UCSD

Lecture : MATH 20A Calculus for Science and Engineering

2021 Fall

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2 Limits

2.1 The limit idea

Zeno's Paradox. Greek philosopher Zeno raised some questions to people:

In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.

Suppose I want to walk through a doorway, and I am standing a meter away from the door. At some point I am at a half meter away, then at another point a quarter of a meter away, and so on. So I can never make it through the doorway.

This can not happen, because we know from real life that quicker runners can always overtake slower ones, and normally people can always pass through a door way if there is no obstacle. What is the problem?

The problem is: the quickest runner can not overtake the slowest, but only within a finite short time. Suppose the pursuer reach the point x_0 whence the pursued started at time t_1. Then the slower ran ahead at x_1. The pursuer spend t_2 time to reach x_1, and the slower used the t_2 time to be ahead at x_2... The pursuer is always behind the slower in the time $t_1 + t_2 + t_3 + ...$ However this is not forever! $t_1 + t_2 + t_3 + ...$ has a finite limit, and the pursuer is only behind before that finite short time (limit). In the second paradox, I am subdividing the one meter into infinite number of smaller steps. Suppose it takes me 1 second to go one meter, so to pass through the infinitely many "small steps", I still need just 1 second (instead of forever).

Example 2.1.1. *What is the value of* $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$*? (Think about it, and we will solve it later.)*

Example 2.1.2. *(Velocity) Suppose a car is running on a straight high way. Let* $s(t)$ *denote the position of the car at time* t*.*

1. The **average velocity** *over a time interval* $[t_0, t_1]$ *is defined as*

$$\frac{\textit{change in position}}{\textit{change in time}} = \frac{\Delta s}{\Delta t} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

2. If you plot the graph of $s(t)$*, we can connect the two points* $(t_0, s(t_0))$ *with* $(t_1, s(t_1))$ *by a straight line. We call the line the* **secant line***. The average velocity then equals the slope of the secant line.*

3. We can use the language of limit to define its **velocity** *at time* t *as follows*

$$\lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}.$$

(For "some bad" functions $s(t)$*, the above limit may not exist, and in that case we cannot define the velocity.)*

4. We can interpret the velocity as the limit of the slopes of the secant lines as $\Delta t \to 0$*. From the picture we see that the limit* $=$ *the velocity* $=$ *the slope of* **the tangent line***.*

Definition 2.1. 1. Let $\{a_1, a_2, a_3, ...\}$ (denoted as $\{a_n\}$) be a sequence of (real) numbers. If there is a number c such that a_n can be arbitrarily close to c for all n large enough, then we say that the sequence $\{a_n\}$ has a limit c. We will write $a_n \to c$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = c$ (reading a_n converges c as n goes to infinity).

2. Let f be a function from the real line (denoted as $\mathbb{R}$) to $\mathbb{R}$. If for some real numbers a, L, we have $f(x) \to L$ along any sequence of $x \to a$, then we say that f has a limit L as $x \to a$. We write $\lim_{x\to a} f(x) = L$.

Example 2.1.3. *Consider sequences* $\{2^{-n}\}$, $\{1, -1, 1, -1, ...\}$. *Do they have limit? If so, what is the limit?*

Example 2.1.4. *Consider the following function* $f = \frac{x-1}{x-1}$. *Since* $x - 1$ *is in the denominator, the function is not defined at* $x = 1$. *But do we have a limit of the function at* $x = 1$*?*

2.2 Limits

Let us first take a look at some functions which are not "super nice" (smooth).

Example 2.2.1. *(Discontinuous functions) 1. Let* $f = 1$ *if* $x \neq 0$, *and* $f = 0$ *if* $x = 0$. *Can you plot the graph of* f*? What is* $\lim_{x\to 0} f(x)$*?*

2. (Heaviside function) $H = 0$ *if* $x \leq 0$, *and* $H = 1$ *if* $x > 0$. *Can you plot the graph of* H*? Does* H *has a limit at* $x = 0$*?*

Definition 2.2. The limit as x decreases in value approaching a (so $x > a$) is called the right-hand limit, denoted as $\lim_{x\to a^+} f(x)$.

The limit as x increases in value approaching a (so $x < a$) is called the left-hand limit, denoted as $\lim_{x\to a^-} f(x)$.

If the left-hand limit does not equal to the right-hand limit, then the limit does not exist. Can you determine the right-hand and left-hand limits of H at $x = 0$?

Sometimes, even the right- (or/and left-) hand limit does not exist.

Example 2.2.2. *Consider the following examples:* $\frac{1}{x}, \ln x$.

If as x approaches a value a (or a^+ or a^-), $f(x)$ tends to ∞ or $-\infty$, since $\pm\infty$ are not numbers, we say that $\lim_{x\to a} f(x)$ (or $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$) does not exist. We can also say that $f(x)$ has an **infinite limit**.

Finally let us introduce one more situation where the limit does not exist. It is called **infinite oscillations**.

Example 2.2.3. *Consider* $\sin\frac{1}{x}$. *What is the value of the function for* $x = \frac{1}{2k\pi+\frac{\pi}{2}}, \frac{1}{2k\pi+\frac{3\pi}{2}}$ *with* k *being integers? Can you plot the graph of the function for* $x > 0$*? What about* $x < 0$*?*

2.3 Basic laws

In order to manipulate limits, we need to be able to apply some simple operations to them.

Proposition 2.3. *Let $f, g : \mathbb{R} \to \mathbb{R}$, and let $a, c \in \mathbb{R}$. Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then*

1. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$;
3. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$;
4. *if* $\lim_{x\to a} g(x) \neq 0$, *then* $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$;
5. *if n is a positive integer,* $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n$. *If p, q are positive integers, and* $f(x) \geq 0$, *then* $\lim_{x\to a}(f(x))^{p/q} = (\lim_{x\to a} f(x))^{p/q}$.

   (We use the notation that $\sqrt[n]{f} = f^{1/n}$ because $(\sqrt[n]{f})^n = f$.)

Example 2.3.1. *Since $\lim_{x\to 4} x = 4$, we know*

$$\lim_{x\to 4} x^2 = (\lim_{x\to 4} x)^2 = 16.$$

2.4 Continuity

Definition 2.4. (Intuitive one) Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. We say that f is continuous if we can draw the graph of f near a without lifting the pencil off the paper.

Definition 2.5. Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. We say that f is **continuous** at a if

1. a is in the domain of f,
2. $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = f(a)$.

We say that f is a **continuous** function if f is continuous on all points of its domain. If f is not continuous at a, we say that f is discontinuous at a.

Example 2.4.1. $\sin x$ *is a continuous function.*

Let us see several functions that are discontinuous.

Example 2.4.2. *(Removable discontinuity) Let*

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x) = 0$ but $f(x) = 1$. So f is discontinuous. However if we redefine f to be equal to 1 at 0, then f would be continuous at 0.

Example 2.4.3. *(Jump discontinuity) Recall H the Heaviside function.*

Example 2.4.4. *(A singularity) Define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1/x^n & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where n is any positive integer. Then $\lim_{x\to 0} f(x)$ does not exist. So f is discontinuous at 0.

Example 2.4.5. *(Infinite oscillation) Let $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = \begin{cases} \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We have seen that $\lim_{x\to 0} f(x)$ does not exist. So f is discontinuous at 0.

The following proposition comes from the law of continuity. We can use it to verify continuity of a function.

Proposition 2.6. *(Laws of continuity) Let $f, g : \mathbb{R} \to \mathbb{R}$, and $a, c \in \mathbb{R}$. Assume that f, g are continuous at a. Then*

$$f + g, cf, fg \quad \text{are continuous at } a,$$

and if $g(a) \neq 0$, then f/g is also continuous at a.

Then as a corollary, since x is a continuous function, by using the property for finite many times, we get that any polynomials are continuous. By polynomials we mean functions of the following form:

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

where $n \geq 0$ is an integer, and a_i with $0 \leq i \leq n$ are numbers (or coefficients).

Moreover, if $p = p(x), q = q(x)$ are two polynomials, then for all x such that $q(x) \neq 0$, we have that $\frac{p(x)}{q(x)}$ is continuous.

Question: 1. If f is discontinuous at a, and g is discontinuous at a, how about $f + g$ and fg?

2. Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinous at all points of $\mathbb{R}$?

Proposition 2.7. *(Continuous functions)*

1. $\cos x, \sin x$ *are continuous on* $\mathbb{R}$*;*
2. *if $a > 0$, then a^x is continuous on $\mathbb{R}$, and $\log_a x$ is continuous for all $x > 0$;*
3. *if $a > 0$, x^a is continuous for all $x \geq 0$.*

Proposition 2.8. *(Composition of functions) If g is continuous at a point $x = a$, and f is continuous at $x = g(a)$. Then $F(x) := f(g(x))$ is continuous at $x = a$.*

If f, g are continuous functions on $\mathbb{R}$. Then $F(x) := f(g(x))$ is also continuous.

Example 2.4.6. *Consider*

$$F(x) = \begin{cases} \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We can let $g(x) = \frac{1}{x}$ *for all* $x \neq 0$ *and* $f(x) = \cos x$. *Then* $F(x) = f(g(x))$. *Since* f *is continuous on* $\mathbb{R}$, *and* g *is continuous for all* $x \neq 0$, *we obtain that* F *is a continuous function for all* $x \neq 0$.

Proposition 2.9. *If* f, g *are continuous functions, then*

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).$$

Example 2.4.7.

$$\lim_{x \to \frac{\pi}{4}} \sin(2x - \frac{\pi}{4}) = \sin(\lim_{x \to \frac{\pi}{4}} (2x - \frac{\pi}{4})) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$

2.5 Indeterminate forms

We know that if a function f is continuous then it is easy compute the limit at one point:

$$\lim_{x \to a} f(x) = f(a).$$

We simply substitute $x = a$. However sometimes we are not allowed to do substitution.

Example 2.5.1. *Consider*

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$$

The function is not defined at $x = 2$ *and when* $x = 2$, *it produces an undefined expression of the form* $0/0$ *(but it is still possible in general to evaluate the limit). However note that*

$$\frac{x^2 - 4}{x - 2} = x + 2 \quad \text{for } x \neq 2,$$

which implies $f = x + 2$ *for all* $x \neq 2$. *Thus the limit is the same as* $\lim_{x \to 2} x + 2 = 4$.

Next, let us present some more difficult examples.

Example 2.5.2. *Calculate*

$$\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}.$$

Solution. The function has the indeterminate form $0/0$ because both the top and the denominator equal 0 for $x = 3$. Let us factor a term $x - 3$ out and cancel it:

$$\frac{x^2 - 4x + 3}{x^2 + x - 12} = \frac{(x - 3)(x - 1)}{(x - 3)(x + 4)} = \frac{x - 1}{x + 4}.$$

So

$$\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12} = \lim_{x \to 3} \frac{x - 1}{x + 4} = \frac{2}{7}.$$

□

Example 2.5.3. *Evaluate*

$$\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}.$$

Solution. It is of indeterminate form $0/0$.

Method 1: Multiply by the conjugate $\sqrt{x}+3$ on both the top and the denominator. See the textbook.

Method 2: Substitution. Let $y=\sqrt{x}$. Then the problem becomes

$$\lim_{y\to 3}\frac{y^2-9}{y-3}=\lim_{y\to 3} y+3=6.$$

□

The following example is slightly different, we evaluate limit of indeterminate form of ∞/∞.

