Lecture: MATH 20D Introduction to Differential Equations

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Scribe:

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1 Introduction

1.1 Background

Definition 1.1. An equation that contains derivatives of unknown functions is called a differential equation.

Example 1.1.1. A falling object. By Newton’s second law:

\[ F = ma \]

where \( F \) denotes an external force, \( m \) denotes mass and \( a \) is the acceleration.

Applying the law to an falling object. Let \( h \) be the height of the object above the ground. We use \( v, a \) as the object’s velocity and acceleration. Then

\[ v = \frac{dh(t)}{dt} = h'(t), \quad a = \frac{d^2h(t)}{dt^2} = h''(t). \]

Now we study the force on the object. There are two forces: gravity and the air resistance. By physics gravity = \( mg \) where \( g \) is the gravity constant, and air resistance=\( \frac{1}{2} \rho Ac|v|^2 \) where \( \rho, A, c \) are air density, cross sectional area and the drag coefficient. Note that the air resistance is proportional to velocity square. By Newton’s second law, we obtain

\[ mh''(t) = -mg + \frac{1}{2} \rho Ac|h'(t)|^2 \]

which is a differential equation.

Definition 1.2. An differential equation always involves the derivative of one variable with respect to another. The former is called a dependent variable and the latter an independent variable.

Definition 1.3. An differential equation involving only derivatives with respect to one independent variable is called an ordinary differential equation (ODE). Otherwise it is called a partial differential equation (PDE).

Definition 1.4. The order is the order of the highest derivatives present in the equation.

Definition 1.5. A linear differential equation is one in which the dependent variable and its derivatives appear in additive combinations of their first powers. More precisely, linear differential equation is of the form:

\[ a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(t)\frac{dy}{dx} + a_0(x)y = f(x). \]

Example 1.1.2. \( \frac{d^{10} y}{dx^{10}} + y + x^{100} + \sin x = 0 \) is linear. While \( \sqrt{y'} + 1 + x = 1, y'' + y' + y^2 = 1 \) are nonlinear.
1.2 Initial Value Problem

Definition 1.6. By an initial value problem for an $n$th-order differential equation

$$F(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}) = 0,$$

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = y_1, \ldots, \frac{d^{n-1} y}{dx^{n-1}}(x_0) = y_{n-1}.$$ 

By an explicit solution, we mean a function $y = y(x)$ such that the above $n$ equalities hold.

1.9 Example 1.2.1. Consider $\frac{dy}{dx} = f(x)$ with initial data $y(0) = 1$. Then the anti-derivatives of $f$ are solutions of the equation. We have

$$y = \int f(x)dx = F(x) + C.$$ 

Use the initial data to solve for $C$: $y(0) = F(0) + C = 1$ implies $y(x) = F(x) + 1 - F(0)$. For example: $\frac{dy}{dx} = 2e^{-x}$ with initial data $y(0) = 1$. The solution $y(x) = -2e^{-x} + 3$.

2 First Order Equations

2.1 Separable Equations

From example 1.2.1 we know that equations of the form $\frac{dy}{dx} = f(x)$ can be solved. More generally, consider the equations of the following form.

Definition 2.1. Consider the equation $\frac{dy}{dx} = f(x, y)$. If $f(x, y) = g(x)p(y)$ for some functions $g, p$, then the differential equation is called separable differential equation.

Method of solving separable equations. Suppose

$$\frac{dy}{dx} = g(x)p(y).$$

Then the implicitly defined solution is

$$\int \frac{1}{p(y)}dy = \int g(x)dx.$$ 

Example 2.1.1. Solve $\frac{dy}{dx} = \frac{x^2-1}{y^3+1}$. (Give implicit solutions.)

Example 2.1.2. Solve for the initial value problem

$$\frac{dy}{dx} = \frac{y - 1}{x^2 + 3}, \quad y(-1) = 0.$$ 

(Compare the problem with Example 2 on textbook page 43.)
Solution. From the equation
\[ \frac{dy}{y - 1} = \frac{dx}{x^2 + 3}, \]
and then
\[ \int \frac{dy}{y - 1} = \int \frac{dx}{x^2 + 3}. \] (1)

Recall \( \frac{d}{dx} \arctan x = \frac{1}{1 + x^2} \). By the chain rule
\[ \frac{d}{dx} \arctan ax = \frac{1}{1 + (ax)^2} \frac{d}{dx}(ax) = \frac{a}{1 + a^2 x^2}. \]

Therefore
\[ \frac{d}{dx} a \arctan ax = \frac{a^2}{1 + a^2 x^2}. \]

Pick \( a = 1/\sqrt{3} \). We get
\[ \frac{d}{dx} \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} = \frac{1/3}{1 + (1/3)x^2} \]
which implies
\[ \int \frac{1/3}{3 + x^2} dx = \int \frac{1/3}{1 + (1/3)x^2} dx = \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C. \]

Now it follows from (1),
\[ \ln |y - 1| = \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C. \]

Then
\[ |y - 1| = \exp \left( \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C \right) = C_1 \exp \left( \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} \right) \]
where \( C_1 := e^C \) and thus \( C_1 > 0 \). Then
\[ y = 1 \pm C_1 \exp \left( \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} \right). \]

Because \( C_1 \) is positive, we can replace \( \pm C_1 \) by \( C_2 \) where \( C_2 \) represents an arbitrary nonzero constant. We have
\[ y = 1 + C_2 \exp \left( \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} \right). \]

Since \( y(-1) = 0 \),
\[ 0 = 1 + C_2 \exp \left( \frac{1}{\sqrt{3}} \arctan \frac{-1}{\sqrt{3}} \right). \]

We get
\[ C_2 = -\exp \left( \frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} \right). \]

The solution is
\[ y = 1 - \exp \left( \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + \frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} \right). \]
Remark 2.2. In the above problem, if without the initial condition, we have obtained that

\[ y = 1 + C_2 \exp\left( \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} \right) \]

are solutions for all \( C_2 \neq 0 \). Note \( C_2 = \pm e^C \) and thus the only constrain is \( C_2 \neq 0 \). However this constrain can be removed. When \( C_2 = 0 \), we get \( y = 1 \), a constant function. It is not hard to verify that \( y = 1 \) is also a solution to \( \frac{dy}{dx} = 0 = \frac{y-1}{x^2+3} \).

