$$

This is called the **Cauchy Product**.

Theorem 8.10 (Differentiation and Integration)**.** *If*

$$f(x) = \Sigma_{n=0}^{\infty} a_n x^n$$

has a positive radius of convergence r, then f is differentiable in the interval $|x| < r$:

$$f'(x) = \Sigma_{n=1}^{\infty} n a_n x^{n-1}.$$

Also f has antiderivatives in $|x| < r$:

$$\int f(x)dx = \Sigma_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C.$$

Remark 8.11. We can replace x in the above theorem by $(x - x_0)$.

If we inductively apply the first part of the theorem, we know that f is nth differentiable for all $n \geq 1$.

Example 8.3.3. *Find the power series for $\frac{1}{1-x}$.*

Solution.

$$\frac{1}{1-x} = 1 + x + x^2 + ... = \Sigma_0^{\infty} x^n. \tag{6}$$

The radius of convergence is 1. □

Example 8.3.4. *Find a power series for each of the following functions:*

$$(a) \quad \frac{1}{1+x^2}, \quad (b) \quad \frac{1}{(x-1)^2}, \quad (c) \quad \arctan x.$$

Solution. Replacing x by $-x^2$ in (6), we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - ... = \Sigma_0^{\infty} (-1)^n x^{2n}. \tag{7}$$

For (b), since $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$, by differentiating (6) twice, we get

$$(\frac{1}{1-x})' = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... = \Sigma_1^n n x^{n-1}.$$

For (c), notice

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt.$$

Therefore we can integrate the series (7) to get

$$\begin{aligned}\arctan x &= \int_0^x \frac{1}{1+t^2} dt \\ &= \Sigma_0^\infty \int_0^x (-1)^n t^{2n} dt \\ &= \Sigma_0^\infty \frac{(-1)^n x^{2n+1}}{2n+1}.\end{aligned}$$

□