Example 2.5.4. *Calculate*

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\sec x}.$$

Solution. Note that $\tan x=\frac{\sin x}{\cos x}$ and $\sec x=\frac{1}{\cos x}$ and $\cos\frac{\pi}{2}=\infty$. So it is of the form ∞/∞. But fortunately we can cancel out the term of $\cos x$. We get

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\sec x}=\lim_{x\to\frac{\pi}{2}}\sin x=1.$$

□

The following example is of the form $\infty-\infty$:

Example 2.5.5.

$$\lim_{x\to 0}\left(\frac{1}{4x}-\frac{1}{x(x+4)}\right).$$

Solution.

$$\frac{1}{4x}-\frac{1}{x(x+4)}=\frac{x+4}{4x(x+4)}-\frac{4}{4x(x+4)}=\frac{x}{4x(x+4)}=\frac{1}{4(x+4)}.$$

So the limit as $x\to 0$ is $\frac{1}{16}$. □

To end the section, can you compute the following symbolic limit and show

$$\lim_{h\to 0}\frac{(h+a)^2-a^2}{h}=2a.$$

2.6 The squeeze theorem

Proposition 2.10. *Let $f, g, h : \mathbb{R} \to \mathbb{R}$. Suppose that $f \le g \le h$ on $\mathbb{R}$ and for some $a, L \in \mathbb{R}$,*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then $\lim_{x\to a} g(x)$ exists and it equals L.

Example 2.6.1. *Use the squeeze theorem to compute $\lim_{x\to 0} x\cos x$.*

Solution. Since $-1 \le \cos x \le 1$, we have $-|x| \le \cos x \le |x|$. Because $\lim_{x\to 0} |x| = \lim_{x\to 0} -|x| = 0$, by the squeeze theorem, $\lim_{x\to 0} x\cos x = 0$. Can you plot the graph of $x\cos x$? □

Proposition 2.11.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

Proof. Let us assume that the following claim holds

$$\cos x \le \frac{\sin x}{x} \le 1 \quad \text{for all} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, x \ne 0. \tag{1}$$

We will prove the claim at the end of this subsection.

Notice that $\cos 0 = 1$. Therefore, by the claim, the squeeze theorem immediately yields $\lim_{x\to 0} \frac{\sin x}{x} = 1$. For the second identity, using

$$\frac{1-\cos x}{x} = \frac{1-\cos x}{x}\frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{(1+\cos x)x} = \frac{\sin x}{(1+\cos x)}\frac{\sin x}{x},$$

we get

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{1+\cos x} = \frac{0}{2} = 0.$$

□

Actually, we can even get $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$. Can you do it from the above proof?

Before proving statement (1), let us look at some examples:

Example 2.6.2. *Let $a \ne 0$. Find $\lim_{x\to 0} \frac{\sin ax}{x}$.*

Solution.

$$\lim_{x\to 0} \frac{\sin ax}{x} = \lim_{x\to 0} \frac{\sin ax}{ax}\frac{a}{1} = a.$$

□

Example 2.6.3. *Find $\lim_{x\to 0} \frac{\tan 3x}{\tan 2x}$. (This is an indeterminate form of $\frac{0}{0}$.)*

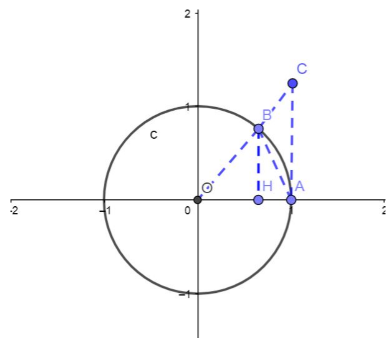

Figure 1: Prop 2.11

Solution. Note that

$$\frac{\tan 3x}{\tan 2x} = \frac{\sin 3x}{\cos 3x}\frac{\cos 2x}{\sin 2x} = \frac{\sin 3x}{3x}\frac{2x}{\sin 2x}\frac{\cos 2x}{\cos 3x}\frac{3}{2}.$$

Hence, using

$$\lim_{x\to 0}\frac{\cos 2x}{\cos 3x} = \frac{1}{1} = 1,$$

and

$$\lim_{x\to 0}\frac{\sin 3x}{3x} = \lim_{x\to 0}\frac{2x}{\sin 2x} = 1$$

by our main proposition, we get $\lim_{x\to 0}\frac{\tan 3x}{\tan 2x} = \frac{3}{2}$. □

Now we present one proof of (1) for $x \in (0, \frac{\pi}{2})$.

Let us suppose that the angle in Figure 1 equals x. Then the arc length of AB equals x (the angle) and $\sin x =$ the length of BH. From the picture, we see that $BH \leq$ the arc length of AB and so $\sin x \leq x$ which yields one inequality in (1) for $x \in (0, \frac{\pi}{2})$.

Next note that the area of the sector OAB equals $\pi \times \frac{x}{2\pi} = \frac{x}{2}$, and the area of the triangle OAC equals $\tan x/2$ (since $AC = \tan x$). Hence we get from the area of the sector $OAB \leq$ the area of the triangle OAC that $x \leq \tan x$ which implies that $\cos x \leq \frac{\sin x}{x}$. We finished the proof.

2.7 Limits at infinity

Definition 2.12. We say $\lim_{x\to\infty} f(x)$ exists if $\lim_{y\to 0^+} f(\frac{1}{y})$ exists.

We say $\lim_{x\to-\infty} f(x)$ exists if $\lim_{y\to 0^-} f(\frac{1}{y})$ exists.

We think of $\lim_{x\to\pm\infty} f(x)$ as describing where f eventually ends up. If $\lim_{x\to\pm\infty} f(x) = L$, we call that $y = L$ is a horizontal asymptote of f.

Example 2.7.1.

$$\lim_{x\to\infty}\frac{1}{x} = \lim_{y\to 0^+} y = 0.$$

Example 2.7.2.

$$\lim_{x\to\infty} 2^x = \lim_{y\to 0^+} 2^{1/y} = 2^{\lim_{y\to 0^+} 1/y} = 2^{\infty} = \infty.$$

$$\lim_{x\to\infty} 2^{-x} = \lim_{y\to 0^+} 2^{-1/y} = 2^{\lim_{y\to 0^+} -1/y} = 2^{-\infty} = 0.$$

$$\lim_{x\to\infty} 2^{-x}\sin x = 0 \quad \textit{by squeeze theorem.}$$

Example 2.7.3.

$$\lim_{x\to\infty} \frac{4x^2+2}{3x^2+x+3} = \lim_{x\to\infty} \frac{4+2x^{-2}}{3+x^{-1}+3x^{-2}} = \frac{4}{3}.$$

$$\lim_{x\to\infty} \frac{4x^2+2}{3x^3+x+3} = \lim_{x\to\infty} \frac{4+2x^{-2}}{3x+x^{-1}+3x^{-2}} = 0.$$

2.8 Intermediate Value Theorem

The Intermediate Value Theorem (I.V.T) is an important property of continuous functions.

Theorem 2.13. *Let $a, b \in \mathbb{R}$ such that $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous (or just $f : \mathbb{R} \to \mathbb{R}$ is continuous). Then f achieves every value between $f(a)$ and $f(b)$ in $[a, b]$. That is for every $y \in \mathbb{R}$ such that it is between $f(a)$ and $f(b)$, there is $x \in [a, b]$ such that $f(x) = y$.*

Example 2.8.1. *Suppose Bob is walking along a straight path. Let $f(t)$ denote the position of Bob at time t. Suppose $f(0) = 0$ and $f(10) = 2$. Then Bob must visit every point of $[0, 2]$ at some time of $[0, 10]$.*

The theorem can be used to show existence of zeros: if a function f is continuous on $[a, b]$, and if $f(a)f(b) < 0$, then f has at least one zero in $[a, b]$ (actually (a, b)).

Example 2.8.2. *There is a number $x \in (\frac{\pi}{2}, \pi)$ such that $2\sin x = x$.*

Solution. To see this, let us consider a function $f(x) := 2\sin x - x$. Then it is not hard to verify that

$$f(\frac{\pi}{2}) = 2\sin(\frac{\pi}{2}) - \frac{\pi}{2} = 2 - \frac{\pi}{2} > 0,$$

and

$$f(\pi) = 2\sin\pi - \pi < 0.$$

By I.V.T., there exists some $x \in (\frac{\pi}{2}, \pi)$ such that $f(x) = 0$. Therefore we have $2\sin x = x$. $\square$

3 Differentiation

3.1 Definition of the derivative

The derivatives can be defined as the slopes of tangent lines.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a nice (smooth) function, and consider the graph of $(x, f(x))$. Let us recall the definition secant lines: $(a, f(a))$ and $(b, f(b))$ are any two points on the graph; the line connecting them is called the second line. The slope equals

$$\frac{f(b) - f(a)}{b - a} \quad \text{difference quotient.}$$

Let us fix a, and then by taking $b \to a$, we get the tangent line at $(a, f(a))$. The slope of the tangent line is the limit of the difference quotient which, if exists, will be defined as the derivative of f at point a.

Definition 3.1. (The derivative at a point) Let f be a function. We say that f is differentiable at a if the following limit exists

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (\text{ or } \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}).$$

If exists, we can denote the limit as $f'(a)$ (or $\frac{df}{dx}(a)$). And $f'(a)$ is called the derivative of f at a. We say that f is differentiable if f is differentiable at all points of its domain.

Example 3.1.1. *If $f(x) \equiv c$ is a constant function, then*

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = 0.$$

If $f(x) = x^2$, then

$$f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{(1 + 2h + h^2) - 1}{h} = 2.$$

If $f(x) = x^{-1}$, then

$$f'(1) = \lim_{h \to 0} \frac{(1 + h)^{-1} - 1}{h} = \lim_{h \to 0} \frac{1 - (1 + h)}{h(1 + h)} = -1.$$

The equation of the tangent line. Assume that f is differentiable at a. Then the tangent line to the graph $y = f(x)$ (or $(x, f(x))$) at point $x = a$ is the line with slope $f'(a)$ which passes through $(a, f(a))$. The corresponding equation of the tangent line is

$$y = f'(a)(x - a) + f(a).$$

Example 3.1.2. *Find the tangent line to the curve $y = x + x^{-1}$ at $x = 3$.*

Solution. Write $f(x) = x + x^{-1}$. Then

$$f'(3) = \lim_{h \to 0} \frac{3 + h + (3+h)^{-1} - 3 - 3^{-1}}{h} = \frac{8}{9}.$$

So the slope is $\frac{8}{9}$. Note that $f(3) = 3 + 3^{-1} = \frac{10}{9}$. Thus the equation for the line is

$$y = \frac{8}{9}(x - 3) + \frac{10}{9}.$$

□

3.2 The derivative as a function

Recall that if f is differentiable at a, then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For any finite (small) h, $\frac{f(a+h)-f(a)}{h}$ is called difference quotient. Actually (in computers) we use difference quotient to approximate derivatives.

Suppose f is differentiable at all points of its domain, then for each x in the domain

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{is a function of } x.$$

We can also write $\frac{d}{dx} f(x)$ or $\frac{df}{dx}(x)$.

Example 3.2.1. *Consider* $f(x) = x^3$.