2.2 Linear Equations

In this section we are going to deal with the equation of the form

\[ \frac{dy}{dx} + P(x)y = Q(x). \] (2)

The key idea: multiply \( \mu(x) \) on both sides of the equation and hope that we can combine the terms \( \mu(x)\frac{dy}{dx}, \mu(x)P(x)y \) by

\[ \mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}(\mu(x)y). \]

Note the RHS

\[ = \mu(x)\frac{dy}{dx} + \mu'(x)y. \]

Then the requirement becomes a equation of \( \mu \):

\[ \mu'(x) = \mu(x)P(x) \]

which is a separable equation. We get

\[ \mu(x) = \exp(\int P(x)dx). \]

With this choice of \( \mu \), the original equation becomes

\[ \frac{dy}{dx}(\mu(x)y) = \mu(x)Q(x) \]

which has the solution

\[ y(x) = \frac{1}{\mu(x)}(\int \mu(x)Q(x)dx + C). \]

This is often referred to as the general solution to (2). \( \mu \) is called the integrating factor to (2).

Example 2.2.1. Find the general solution to

\[ \frac{1}{x} \frac{dy}{dx} - \frac{3y}{x^2} = x^3 \cos x. \]
Solution. **Step one.** Write the equation into the standard form. Multiplying $x$ on both sides, we get

$$\frac{dy}{dx} - \frac{3y}{x} = x^4 \cos x.$$  

**Step two.** Calculating the integrating factor:

$$\exp\left(\int \frac{-3}{x} \, dx\right) = \exp(-3 \ln |x| + C).$$

Since we only need one integrating factor, let us select $C = 0$ and suppose for the moment $x > 0$. Then

$$\mu(x) := \exp(-3 \ln x) = \frac{1}{x^3}.$$  

**Step three.** Multiply the integrating factor and do the computation. We have

$$\frac{1}{x^3} \frac{dy}{dx} - \frac{3}{x^4} = x \cos x,$$

so

$$\frac{d}{dx} \left(\frac{1}{x^3} y\right) = x \cos x.$$

Integrate both sides:

$$\frac{y}{x^3} = \int x \cos x \, dx.$$  

Apply integration by parts and then we get

$$\frac{y}{x^3} = \int x \, d \sin x = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$  

Thus the general solution is

$$y = x^4 \sin x + x^3 \cos x.$$  

**Example 2.2.2.** For the initial value problem

$$y' = \sqrt{1 + \cos^2 x} - y, \quad y(1) = 4,$$

find $y(2)$.  

**Solution.** The standard form of the equation is

$$y' + y = \sqrt{1 + \cos^2 x}.$$  

Then

$$\mu(x) = \exp\left(\int 1 \, dx\right) = e^{x + C}.$$
Take $C = 0$. Multiplying $\mu(x)$ on both sides of the equation gives
\[ e^x y' + e^x y = e^x \sqrt{1 + \cos^2 x}. \]

Then
\[ \frac{d}{dx}(e^x y) = e^x \sqrt{1 + \cos^2 x} \]
and we have
\[ e^x y = \int e^x \sqrt{1 + \cos^2 x} \, dx. \]

However this indefinite integral cannot be expressed in finite terms with elementary functions.
So instead of indefinite integral, let us do definite integral for $x$ over $[1, 2]$. Then
\[ (e^x y) \bigg|_{x=1}^{x=2} = \int_1^2 e^x \sqrt{1 + \cos^2 x} \, dx. \]
By the initial data
\[ (e^x y) \bigg|_{x=1}^{x=2} = e^2 y(2) - e y(1) = e^2 y(2) - 4e. \]
Hence
\[ y(2) = e^{-2}(4e + \int_1^2 e^x \sqrt{1 + \cos^2 x} \, dx) \]
which is the answer. We can use computer or the Simpson’s rule to approximate the value.

### 2.3 Exact Equations

Consider the following general first order equation:
\[ \frac{dy}{dx} = f(x, y). \]  \hspace{1cm} (3) \hspace{1cm} \text{\textbf{2.2}}

**Definition 2.3.** For any constant $C$, $F(x, y) = C$ is said to be an implicit solution of (3) if
\[ f(x, y) = -\frac{\partial_x F}{\partial_y F}. \]  \hspace{1cm} (4) \hspace{1cm} \text{\textbf{2.4}}

where $\partial_x F = \frac{\partial F}{\partial x}$ and $\partial_y F = \frac{\partial F}{\partial y}$.

**Remark 2.4.** The derivation of (4). View $y$ as a function of $x$, and then
\[ \frac{d}{dx} F(x, y) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0, \]
which is the same as
\[ \frac{dy}{dx} = -\frac{\partial_x F}{\partial_y F}. \]
Suppose
\[ f(x, y) = -\frac{M(x, y)}{N(x, y)}. \]
For instance, we can pick \( M(x, y) = -f(x, y) \) and \( N(x, y) = 1 \). Now let us rewrite (3) as
\[ M(x, y)dx + N(x, y)dy = 0. \]
The advantage of this notation is that we don’t really distinguish the role of dependent variable (\( x \)) and independent variable (\( y \)).

**Definition 2.5.** \( M(x, y)dx + N(x, y)dy \) is called a **differential form**. The differential form is said to be **exact** if there is a function \( F(x, y) \) such that
\[ \partial_x F(x, y) = M(x, y) \text{ and } \partial_y F(x, y) = N(x, y). \]
In such a case, we can write
\[ dF = M(x, y)dx + N(x, y)dy \]
which is called the **total differential** of \( F \), and the equation
\[ dF = M(x, y)dx + N(x, y)dy = 0 \]
is called an **exact equation**.

**Example 2.3.1.** Consider
\[ (2xy^2 + 1)dx + (2x^2y)dy = 0. \]
It is an exact equation. Because \( F = x^2y^2 + x \) satisfies
\[ dF = (2xy^2 + 1)dx + (2x^2y)dy = 0. \]
Then \( F(x, y) = C \), which is
\[ x^2y^2 + x = C \]
is the general implicit solution to the original equation.

**Theorem 2.6.** **Test for Exactness.** The differential form \( M(x, y)dx + N(x, y)dy \) is exact if and only if
\[ \frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y). \]

**Example 2.3.2.** Solve
\[ (2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0. \]
**Solution. Step one. Check exactness.** Here \( M = 2xy - \sec^2 x \) and \( N = x^2 + 2y \). Because

\[
\partial_y M = 2x = \partial_x N,
\]

the equation is exact.