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2 h + 3x h^2 + h^3 - x^3}{h} = 3x^2.$$

Here we used a formula for $(a+b)^3$. More generally, we can consider $f(x) = x^n$ where n is a positive integer. We need the following

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \quad \text{where } \binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

we define $0! = 1$, $n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$. Using this,

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{\left(\sum_{k=0}^{n} \binom{n}{k} x^k h^{n-k}\right) - x^n}{h} = \lim_{h \to 0} \sum_{k=0}^{n-1} \binom{n}{k} x^k h^{n-k-1}$$
$$= \binom{n}{n-1} x^{n-1} = n x^{n-1}.$$

We obtain a formula $(x^n)' = n x^{n-1}$. Actually this formula holds for all $n \in \mathbb{R}$.

Example 3.2.2.
$$(x^{\frac{2}{3}})' = \frac{2}{3}x^{-\frac{1}{3}}, \quad (x^{-2})' = -2x^{-3}\,(x \neq 0).$$

e **is a "magical" number.** The number e, sometimes called the natural number, or Euler's number, approximately equals to 2.71828.

Example 3.2.3.
$$\lim_{h\to 0}\frac{e^h-1}{h} = 1, \quad \lim_{n\to\infty}(1+\frac{1}{n})^n = e, \quad e = \sum_{k=0}\frac{1}{k!} \quad (0! = 1),$$

$e^{i\pi} = -1$ *where i is the imaginary unit, and this formula is called Euler's formula.*

Using the first identity, we can show that $(e^x)' = e^x$:
$$(e^x)' = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

Proposition 3.2. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on (a, b), then f is continuous on (a, b).*

Example 3.2.4. $f(x) = |x + a|$ *is continuous, but it is not differentiable at $x = -a$.*

Example 3.2.5. $f(x) = x^{\frac{1}{3}}$ *is not differentiable at $x = 0$.*

Proof. Note
$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^{\frac{1}{3}}-0}{h} = \lim_{h\to 0}h^{-\frac{2}{3}}$$
which does not exist. So $f(x)$ is not differentiable at $x = 0$.

When $x \neq 0$, we see that $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. $f'(x)$, as a function of x, has a singularity at $x = 0$. If you plot the graph of $f(x)$, it looks like a cusp. □

Example 3.2.6. *The following function is continuous but not differentiable (at $x = 0$):*
$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We can use squeeze theorem to find that $\lim_{x\to 0} f(x) = 0$ since
$$-|x| \leq f(x) \leq |x|.$$

Now we try to compute the derivative
$$\lim_{h\to 0}\frac{h\sin\frac{1}{h}-0}{h} = \lim_{h\to 0}\sin\frac{1}{h}$$

which does not exist.

Question: If
$$f(x) = \begin{cases} x^2\sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

then f becomes differentiable at $x = 0$. Can you prove it?

3.3 Product and quotient rules

Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions, and let $c \in \mathbb{R}$.

Proposition 3.3.

$$(cf(x))' = cf'(x), \quad (f(x) \pm g(x))' = f'(x) \pm g'(x),$$

$$\textit{product rule:} \qquad (f(x)g(x))' = f'(x)g(x) + f(x)g'(x),$$

$$\textit{quotient rule:} \qquad (\frac{f(x)}{g(x)})' = \frac{g(x)f'(x) - g'(x)f(x)}{g(x)^2}.$$

Example 3.3.1. *Compute*

$$\frac{d}{dx}(x^4 + 3x^2 + 2x^{\frac{1}{2}} + 1).$$

Solution.

$$(x^4 + 3x^2 + 2x^{\frac{1}{2}} + 1)' = (x^4)' + 3(x^2)' + 2(x^{\frac{1}{2}})' = 4x^3 + 6x + x^{-\frac{1}{2}}.$$

□

Example 3.3.2. *Let* $f(x) = x^3$, $g(x) = e^x$. *We have*

$$f'(x) = 3x^2, \quad g(x)' = e^x.$$

Then

$$(x^3 a^x)' = (f(x)g(x))' = f'(x)g(x) + f(x)g'(x) = e^x(x^3 + 3x^2).$$

Example 3.3.3. *Let* $f(x) = \frac{x}{x^2+1}$. *Then*

$$f'(x) = \frac{(x^2+1)(x)' - (x^2+1)'x}{(x^2+1)^2} = \frac{x^2 + 1 - 2x}{(x^2+1)^2} = \frac{(x-1)^2}{(x^2+1)^2}.$$

Now let us talk about the proofs of the product rule and the quotient rule.

Proof. (of the product rule and the quotient rule)

We use definitions. Let us first compute

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x) \\ &= (f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x)). \end{aligned}$$

So

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \lim_{h \to 0} \frac{(f(x+h) - f(x))}{h} g(x+h) + f(x) \frac{(g(x+h) - g(x))}{h} \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

To prove the quotient rule, let us consider the case when $f(x) \equiv 1$. Notice

$$\frac{1}{g(x+h)} - \frac{1}{g(x+h)} = \frac{g(x) - g(x+h)}{g(x)g(x+h)}.$$

So

$$\lim_{h \to 0} \frac{1}{h}(\frac{1}{g(x+h)} - \frac{1}{g(x+h)}) = \lim_{h \to 0} \frac{g(x) - g(x+h)}{h} \frac{1}{g(x)g(x+h)} = \frac{-g'(x)}{g(x)^2}.$$

For general $f(x)$, we use product rule:

$$\begin{aligned}(\frac{f(x)}{g(x)})' &= (f(x)\frac{1}{g(x)})' = f'(x)\frac{1}{g(x)} + f(x)(\frac{1}{g(x)})' \\ &= \frac{f'(x)}{g(x)} + \frac{-f(x)g'(x)}{g(x)^2} = \frac{g(x)f'(x) - g'(x)f(x)}{g(x)^2}.\end{aligned}$$

□

3.4 Rate of changes

Recall

$$f'(x) \sim \frac{f(x+h) - f(x)}{h}.$$

$f'(x)$ describes the rate of changes of $f(x)$ with respect to x.

Example 3.4.1. *Suppose the radius of a circle is increasing at a rate of* 1 *meter per second. What is the rate of change of the area of the circle, when the radius is* 3 *meters?*

Let r be the radius of the circle, and so the area is $A = A(r) = \pi r^2$. So, considering r to be a function of t and since $\frac{dr}{dt} = 1$, we have $A = A(r(t)) = \pi r(t)^2$. So

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r$$

Here we used the product rule to get $(r(t)^2)' = 2r(t)\frac{dr(t)}{dt}$. When $r = 3$, we therefore have $\frac{dA}{dt} = 6\pi$.

Example 3.4.2. *Suppose a vertical cylindrical tank of radius* 3 *meters is draining water at a rate of* 3000 *liters per minute. How fast is the level of water dropping? (*1000 *liters is one cubic meter.)*

Let h be the height of water in the tank. The current volume of water in the tank is $V = \pi r^2 h = 9\pi h$. Since 1000 liters is one cubic meter, we have $\frac{dV}{dt} = -3$. We have

$$-3 = \frac{dV}{dt} = \frac{d\pi r^2 h}{dt} = 9\pi \frac{dh}{dt}.$$

This says that $\frac{dh}{dt} = -\frac{1}{3\pi}$.

Example 3.4.3. *A* 10 *feet ladder is leaning against a wall. While touching both the wall and the ground, the bottom of the ladder is sliding away from the wall at a rate of* 1 *foot per second. What is the speed of the top of the ladder when the top is* 6 *feet off of the ground?*

Let h be distance of the top of the ladder from the ground, and let r be the length from the bottom of the ladder to the wall. The ladder makes a right triangle with legs r, h and hypotenuse 10. So $r^2 + h^2 = 10^2 = 100$. Differentiating this equation with respect to t, we have

$$2rr' + 2hh' = 0, \quad \text{and so } h' = -\frac{rr'}{h}.$$

When $h = 6$, we have $r = 8$. Also, since it is given that $r' = 1$, we have $h' = -\frac{8}{6} = -\frac{4}{3}$ per second.

3.5 Higher derivatives

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. If $f(t)$ represents the position of an object at time t, then $f'(t)$ is the velocity of the object at time t. We can also consider the derivative of $f'(x)$ (since $f'(x)$ is also a function, which is known as acceleration. We denote this second derivative of f as $f''(t)$.

Definition 3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose f' is differentiable . Then we define the second derivative of f at any $x \in \mathbb{R}$ by

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d^2}{dx^2}f(x).$$

Similarly we can define the third derivative $f'''(x) = \frac{d^3}{dx^3}f(x)$, the fourth derivative $f^{(4)}(x) = \frac{d^4}{dx^4}f(x)$, and the n^{th} derivative $f^{(n)}(x) = \frac{d^n}{dx^n}f(x)$.

Example 3.5.1. *Let* $f(x) = x^3 - 3x + \frac{1}{x}$. *Then*

$$f'(x) = 3x^2 - 3 - x^{-2}, \quad f''(x) = 6 + 2x^{-3}, \quad f'''(x) = -6x^{-4}.$$

Example 3.5.2. *Suppose that Bob throws a baseball vertically in the air with initial upward velocity* v_0 *and initial position* r_0*. Then by the Newton's Law (we neglect air friction), the position of the baseball at time* t *is*

$$r(t) = r_0 + tv_0 - \frac{gt^2}{2}$$

where g *is the gravitational acceleration. Then the velocity of the baseball at time* t *is* $r'(t) = v_0 - gt$*, and the acceleration is a constant* $r''(t) = g$*.*

(Newton's second law says that $F = ma$ *i.e. forces equals the mass times the acceleration.)*

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Since f' represents the rate change of f, then

$$f' > 0 \text{ corresponds to } f \text{ is increasing,}$$
$$f' < 0 \text{ corresponds to } f \text{ is decreasing.}$$

Similarly if $f'' \geq 0$, then f' is increasing, and the graph of f would be convex (we say that f is convex). If $f'' \leq 0$, then f' is decreasing, and the graph of f would be concave (we say that f is concave).

3.6 Trignometric functions

About $\sin$ function:

$$\sin x = -\sin(-x) = -\sin(x+\pi), \quad \sin 0 = 0,$$
$$\sin\frac{\pi}{6} = \frac{1}{2}, \quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

About $\cos$ function:

$$\cos x = \cos(-x) = -\cos(x+\pi), \quad \cos 0 = 1,$$
$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \sin\frac{\pi}{3} = \frac{1}{2}.$$

Moreover

$$\cos x = \sin(\frac{\pi}{2} - x), \quad \cos^2 x + \sin^2 x = 1.$$

We also define

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Proposition 3.5.