**Step two. View \( y \) as a constant and solve for \( F \) as a function of \( x \).** By exactness, there is \( F \) such that

\[
\partial_x F = M \quad \text{and} \quad \partial_y F = N.
\]

Then for some \( g(y) \),

\[
F(x, y) = \int M(x, y) \, dx + g(y) = \int (2xy - \sec^2 x) \, dx + g(y) = x^2y - \tan x + g(y).
\]

**Step three. Solve for \( g \).** Now view \( y \) as a variable and \( x \) as a constant. Since \( \partial_y F = N \),

\[
N(x, y) = x^2 + 2y = \partial_y(x^2y - \tan x + g(y)).
\]

Then

\[
x^2 + 2y = x^2 + g'(y)
\]

which gives

\[
g = y^2.
\]

And we have

\[
F(x, y) = x^2y - \tan x + y^2.
\]

The general solutions are given implicitly by

\[
x^2y - \tan x + y^2 = C.
\]

\[
\square
\]

### 2.4 Integrating factors

**Definition 2.7.** If the equation

\[
M(x, y) \, dx + N(x, y) \, dy = 0
\]

is not exact, but for some function \( \mu(x, y) \) the equation

\[
\mu(x, y)M(x, y) \, dx + \mu(x, y)N(x, y) \, dy = 0
\]

is exact, then \( \mu \) is called an **integrating factor** of (5).
In general finding integrating factors is a hard problem. In view of Theorem 2.6, \( \mu \) is an integrating factor if it satisfies
\[
\frac{\partial}{\partial y} [\mu(x, y)M(x, y)] = \frac{\partial}{\partial x} [\mu(x, y)N(x, y)].
\]
This reduces to the equation
\[
M\frac{\partial y}{\partial y} \mu - N\frac{\partial x}{\partial x} \mu = (\frac{\partial x}{\partial x} N - \frac{\partial y}{\partial y} M)\mu. \tag{6} \]

Unfortunately solving this equation is as hard as solving the original equation. There are, however, two exceptions.

If
\[
\frac{1}{N}(\frac{\partial y}{\partial y} M - \frac{\partial x}{\partial x} N) \tag{7}
\]
is only a function of \( x \), then we can assume that \( \mu \) is also only a function of \( x \). In such a case (6) is reduced to
\[
\frac{d}{dx} \mu = \frac{\mu}{N}(\frac{\partial y}{\partial y} M - \frac{\partial x}{\partial x} N), \tag{8}
\]
which is a separable differential equation and can be solved.

**Theorem 2.8.** If (7) only depends on \( x \) (not \( y \)), then
\[
\mu = \mu(x) = \exp \left( \int \frac{1}{N}(\frac{\partial y}{\partial y} M - \frac{\partial x}{\partial x} N) dx \right)
\]
is an integrating factor for (5). Similarly if
\[
\frac{1}{M}(\frac{\partial x}{\partial x} N - \frac{\partial y}{\partial y} M)
\]
only depends on \( y \), then
\[
\mu = \mu(y) = \exp \left( \int \frac{1}{M}(\frac{\partial x}{\partial x} N - \frac{\partial y}{\partial y} M) dx \right)
\]
is an integrating factor for (5).

**Example 2.4.1.** Solve
\[
(2x^2 + y)dx + (x^2y - x)dy = 0.
\]

**Solution.** A quick inspection shows that this equation is neither separable nor linear. Let us try the method given in the above theorem. Notice
\[
\frac{\partial}{\partial y} M = 1 \neq (2xy - 1) = \frac{\partial}{\partial x} N.
\]

Then the equation is not exact. We compute
\[
\frac{1}{N}(\frac{\partial y}{\partial y} M - \frac{\partial x}{\partial x} N) = \frac{1 - (2xy - 1)}{x^2y - x} = \frac{-2}{x}
\]
which is a function of only \( x \). So an integrating factor is given by
\[
\mu(x) = \exp \left( \int \frac{-2}{x} \, dx \right) = x^{-2}.
\]

After multiplying \( \mu = x^{-2} \) on both sides of the equation, we get an exact equation
\[
(2 + yx^{-2}) \, dx + (y - x^{-1}) \, dy = 0.
\]
Suppose it is the total differential of \( F \). Then \( \partial_x F = 2 + yx^{-2} \) which says that
\[
F(x, y) = 2x - yx^{-1} + g(y)
\]
for some function \( g(y) \). In view of \( \partial_y F = y - x^{-1} \), we have
\[
F(x, y) = 2x - yx^{-1} + \frac{y^2}{2}
\]
and
\[
F(x, y) = 2x - yx^{-1} + \frac{y^2}{2} = C
\]
are the solutions.

Sometimes, we don’t really need to aim at making the whole differential form as a total differential of a function \( F \).

**Example 2.4.2.** Solve
\[
\frac{dy}{dx} = \frac{y + x^2 \cos x}{x}.
\]

**Solution.** We can rewrite the equation as
\[
xdy - ydx = x^2 \cos x \, dx.
\]
By the method we have just introduced, the integrating factor for the differential form \( xdy - ydx \) equals
\[
\mu = \mu(x) = x^{-2}.
\]
Multiplying \( \mu \), we get
\[
\frac{xdy - ydx}{x^2} = \cos x \, dx.
\]
Since
\[
\frac{d}{dx} \frac{y}{x} = \frac{xdy - ydx}{x^2}, \quad \cos x \, dx = d \sin x,
\]
the equation becomes
\[
\frac{d}{dx} \frac{y}{x} = d \sin x.
\]
The solutions are
\[
\frac{y}{x} + \sin x = C.
\]
There are some useful total differential formulas:

\begin{align*}
y \, dx + x \, dy &= d(xy), \\
x \, dx + y \, dy &= d\sqrt{x^2 + y^2}, \\
\frac{x \, dx + y \, dy}{x^2 + y^2} &= \frac{1}{2} d \ln(x^2 + y^2),
\end{align*}

\begin{align*}
y \, dx - x \, dy &= \frac{d}{y}, \\
\frac{y \, dx - x \, dy}{x^2 + y^2} &= d \arctan \frac{x}{y}, \\
\frac{x \, dx - y \, dy}{x^2 + y^2} &= \frac{1}{2} d \ln(x^2 + y^2), \\
\end{align*}

\text{etc.}

3 Linear Second-Order Equations

3.1 Homogeneous Linear Equations

In this section let us study the linear second-order constant-coefficient differential equation

\[ ay'' + by' + cy = f(x). \]  (9)

First we study the **homogeneous** case when \( f(x) = 0. \)

Let us try a spatial solution of the form \( e^{\lambda x}. \) Substitute \( y = e^{\lambda x} \) into the equation

\[ ay'' + by' + cy = 0. \]  (10)

We get

\[ e^{\lambda x} (a\lambda^2 + b\lambda + c) = 0. \]

Thus if \( \lambda \) is a solution to

\[ a\lambda^2 + b\lambda + c = 0, \]  (11)

which is called the **characteristic equation** (or the **auxiliary equation**), then \( e^{\lambda x} \) is a solution to (10).

**Example 3.1.1.** *Find the solutions to*

\[ y'' + 2y' - y = 0. \]

*Find the solution to the equation with initial values \( y(0) = 0, y'(0) = -1. \)*

**Solution.** First let us solve the characteristic equation

\[ \lambda^2 + 2\lambda - 1 = 0. \]

We get the roots are

\[ \lambda_1 = -1 + \sqrt{2}, \quad \lambda_2 = -1 - \sqrt{2}. \]

Therefore we find two special solutions

\[ y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}. \]
Since the equation is linear, any functions of the following form are solutions
\[ y := C_1 y_1 + C_2 y_2 = C_1 e^{(-1+\sqrt{2})x} + C_2 e^{(-1-\sqrt{2})x} \]
where \( C_1, C_2 \) are any constants.

Now we solve for the initial value problem. By the condition
\[
\begin{align*}
y(0) &= C_1 + C_2 = 0, \\
y'(0) &= C_1 (-1 + \sqrt{2}) + C_2 (-1 - \sqrt{2}) = -1.
\end{align*}
\]
We obtain \( C_1 = -C_2 \) from the first equality. And then from the second, we have
\[ -1 = C_1 (-1 + \sqrt{2}) - C_1 (-1 - \sqrt{2}) = C_1 2\sqrt{2}. \]
We have
\[ C_1 = -\frac{\sqrt{2}}{4}, \quad C_2 = \frac{\sqrt{2}}{4}. \]
Thus
\[ y = -\frac{\sqrt{2}}{4} e^{(-1+\sqrt{2})x} + \frac{\sqrt{2}}{4} e^{(-1-\sqrt{2})x}. \]

Now let us answer the question that how many solutions are there. We need the following definition.

**Definition 3.1.** A pair of functions \( y_1(x) \) and \( y_2(x) \) is said to be **linearly independent** on the interval \( I \) if NEITHER of them is a constant multiple of the other on \( I \). We say that they are **linearly dependent** is one of them is a constant multiple of the other.

**Lemma 3.2.** The **Wronskian** of \( y_1(x), y_2(x) \) is defined to be
\[ W(y_1, y_2) := y_1 y_2' - y_2 y_1'. \]
\( y_1, y_2 \) are linearly dependent if and only if \( W(y_1, y_2) = 0 \).

**Theorem 3.3.** If \( y_1, y_2 \) are two solutions to (10) and they are linearly independent on \( \mathbb{R} \), then
\[ \{C_1 y_1 + C_2 y_2 \quad \text{with} \quad C_1, C_2 \in \mathbb{R} \} \]
are all the solutions to (10). In particular, if the characteristic equation (11) has two different real roots \( \lambda_1, \lambda_2 \), then all the solutions to (10) are of the form
\[ C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}, \]
where \( C_1, C_2 \) are constants.
The theorem is useful since it tells us that to find all solutions to (10), we only need to find two linearly independent particular solutions. And we have already shown the way to find two solutions if the characteristic function has two different real roots. Now we consider the case if the characteristic function has only one repeated real root.

**Theorem 3.4.** It the characteristic function has only one repeated real root \( \lambda \), then both \( y_1(x) = e^{\lambda x} \) and \( y_2(x) = xe^{\lambda x} \) are solutions to (10). In such a case

\[
y(x) = C_1 e^{\lambda x} + C_2 xe^{\lambda x}
\]

with \( C_1, C_2 \in \mathbb{R} \), are the general solutions.

**Example 3.1.2.** Find a solution to the initial value problem

\[
y'' + 4y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 3.
\]

**Solution.** The corresponding characteristic equation is

\[
\lambda^2 + 4\lambda + 4 = 0
\]

which has a repeated root \( \lambda = -2 \). Hence the general solutions to the differential equation are

\[
y(x) = C_1 e^{-2x} + C_2 xe^{-2x}.
\]

Using the initial data, we can solve for the value of \( C_1, C_2 \). We get

\[
y(x) = e^{-2x} + 5xe^{-2x}.
\]

Actually the same idea applies to high order equations.

**Example 3.1.3.** Find a general solution to

\[
y''' + y'' - 5y' + 3y = 0.
\]

**Solution.** The corresponding characteristic equation is

\[
0 = \lambda^3 + \lambda^2 - 5\lambda + 3 = (\lambda - 1)^2(\lambda + 3).
\]

Then \( \lambda = 1 \) is a root with multiplicity 2 and \( \lambda = -3 \) is another root. It is not hard to check that \( y_1 = e^x, y_2 = xe^x \) and \( y_3 = e^{-3x} \) are solutions. The general solutions are then given by

\[
y(x) = C_1 e^x + C_2 xe^x + C_3 e^{-3x}.
\]

\[ \square \]
### 3.2 Complex roots

Consider the equation
\[ ay'' + by' + cy = 0 \]
and its characteristic function
\[ a\lambda^2 + b\lambda + c = 0. \]
Suppose there are two complex roots to the characteristic function:
\[ \lambda_1 = \alpha + \beta i, \quad \lambda_2 = \alpha - \beta i. \]
Theorem 3.3 implies that all the solutions to the equation are of the form
\[ C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}. \]

Notice here we have exponential function valued at a complex number. We introduce the well-known **Euler’s formula**:
\[ e^{i\theta} = \cos \theta + i \sin \theta. \]
Hence we have
\[
\begin{align*}
  e^{(\alpha+i\beta)x} &= e^{\alpha x} e^{i\beta x} = e^{\alpha x} \cos(\beta x) + ie^{\alpha x} \sin(\beta x), \\
  e^{(\alpha-i\beta)x} &= e^{\alpha x} e^{-i\beta x} = e^{\alpha x} \cos(\beta x) - ie^{\alpha x} \sin(\beta x).
\end{align*}
\]
Now we pick \( C_1 = \frac{1}{2}, C_2 = \frac{1}{2} \) in (12), and then \( C_1 = -\frac{i}{2} \) and \( C_2 = \frac{i}{2} \). We get respectively that the real and complex parts of the above particular solutions, which are
\[
\begin{align*}
  e^{\alpha x} \cos(\beta x), & \quad e^{\alpha x} \sin(\beta x),
\end{align*}
\]
are two linearly independent solutions to the original equations.

In view of Theorem 3.3, we have the following theorem.

**Theorem 3.5.** If the characteristic equation has two complex conjugate roots \( \alpha \pm i \beta \), then the general real solutions to the equations are
\[ y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x \]
where \( c_1, c_2 \) are arbitrary real numbers.

**Example 3.2.1.** Find the general solutions to
\[ 36y'' - 12y' + 37y = 0. \]

**Solution.** The corresponding characteristic equation is
\[ 36\lambda^2 - 12\lambda + 37 = 0. \]
The roots are
\[ \lambda_{1,2} = \frac{1}{6} \pm i. \]
Therefore
\[ e^{x/6} \cos x, \quad e^{x/6} \sin x \]
are two linearly independent solutions. By Theorem 3.5, the general real solutions are
\[ y(x) = c_1 e^{x/6} \cos x + c_2 e^{x/6} \sin x \]
with \( c_1, c_2 \in \mathbb{R} \).