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x.$$

To prove the proposition, we need the following formulas:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b,$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

Proof. By the definition,

$$\begin{aligned}(\sin x)' &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x (\lim_{h\to 0} \frac{\sin h}{h}) = \cos x.\end{aligned}$$

For the second part, we apply the first formula and the relation between $\sin$ and $\cos$. Notice

$$\cos x = \sin(\frac{\pi}{2} - x) = -\sin(x - \frac{\pi}{2}) = \sin(x + \frac{\pi}{2}).$$

Therefore

$$(\cos x)' = (\sin(x + \frac{\pi}{2}))' = \cos(x + \frac{\pi}{2}) = -\sin x.$$

$\square$

Example 3.6.1.

$$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x.$$

Proposition 3.6.

$$(\cot x)' = -\csc^2 x, \quad (\sec x)' = \sec x \tan x, \quad (\csc x)' = -\csc x \cot x.$$

Example 3.6.2.

$$\frac{d^2}{dx^2} \tan x = \frac{d}{dx}(\frac{1}{\cos^2 x}) = \frac{2\sin x}{\cos^3 x}.$$

$$\frac{d^3}{dx^3} \tan x = \frac{d}{dx}(\frac{2\sin x}{\cos^3 x}) = 2\frac{\cos^3 x \cos x - 3\cos^2 x(-\sin x)\sin x}{\cos^6 x} = \frac{2\cos^2 x + 3\sin^2 x}{\cos^4 x}.$$

3.7 Chain rule

If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable, then

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Writing $u = g(x)$ and $y = f(g(x)) = f(u)$, we can also write

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Here we should not use the notation y', because then it is not clear that we want to differentiate y with respect to x.

Example 3.7.1. *Suppose* $f(x) = x^3$ *and* $g(x) = x^{\frac{1}{2}} - 3$. *Then*

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = 3u^2|_{u=g(x)}g'(x) = 3(x^{\frac{1}{2}} - 3)^2\frac{1}{2}x^{-\frac{1}{2}} = \frac{3}{2}(x^{\frac{1}{2}} - 3)^2 x^{-\frac{1}{2}}.$$

Proof. (of the Chain rule)

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} \\ &= f'(g(x))g'(x).\end{aligned}$$

$\square$

Example 3.7.2. *Calculate the derivative of $y = e^{1+\sin x}$.*

Solution. Let $f(u) = e^u$ and $u = g(x) = 1 + \sin x$. Then $\frac{d}{du} f(u) = e^u$ and $\frac{d}{dx} g(x) = \cos x$. So by Chain rule:

$$\frac{d}{dx} e^{1+\sin x} = f'(g(x))g'(x) = e^{1+\sin x} \cos x.$$

□

Example 3.7.3. *A spherical balloon has a radius r that is increasing at a rate of 3 cm/s. At what rate is the volume V of the balloon increasing when $r = 10$ cm?*

Solution. Since $V = V(t)$ is the volume, the rate that V increases corresponds to $\frac{d}{dt} V(t)$. It is known that $r = r(t)$ increases at a rate of 3, and so $\frac{d}{dt} r(t) = 3\ cm/s$. We also know that $V = \frac{4}{3}\pi r^3$. By Chain rule,

$$\frac{d}{dt} V = \frac{dV}{dr} \times \frac{dr}{dt} = \frac{d}{dr}(\frac{4}{3}\pi r^3) \times 3 = 4\pi r^2.$$

Therefore when $r = 10$, the rate is $1200\pi\ cm^3/s$. □

As a corollary of the Chain rule, we can show the following.

Proposition 3.7. *(Some rules) Let g be a differentiable function, and n, a, b are some constants. Then*

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} g'(x), \quad \frac{d}{dx} e^{g(x)} = e^{g(x)} g'(x), \quad \frac{d}{dx} g(ax + b) = a g'(ax + b).$$

Example 3.7.4. *Compute $\frac{d}{dx}\sqrt{1 + \sqrt{x^2 + 1}}$.*

Solution.

$$\begin{aligned}
\frac{d}{dx}\sqrt{1 + \sqrt{x^2 + 1}} &= \frac{d}{dx}(1 + (1 + x^2)^{\frac{1}{2}})^{\frac{1}{2}} \\
&= \frac{1}{2}(1 + (1 + x^2)^{\frac{1}{2}})^{-\frac{1}{2}} \frac{d}{dx}(1 + (1 + x^2)^{\frac{1}{2}}) \\
&= \frac{1}{2}(1 + (1 + x^2)^{\frac{1}{2}})^{-\frac{1}{2}} \frac{1}{2}(1 + x^2)^{-\frac{1}{2}} \frac{d}{dx}(1 + x^2) \\
&= (1 + (1 + x^2)^{\frac{1}{2}})^{-\frac{1}{2}} \frac{1}{2}(1 + x^2)^{-\frac{1}{2}} x.
\end{aligned}$$

□

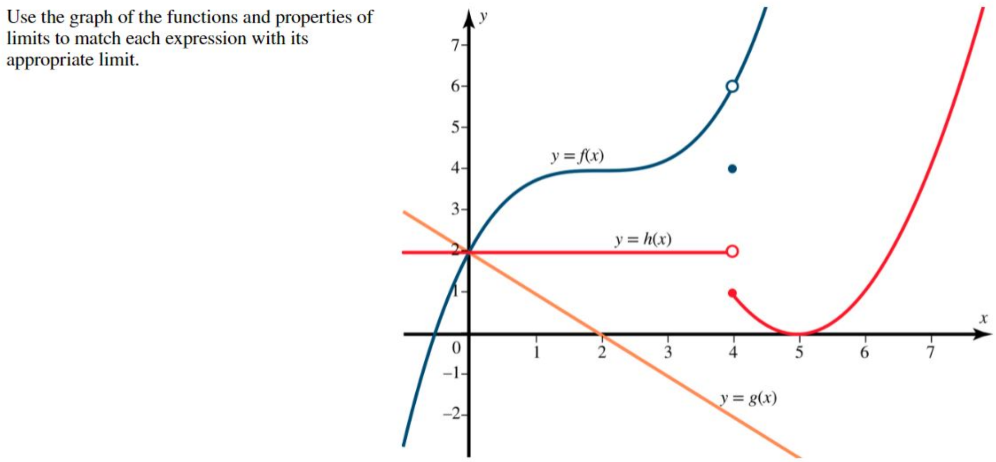

Figure 2: Ex 3.7.5

Review

Example 3.7.5. *Find*

$$\lim_{x\to 4^-} \frac{f(x)+g(x)}{h(x)}, \quad \lim_{x\to 4} f(x)+2g(x), \quad \lim_{x\to 4^+} h(x)-g(x),$$

$$\lim_{x\to 4^+} g(x)^2 - h(x), \quad \lim_{x\to 4} h(x)(4-x), \quad \lim_{x\to 4} h(x)+\lfloor x \rfloor.$$

Example 3.7.6. *Find*

$$\lim_{x\to 0} \frac{1-\cos(x^2)}{x}, \quad \lim_{x\to 0} \frac{1-\cos x}{x^2}.$$

Hint: $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Example 3.7.7. *Suppose that Bob throws a baseball vertically in the air with initial upward velocity v_0 and initial position r_0. Then by the Newton's Law (we neglect air friction), the position of the baseball at time t is*

$$r(t) = r_0 + tv_0 - \frac{gt^2}{2}$$

where g is the gravitational acceleration. Then the velocity of the baseball at time t is $r'(t) = v_0 - gt$, and the acceleration is a constant $r''(t) = g$.

(Newton's second law says that $F = ma$ i.e. forces equals the mass times the acceleration.)

Let $v_0 = 2$ and $r_0 = 10$. Find the velocity of the ball when it hits the ground. Find the highest place where the ball can reach.

Example 3.7.8. *Find $\frac{d}{dx}\frac{e^x}{f(x)g(x)}$.*

Example 3.7.9. *Find $f^{(n)}(x)$ where $f(x) = x^6$.*

3.8 Implicit differentiation

Consider a curve which satisfies some equation of x, y. For example, straight line: $y = ax + b$, or a unit circle $x^2 + y^2 = 1$. We want to find out $\frac{dy}{dx}$ (from which we get slopes of tangent lines). But it is possibly very complicated (or even impossible) to represent y by x. For example, the equation can by $y^4 + xy = x^3 - x + 2 + e^{yx}$. In this case we say that y is defined implicitly as a function of x. Then to find $\frac{dy}{dx}$, we think of y as a function of x, and we take derivative of both sides of the equation.

For example, let us consider the unit circle. Then

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

implies

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0.$$

We get $2x + 2y\frac{dy}{dx} = 0$ by Chain rule. Therefore, we have $y' = -\frac{x}{y}$.

Example 3.8.1. *Consider* $x^3 + y \sin x + y^3 = 1$. *Find the tangent line at* $(x, y) = (1, 0)$.

Solution. First we can check that $(x, y) = (1, 0)$ is indeed a point on $x^3 + y \sin x + y^3 = 1$. Let us now find $\frac{dy}{dx}$. Differentiating the equation with respect to x on both sides of the equation yields

$$3x^2 + \sin x \frac{dy}{dx} + y \cos x + 3y^2 \frac{dy}{dx} = 0.$$

After simplification, we get

$$\frac{dy}{dx} = -\frac{y \cos x + 3x^2}{\sin x + 3y^2}.$$

So at $(x, y) = (1, 0)$,

$$\frac{dy}{dx}|_{(x,y)=(1,0)} = -\frac{3}{\sin 1}.$$

And so the slope of the tangent line is $-\frac{3}{\sin 1}$. Because the tangent line passes through the point $(1, 0)$, the equation of the tangent line is

$$y = -\frac{3}{\sin 1}(x - 1).$$

$\square$

Let $y = f(x)$ be a function. Then the inverse function of $f(x)$, denoted as $x = f^{-1}(y)$ is a function such that

$$f(f^{-1}(y)) = y, \quad f^{-1}(f(x)) = x.$$

For example the inverse function of $y = 2x + 3$ is $x = \frac{1}{2}(y - 3)$. Since we often use x as the free variable, we can also write the inverse function of $y = 2x + 3$ as $\frac{1}{2}(x - 3)$.

Suppose that the graph of $f(x)$ is known, can you plot the graph of f^{-1}?

We can apply the chain rule to differentiate $f^{-1}(f(x))$. Since $f^{-1}(f(x)) = x$, by differentiating both sides with respect to x, we get

$$(f^{-1})'(f(x))f'(x) = 1$$

which yields

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))} \tag{2}$$

where $y := f(x)$ (and so $x = f^{-1}(y)$ by the definition).

We use $\sin^{-1} x$ and $\cos^{-1} x$ to denote the inverse function of $\sin\theta$ and $\cos\theta$, respectively. So we have

$$\sin^{-1} x = \theta \quad \text{is the same as} \quad \sin\theta = x,$$

or

$$\sin^{-1}(\sin\theta) = \theta, \quad \text{and } \sin(\sin^{-1} x) = x.$$

Now let us use implicit differentiation to find the derivatives of $\sin^{-1}$ and $\cos^{-1}$.

Theorem 3.8.

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

(We need $x \in (-1, 1)$.)

The proof can be found in the textbook. You can also apply (2) to give a different proof.

Theorem 3.9.

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}.$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}.$$

(For the last two formulas, we need $|x| > 1$.)

In the following example, let us compute higher order derivatives implicitly.

Example 3.8.2. *Find $\frac{d^2y}{dx^2}$ for $x^2 + 4y^2 = 7$.*

Solution. First let us find $\frac{dy}{dx}$. By differentiating the equation with respect to x on both sides, we get

$$2x + 8yy' = 0 \tag{3}$$

which implies that $y' = -\frac{x}{4y}$. Now we differentiate (3) again with respect to x on both sides, and we get

$$2 + 8(y')^2 + 8yy'' = 0.$$

Plugging in $y' = -\frac{x}{4y}$, we obtain

$$y'' = -\frac{4y^2 + x^2}{16y^3} = -\frac{7}{16y^3}.$$

□

3.9 Exponential and Logarithmic functions

The general exponential function is $f(x) = a^x$ where $a > 0$. The Logarithmic function is the inverse function of the exponential function: $f(x) = \log_a x$. So $y = a^x$ is equivalent to $\log_a y = x$.