**Example 3.2.2.** Find the general solution to
\[ y^{(4)} + 13y'' + 36y = 0. \]

**Solution.** The corresponding characteristic equation is
\[ \lambda^4 + 13\lambda^2 + 36 = 0. \]
Since
\[ \lambda^4 + 13\lambda^2 + 36 = (\lambda^2 + 4)(\lambda^2 + 9), \]
the roots are
\[ \lambda_{1,2} = \pm 2i, \quad \lambda_{3,4} = \pm 3i. \]
Thus
\[ \cos 2x, \quad \sin 2x, \quad \cos 3x, \quad \sin 3x \]
are four linearly independent solutions. It follows from Theorem 3.5 that the general solutions are
\[ y(x) = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x. \]

### 3.3 Undetermined Coefficients

In this section let us attack the nonhomogeneous equation with constant coefficients:

\[ ay'' + by' + cy = f(x). \]  \hspace{1cm} (13)  \hspace{1cm} 3.3

In the following example, we first find one solution to the equation. We usually call any one solution as a **particular solution**.

**Example 3.3.1.** Find a particular solution to
\[ y'' + 3y' + 2y = 3x. \]
Solution. Let us try \( y = ax + b \). Plug in \( y = ax + b \) into the equation, we get

\[
\text{the RHS } = 3a + 2ax + 2b.
\]

In order to have the RHS the same as the LHS, we need

\[
3a + 2b + 2ax = 3x
\]

which gives \( a = \frac{3}{2} \) and \( b = -\frac{9}{4} \). Thus \( y = \frac{3}{2}x - \frac{9}{4} \) is one solution.

Remark 3.6. The undetermined coefficient method:

1. If \( f = \sum_{i=0}^{n} a_i x^i \), then we try

\[
y(x) = \sum_{i=0}^{n} b_i x^i
\]

and solve for \( b_i \) to get one particular solution. If not working, we need to try

\[
y(x) = x(\sum_{i=0}^{n} b_i x^i), \quad \text{if } 0 \text{ is a single root of the char. eqn.}
\]

\[
y(x) = x^2(\sum_{i=0}^{n} b_i x^i), \quad \text{if } 0 \text{ is a double root of the char. eqn.}
\]

2. If \( f = (a_0 + a_1 x)e^{rx} \), then we try

\[
y(x) = (b_0 + b_1 x)e^{rx}
\]

and solve for \( b_0, b_1 \) to get one particular solution. Sometimes we need to try

\[
y(x) = x(b_0 + b_1 x)e^{rx}, \quad \text{if } r \text{ is a single root of the char. eqn.}
\]

\[
y(x) = x^2(b_0 + b_1 x)e^{rx} \quad \text{if } r \text{ is a double root of the char. eqn.}
\]

3. If \( f = a_0 \cos x \) or \( a_0 \sin x \) or \( a_0 \cos x + a_1 \sin x \), then we try

\[
y(x) = b_0 \cos x + b_1 \sin x
\]

and solve for \( b_0, b_1 \) to get one particular solution. Again sometimes we need

\[
b_0 x \cos x + b_1 x \sin x \quad \text{or } b_0 x^2 \cos x + b_1 x^2 \sin x.
\]

Example 3.3.2. Find a particular solution to

\[
y'' - 4y' + 4y = e^{2t} + 5e^{-3t}.
\]

Solution. Let us consider the following two equations separately:

\[
y'' - 4y' + 4y = e^{2t}, \quad \text{(14)}
\]

\[
y'' - 4y' + 4y = 5e^{-3t}. \quad \text{(15)}
\]
For the first equation try
\[ y_1(t) = ae^{2t} \]
and we find
\[ y_1'' - 4y_1' + 4y_1 = 0 \]
for all \(a\). Also it is not possible to have \(y_1(t) = ate^{2t}\) being a solution. Let us try \(y_1(t) = at^2e^{2t}\). Since
\[ y_1' = 2ate^{2t} + 2a^2e^{2t}, \]
\[ y_1'' = 2ae^{2t} + 8ate^{2t} + 4at^2e^{2t}. \]
We get
\[ y_1'' - 4y_1' + 4y_1 = 2ae^{2t}, \]
which implies \(a = \frac{1}{2}\) and \(y_1 = \frac{1}{2}t^2e^{2t}\) is one solution to (14).

Now we try \(y_2 = be^{-3t}\) for the equation (15). We get
\[ y_2'' - 4y_2' + 4y_2 = (9b + 12b - 4b)e^{-3t} = 17be^{-3t}. \]
Thus \(y_2 = \frac{5}{17}e^{-3t}\) is a solution to (15).

We claim that
\[ y = y_1 + y_2 = \frac{1}{2}t^2e^{2t} + \frac{5}{17}e^{-3t} \]
is a particular solution to the original equation. This is because
\[ (y_1 + y_2)'' - 4(y_1 + y_2)' + 4(y_1 + y_2) = y_1'' - 4y_1' + 4y_1 + y_2'' - 4y_2' + 4y_2 = e^{2t} + 5e^{-3t}. \]

\[ \square \]

**Example 3.3.3.** Find the form for a particular solution to
\[ y'' - 2y' + 3y = 2te^t \sin t. \]

**Solution.** Try
\[ y_p(t) = (a_0 + a_1t)e^t \cos t + (b_0 + b_1t)e^t \sin t. \]

\[ \square \]

### 3.4 General solutions to Nonhomogeneous equation

**Theorem 3.7** (Superposition Principle). If \(y_1, y_2\) are respectively solutions to
\[ ay'' + by' + cy = f_1(x), \quad ay'' + by' + cy = f_2(x), \]
then \(k_1y_1 + k_2y_2\) is a solution to
\[ ay'' + by' + cy = k_1f_1(x) + k_2f_2(x). \]
Theorem 3.8. Suppose \( y_p \) is a particular solution to

\[
ay'' + by' + cy = f(x)
\]

and \( y_1, y_2 \) are two linearly independent solutions to the homogeneous equation

\[
ay'' + by' + cy = 0.
\]

Then the general solutions to (16) are (all solutions are of the following form)

\[
y_p + C_1 y_1 + C_2 y_2
\]

with \( C_1, C_2 \in \mathbb{R} \).

Example 3.4.1. Given that \( y_p(x) = x^2 \) is a particular solution to

\[
y'' - y = 2 - x^2,
\]

find a solution satisfying \( y(0) = 1, y'(0) = 0 \).

Solution. The corresponding homogeneous equation,

\[
y'' - y = 0,
\]

has the associated auxiliary equation \( \lambda^2 - 1 = 0 \). The equation has \( \lambda = \pm 1 \) two different real roots. So \( y_1 = e^{\pm x} \) are two linearly independent solutions to the homogeneous equation. We find that the general solutions to the original nonhomogenous equation are

\[
y(x) = x^2 + C_1 e^x + C_2 e^{-x}.
\]

To meet the initial conditions, set

\[
y(0) = C_1 + C_2 = 1,
\]

\[
y'(0) = C_1 - C_2 = 0,
\]

which yields \( C_1 = C_2 = \frac{1}{2} \). The answer is

\[
y(x) = x^2 + \frac{1}{2}(e^x + e^{-x}) = x^2 + \cosh x.
\]

Example 3.4.2. A mass-spring system is driven by a sinusoidal external force \( 5 \sin t + 5 \cos t \). The equation is

\[
my'' + by' + ky = F_{\text{ext}}(t).
\]

The mass \( m \) equals 1, the spring constant \( k = 2 \) and the damping coefficient \( b = 2 \). If the mass is initially located at \( y(0) = 1 \) with velocity \( y'(0) = 2 \), find its equation of motion.
**Solution.** Since $F_{\text{ext}}(t) = 5 \sin t + 5 \cos t$, let us use the method of undetermined coefficients and try

$$y_p = A \sin t + B \cos t.$$ 

Plugging in $y_p$ into the equation and solving for $A, B$ gives

$$y_p = 3 \sin t - \cos t.$$ 

The associated homogeneous equation is

$$y'' + 2y' + 2y = 0.$$ 

Solve for the corresponding characteristic equation and we get two complex roots: $-1 \pm i$. Therefore the general solutions to the homogeneous equation are

$$C_1 e^{-t} \cos t + C_2 e^{-t} \sin t.$$ 

The general solutions to the original nonhomogenous equation are

$$C_1 e^{-t} \cos t + C_2 e^{-t} \sin t + 3 \sin t - \cos t.$$ 

Finally the initial data implies that $C_1 = 2, C_2 = 1$, and thus the solution is

$$y(t) = 2 e^{-t} \cos t + e^{-t} \sin t + 3 \sin t - \cos t.$$ 

\[ \square \]

### 3.5 Variation of Parameters

**Variation of parameters** is a more general method (comparing to the undetermined coefficients method) to find a particular solution to a linear second order equation.

Consider

$$a y'' + b y' + c y = f(x) \tag{17} \label{eq:3.5.1}$$

and suppose $y_1(x), y_2(x)$ are two linearly independent solutions to

$$a y'' + b y' + c y = 0.$$ 

Then we know that $C_1 y_1(x) + C_2 y_2(x)$ are all solutions to the homogeneous equation. The variation of parameters method is the strategy to replace the constants $C_1, C_2$ by functions i.e. we seek a solution of the form

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x).$$

Let us formally plug $y_p$ into the RHS of the equation (17). Direct computation yields

$$y_p' = v_1' y_1 + v_2' y_2 + v_1 y_1' + v_2 y_2'.$$
Since \( v_1, v_2 \) give us two much freedom, it is hard to find the right ones. We impose the requirement

\[ v_1'y_1 + v_2'y_2 = 0. \]

Such requirement is good, because it also simplifies the expression of \( y'_p \):

\[ y'_p = v_1'y_1 + v_2'y_2. \]

And then

\[ y''_p = v_1'y_1 + v_2'y_2 + v_1'y_1'' + v_2'y_2''. \]

Substituting \( y_p, y'_p, y''_p \) into (17), we find

\[
\begin{align*}
f(x) &= ay''_p + by'_p + cy_p \\
&= a(v_1'y_1'' + v_2'y_2'') + b(v_1'y_1' + v_2'y_2') + c(v_1(x)y_1(x) + v_2(x)y_2(x)) \\
&= a(v_1'y_1' + v_2'y_2') + v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) \\
&= a(v_1'y_1' + v_2'y_2').
\end{align*}
\]

To summarize, if we can find \( v_1, v_2 \) that satify

\[
\begin{align*}
a(v_1'y_1' + v_2'y_2') &= f(x), \\
v_1'y_1 + v_2'y_2 &= 0,
\end{align*}
\]

then \( y_p \) will be a particular solution.

Also notice in the above argument, we do not really used that \( a, b, c \) are constants. Indeed the method applies to non-constant coefficients equation!

**Example 3.5.1.** Find a general solution on \((-\pi/2, \pi/2)\) to

\[ y'' + y = \tan t. \]

**Solution.** Consider \( y_p \) of the form

\[ y_p = v_1y_1 + v_2y_2 \]

where \( v_1, v_2 \) are two functions and \( y_1, y_2 \) solves the homogeneous equation. So \( y_1 = \cos t \) and \( y_2 = \sin t \).

By the method of variation of parameter, we need \( v_1, v_2 \) to satisfy

\[
\begin{align*}
v_1'y_1' + v_2'y_2' &= \tan t, \\
v_1'y_1 + v_2'y_2 &= 0,
\end{align*}
\]

this is

\[
\begin{align*}
- v_1' \sin t + v_2' \cos t &= \tan t, \\
v_1' \cos t + v_2' \sin t &= 0.
\end{align*}
\]
This is a linear system for $v'_1, v'_2$. We get

$$v'_1 = - \tan t \sin t, \quad v'_2 = \sin t.$$  

After integrating

$$v_1 = - \int \tan t \sin t \, dt = - \int \frac{\sin^2 t}{\cos^2 t} \, dt \sin t = \sin t - \ln |\sec t + \tan t| + C,$$

$$v_2 = \int \sin t \, dt = - \cos t + C.$$  

We only need one particular solution, so pick $C = 0$ in the above. We obtain

$$y_p(t) = v_1y_1 + v_2y_2 = (\sin t - \ln |\sec t + \tan t|) \cos t - \cos t \sin t.$$  

The general solutions are

$$y = C_1 \cos t + C_2 \sin t - (\cos t) \ln |\sec t + \tan t|).$$

\[\square\]

Remark 3.9. The formula for $v_1, v_2$ is

$$v_1 = \int -\frac{f y_2}{aW(y_1, y_2)} \, dx, \quad v_1 = \int \frac{f y_1}{aW(y_1, y_2)} \, dx.$$  

Example 3.5.2. Find a particular solution of the variable coefficient linear equation

$$t^2 y'' - 4ty' + 6y = 4t^3$$  

given $y_1 = t^2$ and $y_2 = t^3$ are solutions to the corresponding homogeneous equation.