Proposition 3.10. *Base change formula:*

$$\log_a x = \frac{\ln x}{\ln a}, \quad a^x = e^{x \ln a}.$$

Power rule for Log function: $\ln x^n = n \ln x$.

Let me only show the first formula. Suppose that $y = \log_a x$, $y_1 = \ln x$, and $y_2 = \ln a$, and the goal is to show that $y = \frac{y_1}{y_2}$. Since Log function is the inverse function of Exp function, have

$$a^y = x, \quad e^{y_1} = x, \quad e^{y_2} = a.$$

This implies that

$$(e^{y_2})^y = x = e^{y_1}$$

which yields $y_2 y = y_1$ because $(e^{y_2})^y = e^{y_2 y}$. This finishes the proof.

Theorem 3.11. *For all $x > 0$,*

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

Proof. Let $y = \ln x$, and then $e^y = x$. Let us differentiate $e^y = x$ on both sides with respect to x. We obtain

$$e^y \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = e^{-1} = \frac{1}{e^y} = \frac{1}{x}.$$

□

Example 3.9.1.

$$(x \ln x)' = x' \ln x + x(\ln x)' = \ln x + 1.$$

Theorem 3.12.

$$\frac{d}{dx} a^x = a^x \ln a, \quad \frac{d}{dx} \log_a x = \frac{1}{x \ln a}.$$

Proof.

$$\frac{d}{dx} \log_a x = \frac{d}{dx} \frac{\ln x}{\ln a} = \frac{1}{x \ln a}.$$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = (\ln a)\, e^{x \ln a} = (\ln a) a^x.$$

□

Example 3.9.2. *Find $f'(x)$ for*

$$f(x) = \frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}, \quad \textit{and} \quad f(x) = x^x.$$

Solution. First, we consider

$$\ln f(x) = 2\ln(x+1) + \ln(2x^2-3) - \frac{1}{2}\ln(x^2+1).$$

Let us take derivative with respective to x:

$$\frac{f'(x)}{f(x)} = \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2-1}.$$

Multiplying $f(x)$, we get

$$f'(x) = \left(\frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2-1}\right)\left(\frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}\right).$$

For the second $f(x)$, since $\ln x^x = x\ln x$, we have $\ln f(x) = x\ln x$. Then, after taking derivative,

$$\frac{f'(x)}{f(x)} = \ln x + 1 \implies f'(x) = (\ln x + 1)x^x.$$

□

Now we introduce Hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

They are called the hyperbolic sine and hyperbolic cosine. It is not hard to check the following property holds.

$$\frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\cosh x = \sinh x.$$

Let us also define

$$\tanh x = \frac{\sinh x}{\cosh x}, \coth x = \frac{\cosh x}{\sinh x}, \operatorname{sech} x = \frac{1}{\cosh x}, \operatorname{csch} x = \frac{1}{\sinh x}.$$

We have the following formulas

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x, \quad \frac{d}{dx}\coth = -\operatorname{csch}^2 x,$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x, \quad \frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x.$$

We can also define the inverse of hyperbolic functions. Let us only consider the inverse function for $\sinh$. We have

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}.$$

To see this, let $y = \sinh^{-1} x$. Then $x = \frac{e^y - e^{-y}}{2}$. Let us apply implicit differentiation and differentiate x on both sides. We get

$$1 = \frac{e^y + e^{-y}}{2} y'.$$

This yields

$$y' = \frac{2}{e^y + e^{-y}} = \frac{1}{\cosh y}.$$

Using that $x = \sinh y$ and $\cosh^2 y - \sinh^2 y = 1$, we get

$$y' = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}.$$

Recall that we have

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

For hyperbolic functions, can you use $\sinh x, \cosh x, \sinh y, \cosh y$ to represent $\sinh(x + y)$?

3.10 Related rates

Example 3.10.1. *(Filling a conical container) Water flows into a conical container at a rate of* 5 *in.*^{3}*/s. Assume that the container has a height of* 4 *in. and a base radius of* 3 *in. Find the rate of the water level is rising when the height is* 3 *in.*

Solution. **Step 1. Identify variables.** Let V represent the volume of the water in the container, and let h be its height.

Step 2. Find an equation. The volume of the conical container is $\frac{1}{3}\pi r^2 d$ where r and d are the base radius and height, respectively, of the conical space. In our case the base is 2 and the height is 4. When there is water in the container and the water is of height h, the part of the container that is not filled up with water is also of conical shape (smaller one), which has a height of $d = 4 - h$. By using similar triangles, the smaller cone has base radius

$$r = 2 \times \frac{d}{4} = \frac{4 - h}{2}.$$

Then the volume of water is

$$V = \frac{1}{3}\pi 2^2 \times 4 - \frac{1}{3}\pi(\frac{4 - h}{2})^2 \times (4 - h) = \frac{16\pi}{3} - \frac{\pi(4 - h)^3}{12}. \tag{4}$$

Step 3. Do differentiation and solve the problem. Since water is flowing into the container, both V and h are functions of t. We know that $\frac{dV}{dt} = 5$. Using this, differentiating (4) on both sides yields

$$\frac{dV}{dt} = -\frac{\pi(3(4 - h)^2)}{12}(-\frac{dh}{dt}) = \frac{\pi(4 - h)^2}{4}\frac{dh}{dt}.$$

So we get, when $h = 3$,

$$\frac{dh}{dt} = \frac{20}{\pi(4-h)^2} = \frac{20}{\pi}.$$

$\square$

Example 3.10.2. *Suppose Bob is standing on the ground, and a plane is flying over him and is h meters above from the ground. Suppose that the plane has velocity V. Let θ be the angle between the ground and line that is connecting Bob and the plane. What is the rate change of the angle θ? (Use θ, h, V to represent the result.)*

Solution. Let y be the horizontal distance between Bob and the plane. Then $\frac{dy}{dt} = V$. Since the plane is h away from the ground, we have

$$y = h \cot \theta.$$

After differentiating both sides with respect to t, we get

$$V = h(-\csc^2 \theta)\frac{d\theta}{dt}.$$

Therefore the rate change of θ is $-\frac{V \sin^2 \theta}{h}$. $\square$

4 Application of the derivatives

4.1 Linear approximation

When we look at a very small domain of a smooth function, it looks like a linear function. Indeed we have the following linear approximation

$$f(x) \approx f(a) + (x-a)f'(a).$$

Here $\approx$ is approximately equal, and x is a point that is close to a.

If $f(x)$ is differentiable at a, then we know that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This implies that, when f is differentiable at a, then $f(x)$ is close to its linearization at a when $|x - a|$ is small.

Definition 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. The linearization of f near a is

$$(L(x) =)L_{a,f}(x) = f(a) + (x-a)f'(a).$$

Example 4.1.1. *Estimate* $\sin(\frac{\pi}{6} + 0.01)$.

Solution. We approximate $\sin x$ by $L(x)$ near $a = \frac{\pi}{6}$. Then

$$L_{\frac{\pi}{6},f} = \sin\frac{\pi}{6} + (x - \frac{\pi}{6})\cos\frac{\pi}{6} = \frac{1}{2} + (x - \frac{\pi}{6})\frac{\sqrt{3}}{2}.$$

We get $\sin(\frac{\pi}{6} + 0.01) \approx L(\frac{\pi}{6} + 0.01) = \frac{1}{2} + (0.01)\frac{\sqrt{3}}{2} \approx 0.50866.$ $\square$

Definition 4.2.

$$\text{percentage error } = \left|\frac{\text{error}}{\text{actual value}}\right| \times 100\%.$$

Example 4.1.2. *Compute the percentage error of the previous example.*

Solution. Using calculators, $\sin(\frac{\pi}{6} + 0.01) \approx 0.50864$. So the percentage error is

$$\approx \left|\frac{0.50864 - 0.50866}{0.5086}\right| \times 100\% \approx 0.004\%.$$

$\square$

Let $y = f(x)$. If we use Δy to denote the change of y, and Δx to denote the change of x, then we have

$$\Delta y \approx f'(x)\Delta x.$$

This is very similar to the linear approximation. Instead of $\Delta y, \Delta x$, let us use dy, dx. We have

$$dy = f'(x)dx.$$

Both the right- and the left-hand sides are called differential forms. You can think of dy, dx the same as $\Delta y, \Delta x$ but usually dy, dx are smaller.

Example 4.1.3. *Suppose a thin metal cable has length $L = 12$ cm when $T = 21°C$. Estimate the change in length when T rises to $24°C$, assuming that $dL = kLdT$ where $k = 1.7 \times 10^{-5}°C$.*

Solution. By the assumption,

$$\frac{dL}{dT}\Big|_{L=12} = kL = 1.7 \times 10^{-5} \times 12 \approx 2 \times 10^{-4}\text{cm}/°C.$$

We know that the change of the temperature is $\Delta T = 24 - 21 = 3$. Then, by the assupmtion again,

$$\Delta L \approx kL\Delta T \approx 6 \times 10^{-4} \text{ cm}.$$

$\square$

The size or error at $x = a$ is

$$E := \text{ error } = |\Delta f - f'(a)\Delta x|.$$

It can be shown that if f is twice differentiable, then

$$E \leq \frac{1}{2}K(\Delta x)^2 \quad \text{where } K := \max\{|f''(x)| \,|\, x \in [a - |\Delta x|, a + |\Delta x|]\}.$$

4.2 Extreme values

The goal is to find the minimum or the maximum value of a function in some bounded interval.

Example 4.2.1. *Suppose that f represents the cost of a moving object. We would like to minimize the cost. Or f may represents energy, or f represents the time spent to accomplish a mission etc. We would like to minimize the energy or the time.*

Definition 4.3. Let f be a function on an interval I, and let $a \in I$. We say that $f(a)$ is the *absolute minimum* of f on I if $f(a) \leq f(x)$ for all $x \in I$. We say that $f(a)$ is the *absolute maximum* of f on I if $f(a) \geq f(x)$ for all $x \in I$.

We refer to *absolute minimum* and *absolute maximum* as *extreme values* (or *extrema*) of f.

The process of finding extrema of f is called *optimization*.

Example 4.2.2. *Let $f(x) = \frac{1}{x}$ where f has domain $(0, \infty)$. Then f has no absolute maximum.*

Let $f(x) = x^4$ where f has domain $[0, 2]$. Then the absolute maximum of f is 16, which occurs at $x = 2$. And the absolutely minimum of f is 0, which occurs at $x = 0$. If we change the domain to $(0, 2]$, then the absolutely minimum of f does not exist.

Questions. Does extrema always exist? If exists, is extrema unique? What conditions do we need to have extremas?

Theorem 4.4. *(Extreme Value Theorem) A continuous function f on a closed and bounded interval $I = [a, b]$ takes on both a minimum and a maximum value on I.*

Sometimes it is also desirable to find extreme values for a function when it is restricted to a small region.

Definition 4.5. (Local Extrema) We say that $f(c)$ is a *local minimum* (resp. *local maximum*) occurring at $x = c$ if $f(c) \leq f(x)$ (resp. $f(c) \geq f(x)$) for all points x in some open interval containing c.

Property. The tangent line (if exists) at a local minimum or a local maximum is horizontal.

Definition 4.6. (Critical point) We say that f has a critical point at c if $f'(c) = 0$ or $f'(c)$ does not exist.

Example 4.2.3. *If $f(x) = x^2$, then $f'(x) = 2x$. So $x = 0$ is the only critical point of f. Observe that $x = 0$ is also the absolute minimum for f with the domain $\mathbb{R}$.*

If $f(x) = |x|$, then $f'(0)$ is undefined. So $x = 0$ is critical. And we observe that it is also the absolute minimum.

If $f(x) = x^3$, then $f'(x) = 3x^2$ which implies that $x = 0$ is the only critical point of f. However it is neither a local maximum nor a local minimum of f.

Theorem 4.7. *If a local minimum or maximum of f occurs at c, then c is a critical point of f.*

Theorem 4.8. *(Extreme Values on Closed Intervals) Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be continuous. Then the extreme values of f on $[a, b]$ occur either at critical points of f or at the endpoints a, b of the interval.*

Example 4.2.4. *Let us find the extreme values of $f(x) = x - \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.*

We have $f'(x) = 1 - \cos x$. So $f'(x) = 0$ gives $\cos x = 1$. When $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos x = 1$ only occur at $x = 0$. So x is the only critical point of f. So the extreme values of f must occur at $\{-\frac{\pi}{2}, \frac{\pi}{2}, 0\}$. Note

$$f(-\frac{\pi}{2}) = -\frac{\pi}{2} + 1, \quad f(\frac{\pi}{2}) = \frac{\pi}{2} - 1, \quad f(0) = 0.$$

So the absolute maximum of f on $[\frac{\pi}{2}, -\frac{\pi}{2}]$ is $\frac{\pi}{2} - 1$ obtained at $x = \frac{\pi}{2}$, and the absolute minimum of f on $[\frac{\pi}{2}, -\frac{\pi}{2}]$ is $-\frac{\pi}{2} + 1$ obtained at $x = -\frac{\pi}{2}$.

Theorem 4.9. *(Rolle's Theorem) Assue that f is continuous on $[a, b]$ and f is differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. From the extreme value theorem, we can let c be an extreme value of f on $[a, b]$. If one extreme value occurs at $c \in (a, b)$, then $f'(c) = 0$ by the critical value theorem. If both maximum and minimum occur at the endpoints a, b, this implies that f is a constant function. Then $f'(x) = 0$ for all $x \in (a, b)$. □

Example 4.2.5. *Verify Rolle's theorem for $f(x) = x^4 - x^2$ on $[-2, 2]$. Note that $f(-2) = f(2) = 12$. So Rolle's theorem implies that $f'(x) = 0$ has solutions in $(-2, 2)$. Actually $f'(x) = 4x^3 - 2x$, and so $x = 0, \pm\frac{1}{\sqrt{2}}$ all satisfy $f'(x) = 0$. So Rolle's theorem is satisfied with three values of c.*

Example 4.2.6. *Show that $f = x^3 + 9x - 4$ has precisely one real root.*

Note that $f(0) = -4$ and $f(1) = 6$. By I.V.T there exists $a \in [0, 1]$ such that $f(a) = 0$. If there are at least two roots say a, b with $a \neq b$, then Rolle's Theorem implies that $f'(c) = 0$ for some $c \in (a, b)$. However $f'(x) = 3x^2 + 9 > 0$. This is not possible.

4.3 Mean value theorem

The mean value theorem is a more general theorem than the Rolle theorem.

Theorem 4.10. *Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that f is differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define

$$g(x) := f(x) - (x - a)\frac{f(b) - f(a)}{b - a},$$

and then it is clear that $g(a) = g(b) = f(a)$. So applying Rolle's theorem to g yields that thre exists $c \in (a, b)$ such that

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

which is the same as $f'(c) = \frac{f(b)-f(a)}{b-a}$. $\square$

Recall that we know that a constant function has derivative that is 0. The inverse is also true.

Corollary 4.11. *Let $f : (a, b) \to \mathbb{R}$ be differentiable with $f'(x) = 0$ for all $x \in (a, b)$. Then there is a constant c such that $f(x) = c$ for all $x \in (a, b)$.*

Proof. Suppose f is not a constant, then there are $a', b' \in (a, b)$ such that $f(a') \neq f(b')$. However the Mean value theorem yields that for some c' in between a', b',

$$f'(c') = \frac{f(b') - f(a')}{b' - a'}$$

which is clearly not 0 by the assumption. This leads to a contradiction. $\square$

Corollary 4.12. *Let $f, g : (a, b) \to \mathbb{R}$ be two differentiable functions. Suppose that $f' = g'$ on (a, b). Then there is a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$.*

Example 4.3.1. *Let us find a function f such that $f'(x) = \cos x$ and $f(0) = 2$.*

Recall that $\frac{d}{dx}(\sin x) = \cos x$. By the above corollary, $f(x) = \sin x + C$ for some C. Since $f(0) = 2$, we obtain $C = 2$. So $f(x) = \sin x + 2$.

In the previous lecture we saw that critical points might be extreme points. Now we show that if $x = c$ is a critical point, and we know the sign of $f'(x)$ around $x = c$, then we can decide whether $f(c)$ is a local minimum or a local maximum.

Theorem 4.13. *If $f'(x) > 0$ for all $x \in (a, b)$, then f is (strictly) increasing on (a, b). If $f'(x) < 0$ for all $x \in (a, b)$, then f is (strictly) decreasing on (a, b).*

(First derivative test) *Suppose $c \in (a, b)$ is a critical point of f. If $f'(x)$ changes its sign from $+$ to $-$, then $f(c)$ is a local maximum. If $f'(x)$ changes its sign from $-$ to $+$, then $f(c)$ is a local minimum.*

Example 4.3.2. *Analyze the critical points and increase/decrease behavior of $f(x) = \cos^2 x + \sin x$ in $(0, \pi)$.*

Solution. First let us find the critical points by solving $f'(x) = 0$. We get

$$-2\cos x \sin x + \cos x = \cos x(1 - 2\sin x) = 0$$

which yields $\cos x = 0$ or $\sin x = \frac{1}{2}$. We get

$$\frac{\pi}{6}, \quad \frac{\pi}{2}, \quad \frac{5\pi}{6}.$$

The four critical points divide the interval into four smaller intervals, and we need to determine the sign of f' in each one.

For $x \in (0, \frac{\pi}{6})$, since $\cos x > 0$ and $\sin x < \frac{1}{2}$, we have $f'(x) > 0$.
For $x \in (\frac{\pi}{6}, \frac{\pi}{2})$, since $\cos x > 0$ and $\sin x > \frac{1}{2}$, we have $f'(x) < 0$.
For $x \in (\frac{\pi}{2}, \frac{5\pi}{6})$, since $\cos x < 0$ and $\sin x > \frac{1}{2}$, we have $f'(x) > 0$.
For $x \in (\frac{5\pi}{6}, \pi)$, since $\cos x < 0$ and $\sin x < \frac{1}{2}$, we have $f'(x) < 0$.

So, applying the first derivative test, for $c = \frac{\pi}{6}, \frac{5\pi}{6}$, f' changes its sign from $+$ to $-$, and so f has a local maximum at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Similarly, f has a local minimum at $\frac{\pi}{2}$. $\square$

4.4 The second derivative and concavity

Definition 4.14. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If f' is increasing in (a, b), then f is concave up (convex). That is if $y(x) = ax + b$ denotes one tangent line to f, then $f(x) \geq y(x)$ for all $x \in (a, b)$.

If f' is decreasing in (a, b), then f is concave down (concave). That is if $y(x) = ax + b$ denotes one tangent line to f, then $f(x) \leq y(x)$ for all $x \in (a, b)$.

Example 4.4.1. *Let $f(x) = x^2$. Then $f'(x) = 2x$ is increasing. So $f(x)$ is concave up (or convex).*

Proposition 4.15. *(Concavity test) Let $f : (a, b) \to \mathbb{R}$ be twice differentiable. If $f'' > 0$ (resp. $f'' < 0$) on (a, b), then f is concave up (resp. concave down) on (a, b).*

Definition 4.16. (Inflection point) Let $f : (a, b) \to \mathbb{R}$ be differentiable, and let $c \in (a, b)$. If f is concave up on one side of c, and is concave down on the other side of c, then c is called an inflection point.

Example 4.4.2. *Let $f(x) = x^3$. Then $f'(x) = 3x^2$ and $f''(x) = 6x$. So f'' changes sign at $x = 0$. $x = 0$ is a reflection point.*

Proposition 4.17. (Second derivative test) *Let $f : (a, b) \to \mathbb{R}$. Let $c \in (a, b)$. Assume that $f'(c), f''(c)$ exist, and $f'(c) = 0$, and f'' is continuous near c.*

1. *If $f''(c) > 0$, then f has a local minimum;*
2. *If $f''(c) < 0$, then f has a local maximum;*
3. *If $f''(c) = 0$, then f may or may not have a local extreme at $x = c$.*

Example 4.4.3. *Consider $f(x) = \frac{x^3}{3} - x - 1$. Identify all local maximum, minimum and inflection point. Identify the region where f is increasing and decreasing. Identify where f is concave up and concave down.*

Solution. By direct computations,

$$f'(x) = x^2 - 1 = (x - 1)(x + 1), \quad f''(x) = 2x.$$

So $x = \pm 1$ are critical points and $x = 0$ is an inflection point.

Since $f''(1) = 2 > 0$, $x = 1$ is a local minimum. Since $f''(-1) = -2 < 0$, $x = -1$ is a local maximum.

Since $f' < 0$ for $x \in (-1, 1)$, f is decreasing on $(-1, 1)$; Since $f' > 0$ for $x < -1$ and $x > 1$, f is decreasing on $(-\infty, -1) \cup (1, \infty)$; Since $f'' > 0$ on $(0, \infty)$, f is concave up in the region; Since $f'' < 0$ on $(-\infty, 0)$, f is concave down in the region.

□

4.5 L'Hopital's rule

Theorem 4.18. *Let f, g be differentiable on (a,b), and let $c \in (a,b)$. If*

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \quad \textit{or} \quad \pm\infty,$$

and $g'(x) \neq 0$ for all $x \in (a,b)$ with $x \neq c$, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists or is infinite, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Remark. We can replace c by $\pm\infty$, c^+ and c^-.

Example 4.5.1. *(Use L'Hopital's rule twice) Find*

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1}.$$

Solution. Notice that

$$e^x - x - 1 = \cos x - 1 = 0 \qquad \text{when } x = 0.$$

Hence by L'Hopital's rule,

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{e^x - 1}{-\sin x}.$$

Since $e^x - 1 = -\sin x = 0$ at $x = 0$, we apply the rule again to get

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{e^x}{-\cos x} = -1.$$

□

Example 4.5.2. *Can you show $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ by L'Hopital's rule?*

Example 4.5.3. *Evaluate $\lim_{x\to 0^+} x^x$.*

Solution. Let us apply $\ln$ on the function. Then

$$\ln x^x = x \ln x = \frac{\ln x}{\frac{1}{x}}.$$

As $x \to 0^+$, the top and the bottom converge to $-\infty$ and ∞. So by L'Hopital's rule

$$\lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

This implies that $\lim_{x\to 0^+} x^x = 1$.