Solution. Let us use the variation of parameter method. Write $y_p = v_1 y_1 + v_2 y_2$. Given $y_1, y_2$, we have

$$W(y_1, y_2) = y_1 y'_2 - y'_1 y_2 = t^4.$$  

Then

$$v_1 = \int -\frac{f y_2}{aW(y_1, y_2)} \, dt = \int -\frac{4t^3 t^3}{t^2(t^4)} \, dt = \int 4 \, dt = 4t + C,$$

$$v_1 = \int \frac{f y_1}{aW(y_1, y_2)} \, dt = \int \frac{4t^3 t^2}{t^6} \, dt = \int \frac{4}{t} \, dt = 4 \ln |t| + C.$$  

Take $C = 0$. The particular solution is

$$y_p = v_1 y_1 + v_2 y_2 = 4t^3(-1 + \ln |t|).$$

\[\square\]
### 3.6 Variable-Coefficient Equations

**Theorem 3.10.** Consider
\[ y' + p(x)y = f(x). \]
If \( p, f \) are continuous on an interval \((a, b)\) that contains \(x_0\), then for any choice of the initial values \(Y_0\), there exists a unique solution \(y(x)\) to the equation satisfying the initial data \(y(x_0) = Y_0\).

**Theorem 3.11.** Consider
\[ y'' + p(x)y' + q(x)y = f(x). \]
If \( p, q, f \) are continuous on an interval \((a, b)\) that contains \(x_0\), then for any choice of the initial values \(Y_0, Y_1\), there exists a unique solution \(y(x)\) to the equation satisfying the initial data
\[ y(x_0) = Y_0, \quad y'(x_0) = Y_1. \]

The following example is one problem in the first midterm.

**Example 3.6.1** (Loss of uniqueness). Consider the initial value problem
\[ \frac{dx}{dt} = \frac{(t + 1)^3}{xt}, \quad x(1) = 0. \]

Using the separable equation method, we get
\[ \frac{1}{2}x^2 = \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 + t - \frac{15}{4}. \]

So there are actually two solutions:
\[ x = \pm \sqrt{\frac{1}{2}t^4 + 2t^3 + 3t^2 + 2t - \frac{15}{2}}. \]

**Explanation why this does not contradict with theorem.**

**Definition 3.12.** A linear second order equation that can be expressed in the form
\[ ax^2y''(x) + bxy'(x) + cy(x) = f(x) \tag{18} \]
where \( a, b, c \) are constants, is called a **Cauchy-Euler**, or **equidimensional equation**.

The idea is to look for solutions of the form \(y = x^r\). Let us explain the idea through the following example.

**Example 3.6.2.** Find two linearly independent solutions to
\[ 3t^2y'' + 11ty' - 3y = 0, \quad t > 0. \]
Solution. Let $y = t^r$. Direct computation yields
\[ y' = rt^{r-1}, \quad y'' = r(r - 1)t^{r-2}. \]
Inserting these into the equation, we get
\[
RHS = 3t^2r(r - 1)t^{r-2} + 11ttr^{r-1} - 3t^r
\]
\[ = 3(r^2 - r)t^r + 11rt^r - 3t^r
\]
\[ = (3r^2 + 8r - 3)t^r.
\]
To have $RHS = 0$, we only need $r$ to be one root of $3r^2 + 8r - 3 = 0$. Hence $r = \frac{1}{3}$ and $-3$. We obtain two linearly independent solutions
\[ y_1 = t^{1/3}, \quad y_2 = t^{-3}. \]

Method for (18): try $y = x^r$. Then we get the associated characteristic equation:
\[ ar^2 + (b - a)r + c = 0. \]
1. the char. eqn. has two real roots $r_1, r_2$. $x^{r_1}, x^{r_2}$ are two solutions.
2. the char. eqn. has one repeated real root $r$. $x^r, x^r \ln x$ are two solutions.
3. the char. eqn. has two complex roots $\alpha \pm \beta i$. $x^\alpha \cos(\beta \ln x), x^\alpha \sin(\beta \ln x)$ are two solutions.

Now let me check that in the case 2, $y = x^r \ln x$ is indeed the solution. We have
\[ y' = rx^{r-1} \ln x + x^{r-1}, \quad y'' = r(r - 1)x^{r-2} \ln x + (2r - 1)x^{r-2}. \]
Plug them into the equation we get
\[
ax^2y'' + bxy' + cy = ar(r - 1)x^r \ln x + a(2r - 1)x^r + brx^r \ln x + bx^r + cx^r \ln x
\]
\[ = (ar^2 + (b - a)r + c)x^r \ln x + (2ar + b - a)x^r
\]
\[ = 0.
\]
Here we used that $r = \frac{b - a}{2a}$ is the repeated root.

Example 3.6.3. Find two linearly independent solutions to
(1) $x^2y'' + 5xy' + 5y = 0$, (2) $x^2y'' + xy' = 0$.

Solution. For (1), the associated characteristic equation is
\[ r^2 + 4r + 5 = 0, \]
which gives two complex roots $r = -2 \pm i$. So the two solutions are
\[ x^{-2} \cos \ln x, \quad x^{-2} \cos \ln x. \]

For (2), the associated characteristic equation: $r^2 = 0$, has a repeated root $r = 0$. So the two solutions are
\[ x^0 = 1, \quad \ln x. \]

\[ \Box \]

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4 Laplace Transforms

4.1 Definition of Laplace Transform

Definition 4.1. Let $f(t)$ be a function on $[0, \infty)$. The Laplace transform of $f$ is the function $F$ defined by

$$F(s) := \int_0^{\infty} e^{-st} f(t) \, dt.$$ 

We usually use the notation $F(s) = \mathcal{L}\{f\}(s)$.

Example 4.1.1. Determine the Laplace transform of

(1) $f(t) = e^{at}, \ t \geq 0$. (2) $g(t) = 2 \chi_{[0,5]}$.

Solution.

$$\mathcal{L}\{f\} = \int_0^{\infty} e^{-st} e^{at} \, dt = \lim_{N \to \infty} \int_0^N e^{(a-s)t} \, dt$$

$$= \lim_{N \to \infty} \left. \frac{e^{(a-s)t}}{a-s} \right|_0^N = \lim_{N \to \infty} \left( \frac{e^{(a-s)N}}{a-s} - \frac{1}{a-s} \right).$$

When $s \leq a$, the limit does not exist. When $s > a$, the limit equals $\frac{1}{s-a}$. So

$$\mathcal{L}\{f\}(s) = \frac{1}{s-a}$$

with the domain $s > a$.

Next

$$\mathcal{L}\{g\} = \int_0^{\infty} e^{-st} 2 \chi_{[0,5]} \, dt = \int_0^5 2e^{-st} \, dt$$

$$= \frac{2}{s} - \frac{2e^{-5s}}{s}$$

when $s \neq 0$. And when $s = 0$, $\mathcal{L}\{g\}(0) = 10$. \hfill \Box

Theorem 4.2 (Linearity of the transform). Let $f_1, f_2$ be functions whose Laplace transform exist for $s \in I$ and let $c_1, c_2$ be constants. Then for $s \in I$,

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_1\}.$$
Definition 4.3. A function $f(t)$ is said to be **piecewise continuous on** $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f$ has a jump discontinuity.