□

Example 4.5.4. *Evaluate* $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x}$.

Solution.

$$\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0.$$

□

Example 4.5.5. *Show that for each positive number* n*,*

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Solution. First let us suppose that n is a positive integer. Then by applying L'Hopital's rule for n times, we get

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

Now for general positive number n, we take $N = \lceil n \rceil$ which is a positive integer. Note that for all $x \geq 1$,

$$0 < \frac{x^n}{e^x} \leq \frac{x^N}{e^x}.$$

Hence by squeeze theorem, $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ holds as well. □

Proof. (of L'Lopital's rule)

Suppose that c is a real number, and $f(c) = g(c) = 0$. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} = \frac{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x\to c} \frac{g(x)-g(c)}{x-c}} = \frac{f'(c)}{g'(c)}.$$

□

Example 4.5.6. **(A very interesting counter-example)** *The necessity of the condition that* $g'(x) \neq 0$ *near* c *can be seen by the following counterexample due to Otto Stolz. Let* $f(x) = x + \sin x \cos x$ *and* $g(x) = f(x)e^{\sin x}$*. Then clearly*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} e^{-\sin x}$$

does not exist. However

$$\frac{f'(x)}{g'(x)} = \frac{2\cos x}{2\cos x + x + \sin x \cos x} e^{-\sin x}$$

which tends to 0 *as* $x \to \infty$*.*

4.6 Graph sketching

Proposition 4.19. *Suppose that f is concave up (convex). Then if $y(x) = ax + b$ denotes one tangent line to f, we have $f(x) \geq y(x)$ for all $x \in (a, b)$.*

Suppose that f is concave down (concave). Then if $y(x) = ax + b$ denotes one tangent line to f, we have $f(x) \leq y(x)$ for all $x \in (a, b)$.

Suppose that we know the signs of f' and f'', then we have an idea about its graph. Then we focus on the transition points: (assuming that f', f'' exist and are continuous)

1. critical points: $f'(x) = 0$;
2. inflection pints: $f''(x) = 0$ and f'' changes its sign.

Example 4.6.1. *Investigate the behavior of*

$$f(x) = 3x^4 - 8x^3 + 6x^2 + 1$$

and sketch its graph.

Solution. Step 1. Determine (possible) transition points. Note

$$f'(x) = 12x^3 - 24x^2 + 12x = 12x(x-1)^2.$$

So $f'(x) = 0$ implies that $x = 0$ and $x = 1$ are the critical points. Next

$$f''(x) = 36x^2 - 48x + 12 = 12(x-1)(3x-1).$$

We get $f''(x) = 0$ at $x = 1, \frac{1}{3}$.

Step 2. Determine the signs of f' and f'', and study the transition points.

Using $f'(x) = 12x(x-1)^2$, we get $f' < 0$ on $(-\infty, 0)$, and $f' > 0$ on $(0,1) \cup (1, \infty)$. Using $f''(x) = 12(x-1)(3x-1)$, we find $f'' > 0$ on $(-\infty, \frac{1}{3}) \cup (1, \infty)$, and $f'' < 0$ on $(\frac{1}{3}, 1)$.

Since f' changes its sign from $-$ to $+$ at $x = 0$, so $x = 0$ is a local minimum. Moreover, f'' indeed changes its signs at $x = 1, \frac{1}{3}$, and so the two points are inflection points. With these information, we can sketch the shape of the graph of f.

Step 3. Find the values of the function at transition points and plot the graph. We can use

$$f(0) = 1, \quad f(\frac{1}{3}) = \frac{38}{27}, \quad f(1) = 2.$$

□

Asymptotes. Horizontal asymptote: $y = L$ if

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

Vertical asymptote: $x = c$ if

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = \pm\infty.$$

Example 4.6.2. *Investigate the behavior of $f(x) = \frac{3x+2}{2x-4}$ and sketch its graph.*

Solution. Step 0. Determine the domain of f.

The domain is $(-\infty, 2) \cup (2, \infty)$ ($\mathbb{R}\backslash\{2\}$).

Step 1. Transition points. By direct computations

$$f'(x) = -\frac{4}{(x-2)^2}, \quad f''(x) = \frac{8}{(x-2)^3}.$$

They are non-zero in the domain. And so there is no critical points and no inflection points.

Step 2. Determine the signs of f' and f''. For $x \in (2, \infty)$, we have $f' < 0$ and $f'' > 0$. While for $x \in (-\infty, 2)$, we have $f' < 0$ and $f'' < 0$.

Step 3. Find values of f, and find asymptotes. First we consider horizontal asymptote.

$$\lim_{x\to\infty} \frac{3x+2}{2x-4} = \lim_{x\to\infty} \frac{3+\frac{2}{x}}{2-\frac{4}{x}} = \frac{3}{2}.$$

The same holds if we replace ∞ by $-\infty$. So there is one horizontal asymptote that is $y = \frac{3}{2}$.

Next since

$$\lim_{x\to 2^+} \frac{3x+2}{2x-4} = \infty$$

and

$$\lim_{x\to 2^-} \frac{3x+2}{2x-4} = -\infty,$$

so $x = 2$ is a vertical asymptote. $\square$

4.7 Applied optimization

As we have discussed before, optimization problems occur in many real world problems. We are going to discuss some problems that we can solve.

Example 4.7.1. *Suppose we want to design a cylindrical can with a minimal amount of material. Then can's volume is 1 liter ($1000\, cm^3$), and it will be made from aluminum of a fixed thickness. What dimension should the can have?*

Solution. Suppose the can has radius r and height h. Then the volume is $\pi r^2 h$ and the surface area (top surface excluded) is $\pi r^2 + 2\pi r h$. So from the assumption we know $\pi r^2 h = 1000$ and we want to minimize $\pi r^2 + 2\pi r h$. To solve the problem, we plug in $h = \frac{1000}{\pi r^2}$ into the second quantity to get

$$f(r) := \pi r^2 + \frac{2000}{r}.$$

Thus the problem is transformed to finding the minimum value of f in the domain $r \in (0, \infty)$.

We solve for the critical value of f:

$$f'(r) = 2\pi r - \frac{2000}{r^2} = 0$$

and this implies that $r = (1000/\pi)^{\frac{1}{3}}$. It is not hard to see that $f'(r) < 0$ when $r < (1000/\pi)^{\frac{1}{3}}$ and $f'(r) > 0$ when $r > (1000/\pi)^{\frac{1}{3}}$. So $r = (1000/\pi)^{\frac{1}{3}}$ is an absolute minimum of $f(r)$. And for this value of r, $h = \frac{1000}{\pi r^2} = (\frac{1000}{\pi})^{\frac{2}{3}}$.

Let us remark here that this optimization problem has no absolute maximum, since

$$\lim_{r\to 0} f(r) = \lim_{r\to\infty} f(r) = \infty.$$

$\square$

Steps (Algorithm):

1. introduce variables, and a function f,
2. identify the domain,
3. find critical points (and end points),
4. choose the largest and the smallest value of f.

Example 4.7.2. *Find two numbers which sum to* 50 *and whose product is a maximum.*

Solution. Suppose $x + y = 50$, and then we want to maximize xy. Let us define

$$f(x) = xy = x(50 - x).$$

Next we look for critical points. We get $f'(x) = -2x + 50$ which yields $x = 25$. Note that $f"(x) > 0$ when $x < 25$, and $f'(x) > 0$ when $x > 25$, we find that f has an absolute maximum at $x = 25$. Thus the maximum value of the product is $f(x) = 25^2 = 625$.

Since $f(x) \to -\infty$ as $x \to \pm\infty$, there is no absolute minimum. $\square$

Example 4.7.3. *(#56 in Exercise 4.7)* **Snell's Law.** *When a light bean travels from a point A above a swimming pool to a point B below the water, it chooses the path that takes the least time. Let v_1 be the velocity of light in air and v_2 the velocity in water (it is known that $v_1 > v_2$). Prove Snell's Law of refraction:*

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Solution. From the figure,

$$h_1 \tan\theta_1 + h_2 \tan\theta_2$$

is a constant (h_1, h_2 are fixed constant as in the figure). From this we can view θ_2 as an implicitly defined function of θ_1 (so once known θ_1, we should know the value of θ_2). By implicit differentiation,

$$h_1 \sec^2\theta_1 + h_2 \sec^2\theta_2 \theta_2' = 0.$$

So

$$\theta_2' = -\frac{h_1 \sec^2\theta_1}{h_2 \sec^2\theta_2}.$$

Next, the travel time of the light is then

$$T = \frac{h_1}{v_1 \cos\theta_1} + \frac{h_2}{v_2 \cos\theta_2}.$$

We can then view T as a function of θ_1 since θ_2 is a function of θ_1. To find the critical point of $T(\theta_1)$, we solve for $T' = 0$:

$$0 = \frac{h_1}{v_1}\sec\theta_1\tan\theta_1 + \frac{h_2}{v_2}\sec\theta_2\tan\theta_2\theta_2' = \frac{h_1}{v_1}\sec\theta_1\tan\theta_1 - \frac{h_1}{v_2}\sec^2\theta_1\sin\theta_2.$$

We get $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$. It remains to check that when θ_1 obtains the value, it is indeed an absolute minimum point. □

Example 4.7.4. *(#57 in Exercise 4.7) A small blood vessel of radius r branches off at an angle θ from a larger vessel of radius R to supply blood along a path from A to B. According to Poiseuille's Law, the total resistance to blood flow is proportional to*

$$T = \frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4},$$

where a and b are two constants as in Figure 29 (in the textbook). Show that the total resistance is minimized when $\cos\theta = (\frac{r}{R})^4$.

Solution. In this problem, let us fix the constants a, b, R, r, and we consider different angle θ. In order to find one angle that minimize the resistance, we need to solve for the critical point of T:

$$T(\theta)' = \frac{b}{R^4}\csc^2\theta - \frac{b}{r^4}\csc\theta\cot\theta.$$

Then $T'(\theta) = 0$ implies that $\cos\theta = (\frac{r}{R})^4$. □

5 Integration

5.1 Computing area & summation notation

Recall that "finding changing rate" motivates "differentiation". Here "computing area" will motivate "integration". Later we will show that it turns out that "integration" is an inverse operator to "differentiation".

Consider $y = f(x)$ where f is a positive function. Let us compute the area under the graph of f for $x \in [a, b]$.

Approximate area by rectangles

1. Divide the interval $[a, b]$ into N subintervals. Then each one has width $\frac{b-a}{N} =: \Delta x$.
2. Suppose the right endpoints of the intervals are $x_0 = a, x_1, \cdots, x_N = b$. Then $x_i = a + i\Delta x$.

3. For each interval $[x_i, x_{i+1}]$, construct rectangles whose height is $f(x_{i+1})$.

4. the area of all rectangles equals

$$R_N = f(x_1)\Delta x + \cdots + f(x_N)\Delta x = (f(x_1) + \cdots + f(x_N))\Delta x$$

   which approximates the area.

Instead of using the right endpoints in 3., we can use left endpoints, or midpoints:

$$L_N := (f(x_0) + \cdots + f(x_{N-1}))\Delta x,$$
$$M_N := (f(\frac{x_0 + x_1}{2}) + \cdots + f(\frac{x_{N-1} + x_N}{2}))\Delta x.$$

Either one of them, we need to deal with the sum of a large number of numbers. So we need to introduce the summation notation.