A function $f(t)$ is said to be **piecewise continuous on** $[0, \infty)$ if $f$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Definition 4.4. A function $f(t)$ is said to be of **exponential order** $\alpha$ if there exist $T, M$ such that

$$|f(t)| \leq Me^{\alpha t}, \quad \text{for all } |t| \geq T.$$  

Example 4.1.2.

$$f(t) = \begin{cases} 
    t, & 0 < t \leq 1, \\
    2, & 1 < t \leq 2, \\
    e^{5t} \sin 2t, & t > 2, 
\end{cases}$$

is a piecewise continuous function in $[0, \infty)$ of exponential order $\alpha$ for all $\alpha \geq 5$.

However,

$$f(t) = e^{2t}, \quad f(t) = e^{3t}$$

are functions of NO exponential order. They grow too fast at $\infty$.

**Theorem 4.5.** If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order $\alpha$, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$. 

---

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<tr>
<td>$\cos at, \ a$ a real constant</td>
<td>$\frac{s}{s^2 + a^2}$</td>
<td>Re($s$) $&gt; 0$</td>
</tr>
<tr>
<td>$e^{kt} \sin at, \ k$ and $a$ real constants</td>
<td>$\frac{a}{(s + k)^2 + a^2}$</td>
<td>Re($s$) $&gt;-k$</td>
</tr>
<tr>
<td>$e^{kt} \cos at, \ k$ and $a$ real constants</td>
<td>$\frac{s + k}{(s + k)^2 + a^2}$</td>
<td>Re($s$) $&gt;-k$</td>
</tr>
</tbody>
</table>
Proof. We need to show that
\[ \int_0^{\infty} e^{-st} f(t) dt \]
exists (converges) for \( s > \alpha \).

Since \( f \) is of exponential order \( \alpha \), there are \( T, M \) such that
\[ |f(t)| \leq Me^{\alpha t}, \quad \text{for } t > T. \]

Hence write
\[ \int_0^{\infty} e^{-st} f(t) dt = \int_0^T e^{-st} f(t) dt + \int_T^{\infty} e^{-st} f(t) dt. \]
The first integral is finite because \( e^{-st} f(t) \) is a bounded function and the domain is finite. As for the second integral, we apply the comparison test. In view of
\[ |f(t)| \leq Me^{\alpha t}, \]
and
\[ \int_T^{\infty} e^{-st} Me^{\alpha t} dt = M \int_T^{\infty} e^{-(s-\alpha)t} dt = \frac{Me^{-(s-\alpha)T}}{s-\alpha} < \infty, \]
the second integral converges for all \( s > \alpha. \)

\[ \square \]

4.2 Properties of the Laplace Transform

**Theorem 4.6 (Translation).** If \( \mathcal{L}\{f\}(s) \) exists for \( s > \alpha \), then
\[ \mathcal{L}\{e^{\beta t} f(t)\}(s) = \mathcal{L}\{f\}(s - \beta) \]
for \( s > \alpha + \beta. \)

**Theorem 4.7 (Derivative).** Let \( f(t) \) be continuous on \([0, \infty)\) and \( f'(t) \) be piecewise continuous on \([0, \infty)\), with both of exponential order \( \alpha \). Then for \( s > \alpha \),
\[ \mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0). \]

**Corollary 4.8 (Integral).** The following equality holds whenever all the terms inside are well-defined
\[ \mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\}(s) = \frac{1}{s} \mathcal{L}\{f(t)\}(s) \]

**Theorem 4.9 (Higher-Order Derivatives).** Let \( f(t), f'(t), ..., f^{(n-1)}(t) \) be continuous on \([0, \infty)\) and \( f^{(n)}(t) \) be piecewise continuous on \([0, \infty)\), with all these functions of exponential order \( \alpha \). Then for \( s > \alpha \),
\[ \mathcal{L}\{f^{(n)}\}(s) = s^n \mathcal{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - f^{(n-1)}(0). \]
Theorem 4.10. [Multiple of a polynomial] Let $F(s) = \mathcal{L}\{f\}(s)$ for $s > \alpha$. Then, for $s > \alpha$,

$$
\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s).
$$

Example 4.2.1. Determine $\mathcal{L}\{e^{at}\sin bt\}$.

Solution. Recall

$$
\mathcal{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2 + b^2}.
$$

Thus, by the translation property,

$$
\mathcal{L}\{e^{at}\sin bt\}(s) = F(s - a) = \frac{b}{(s - a)^2 + b^2}.
$$

Example 4.2.2. Determine $\mathcal{L}\{t\sin bt\}$.

Solution. We know

$$
\mathcal{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2 + b^2}.
$$

It follows from Theorem 4.10,

$$
\mathcal{L}\{t\sin bt\} = -\frac{dF}{ds}(s) = \frac{2bs}{(s^2 + b^2)^2}.
$$

Example 4.2.3. Determine $\mathcal{L}\{\sin^2 t + e^{3t}t^2\}$.

Solution. Since $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$,

$$
\mathcal{L}\{\sin^2 t\} = \frac{1}{2} \mathcal{L}\{1\} - \frac{1}{2} \mathcal{L}\{\cos 2t\} = \frac{1}{2s} - \frac{s}{2(s^2 + 4)}.
$$

Next,

$$
\mathcal{L}\{e^{3t}t^2\} = \frac{d^2}{ds^2} \mathcal{L}\{e^{3t}\}(s) = \frac{d^2}{ds^2} \frac{1}{s - 3} = \frac{2}{(s - 3)^3}.
$$

So

$$
\mathcal{L}\{\sin^2 t + e^{3t}t^2\} = \frac{1}{2s} - \frac{s}{2(s^2 + 4)} + \frac{2}{(s - 3)^3}.
$$

Example 4.2.4. Prove Theorem 4.9 for $n = 3$. 

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Solution. Apply Theorem 4.7 for $f$ and $f'$:

$$\mathcal{L}\{f'(s)\} = s\mathcal{L}\{f\}(s) - f(0),$$

$$\mathcal{L}\{f''(s)\} = s\mathcal{L}\{f'(s)\} - f'(0).$$

Applying Theorem 4.7 for $f'$ and $f''$, we get

$$\mathcal{L}\{f^{(3)}\}(s) = s\mathcal{L}\{f''\}(s) - f''(0).$$

Use the above three:

$$\mathcal{L}\{f^{(3)}\}(s) = s\mathcal{L}\{f''\}(s) - f''(0)$$

$$= s(s\mathcal{L}\{f'(s)\} - f'(0)) - f''(0)$$

$$= s^2\mathcal{L}\{f'(s)\} - sf'(0) - f''(0)$$

$$= s^3\mathcal{L}\{f(s)\} - s^2f(0) - sf'(0) - f''(0).$$

☐