Definition 5.1. (*summation notation*) Let $\{x_1, \cdots, x_n\}$ be a set of numbers. We define

$$\sum_{i=1}^{n} x_i := x_1 + \cdots + x_n.$$

Example 5.1.1. *(Power sums)*

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \qquad \sum_{i=a}^{b} i = \frac{(b-a+1)(a+b)}{2},$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Proposition 5.2. *Let $\{x_n\}_n$ and $\{y_n\}_n$ be two sets of n numbers, and let $c \in \mathbb{R}$. Then*

$$\sum_{i=1}^{n}(x_i \pm y_i) = (\sum_{i=1}^{n} x_i) \pm (\sum_{i=1}^{n} y_i), \qquad \sum_{i=1}^{n} cx_i = c(\sum_{i=1}^{n} x_i).$$

Example 5.1.2.

$$\sum_{i=1}^{n}(3i^2 - i) = 3\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i = 3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)(2n+1) - n(n+1)}{2} = n^2(n+1).$$

Example 5.1.3. *Let N be a positive integer. Find R_N for $f(x) = x^2$ on $[0, 4]$.*

Solution. Let $\Delta x := \frac{4}{N}$, and $x_i := i\Delta x$. Then

$$\begin{aligned} R_N &= \sum_{i=1}^{N} f(x_i)\Delta x = \frac{4}{N}\left(\sum_{i=1}^{N}(\frac{4i}{N})^2\right) = \frac{64}{N^3}\left(\sum_{i=1}^{N} i^2\right) \\ &= \frac{64}{N^3}\frac{N(N+1)(2N+1)}{6} = \frac{32}{3}(1+\frac{1}{N})(2+\frac{1}{N}). \end{aligned}$$

□

From this computation, we know that the area between the graph of f and the x-axis for $x \in [0, 4]$ is

$$\lim_{N\to\infty} R_N = \frac{64}{3}.$$

We can also compute L_N:

$$\begin{aligned} L_N &= \sum_{i=0}^{N-1} f(x_i)\Delta x = \frac{4}{N}\left(\sum_{i=0}^{N-1}(\frac{4i}{N})^2\right) = \frac{64}{N^3}\left(\sum_{i=0}^{N-1} i^2\right) \\ &= \frac{64}{N^3}\frac{(N-1)N(2N-2+1)}{6} = \frac{32}{3}(1-\frac{1}{N})(2-\frac{1}{N}), \end{aligned}$$

which again converges to $\frac{32}{3}$ as $N \to \infty$.

5.2 The definite integral

In the previous section we used L_N, R_N, M_N (which are called the Riemann sums) to approximate the area under the graph of f on a closed interval.

Definition 5.3. (Riemann sums) Let $a = x_0 < x_1 < \cdots < x_n = b$. Let $c_1 \in [x_0, x_1], \cdots, c_n \in [x_{n-1}, x_n]$. A general Riemann sum is any sum of the form

$$\sum_{i=1}^{n}(x_i - x_{i-1})f(c_i) \quad (= \sum_{i=1}^{n} \Delta x_i f(c_i) \text{ where } \Delta x_i := x_i - x_{i-1})$$

Remark 5.4. 1. $a = x_0 < x_1 < \cdots < x_n = b$ is called a *partition* of size n (denoted as P). The subintervals are not required to have the same length.

2. $\{c_i\}$ (denoted as a vector c) are called *sample points*.
3. We do not require $f(c_i) \geq 0$. If $c_i < 0$, the rectangle is below the x-axis.
4. We use the notation $R(f, P, c) := \sum_{i=1}^{n}(x_i - x_{i-1})f(c_i)$.

Definition 5.5. Given a partition P, the maximum width of the rectangles is denoted by

$$\|P\| = \max_{i=1,\cdots,n}(x_i - x_{i-1})$$

(If $\|P\|$ is small, the partition is fine and then we expect better area approximation.)

Definition 5.6. (The definite integral) Let $f : [a, b] \to \mathbb{R}$. If the following limit exists, we say that f is integrable on $[a, b]$,

$$\int_a^b f(x)dx := \lim_{\|P\|\to 0} \sum_{i=1}^{n} (x_i - x_{i-1}) f(c_i).$$

Remark 5.7. By the limit $\lim_{\|P\|\to 0} \sum_{i=1}^{n} (x_i - x_{i-1}) f(c_i) = I$ we mean: for any small $\epsilon > 0$, there is $\delta = \delta(\delta) > 0$ such that for all P as long as $\|P\| < \delta$, we have

$$\left| \lim_{\|P\|\to 0} \sum_{i=1}^{n} (x_i - x_{i-1}) f(c_i) - I \right| < \epsilon$$

for all choices of $\{c_i\}$ such that $c_i \in [x_{i-1}, x_i]$.

Remark 5.8. $\int_a^b f(x)dx$ is called the integral of f on $[a, b]$. The function f is called the integrand. The endpoints a, b of $[a, b]$ are called the limits of integration. In the integral, the variable x is just a place holder, which has no intrinsic meaning. For example, it is the same to write $\int_a^b f(s)ds$.

Remark 5.9. (Geometric interpretation) If $f \geq 0$, then $\int_a^b f(x)dx$ represents the area under the graph of f. In general, $\int_a^b f(x)dx$ represents the signed area under the graph of f. That is, the area enclosed by f lying above the x-axis, minus the area enclosed by f lying below x-axis.

Example 5.2.1. *Let $f(x) = x$. Then $\int_{-1}^1 f(x)dx = 0$. This is because the area of f above the x-axis is a triangle of area $\frac{1}{2}$, and the area of f below the x-axis is also of $\frac{1}{2}$.*

Theorem 5.10. *(Continuous functions on closed intervals are integrable) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $\int_a^b f(x)dx$ exists.*

Example 5.2.2. *In the theorem "closed interval" is crucial. $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$. However $\int_0^1 f(x)dx$ does not exist. Because from the geometry (will be done in class)*

$$\int_0^1 f(x)dx \geq \frac{1}{n} \times n + \frac{1}{n} \times \frac{n}{2} + \frac{1}{n} \times \frac{n}{3} + \cdots + \frac{1}{n} \times 1 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

which can be infinitely large as $n \to \infty$.

Since a composition of continuous functions is continuous, and a product of continuous functions is continuous, we have the following corollary.

Corollary 5.11. *Let f, g be continuous on $[a, b]$. Then $\int_a^b f(g(x))dx$, and $\int_a^b f(x)g(x)dx$ exist.*

Proposition 5.12. *Suppose f, g are continuous on $[a, c]$ and $c > b > a$.*

(1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ *(so $\int_a^a f(x)dx = 0$);*

(2) $\int_a^b kdx = k(b - a)$*;*

(3) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$*;*

(4) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$*;*

(5) $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$*;*

(6) if $f \geq g$*, then* $\int_a^b f(x)dx \geq \int_a^b g(x)dx$*;*

(7) $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$*.*

Example 5.2.3. *Let us apply (2)(6) to approximate* $\int_0^2 \sqrt{1+\sin x}dx$*.*

Solution. Since $-1 \leq \sin x \leq 1$, $0 \leq \sqrt{1+\sin x} \leq \sqrt{2}$. We obtain

$$0 \leq \int_0^2 \sqrt{1+\sin x}dx \leq \int_0^2 \sqrt{2}dx = 2\sqrt{2}.$$

□

5.3 The indefinite integral

In the previous lecture we discussed the definite integral, we know that continuous functions are integrable on a closed interval. However it is not convenient to use the definition to compute the value of integration. To solve the problem, we need indefinite integral.

Definition 5.13. (Antiderivative, indefinite integral) Let $f : \mathbb{R} \to \mathbb{R}$. We say that F is an *antiderivative* of f if F is differentiable and $F'(x) = f(x)$ for all x in the domain. We write

$$\int f(x)dx = F(x) + C.$$

We refer to $\int f(x)dx$ as the *indefinite integral* of f.

Remark 5.14. Here we have C, because if $F(x)$ is an antiderivative, then so is $F(x) + C$.

Let F, G be two antiderivatives of f. So $F' = G' = f$. Then we know, by Corollary 4.12, there exists a constant C such that $F(x) = G(x) + C$. So if one obtains one antiderivative F of f, then all antiderivatives of f is given by

$$F(x) + C, \quad \text{with } C \in \mathbb{R}.$$

The indefinite integral is not associated to any interval. And it is not a number. It is actually a class of functions.

Example 5.3.1. $f(x) = x^2$*. Then the set of all antiderivatives of* f *is given by* $F(x) = \frac{1}{3}x^3 + C$ *with* $C \in \mathbb{R}$*.*

Example 5.3.2. *Suppose one throws a ball straight up at a velocity of v_0 m/s with initial position s_0 meters. The only acceleration that acts on the ball is a constant $g \approx -9.8$ m/s (gravity constant) (the minus sign is due to that we take pointing up as the positive direction).*

Solution. Let v be the velocity. So $v' = g$. We get $v(t) = gt + C = -9.8t + C$. Since $v(0) = v_0$, we get $v(t) = -9.8t + v_0$.

If we take antiderivative again, the position of the ball $s(t)$ satisfies

$$s(t) = \int v(t)dt = -\frac{9.8}{2}t^2 + v_0 t + C$$

for some C. Using that $s(0) = s_0$, we get $s(t) = -4.9t^2 + v_0 t + s_0$. $\square$

Proposition 5.15. *(Indefinite integral of powers of x) When $n \neq -1$, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$. When $n = -1$, $\int \frac{1}{x} dx = \ln|x| + C$ for $x \neq 0$.*

Example 5.3.3.

$$\int x^3 dx = \frac{x^4}{4} + C, \qquad \int 2x^{\frac{3}{2}} dx = \frac{4}{5}x^{\frac{5}{2}} + C.$$

Proposition 5.16. *Let $c \in \mathbb{R}$ and suppose $f, g : \mathbb{R} \to \mathbb{R}$ have antiderivatives. Then*

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx, \quad \int cf(x)dx = c\int f(x)dx.$$

Example 5.3.4.

$$\int \sin x dx = -\cos x + C, \quad \int \cos x dx = \sin x + C,$$

$$\int (\sec x)^2 dx = \tan x + C, \quad \int \sec x \tan x dx = \sec x + C.$$

$$\int e^x dx = e^x + C, \quad \int e^{cx+d} dx = \frac{1}{c}e^{cx+d} + C \quad (c \neq 0).$$

Example 5.3.5. *Find*

$$\int 12e^{7-3x} + 2x dx.$$

5.4 The fundamental theorem of calculus

Recall $\int_a^b f(x)dx$. Consider

$$F(x) = \int_a^x f(t)dt.$$

Then if we vary x, this $F(x)$ can be viewed as a function of x. This also has a geometric interpretation.

Now we can finally describe the precise manner in which differentiation and integration are inverse operators to each other.

Theorem 5.17. *(Fundamental Theorem of Calculus) (i) Let $f : [a,b] \to \mathbb{R}$ be differentiable. Assume also that $f' : [a,b] \to \mathbb{R}$ is continuous. Then*

$$\int_a^b f'(x)dx = f(b) - f(a).$$

(ii) Let $f : [a,b] \to \mathbb{R}$ be continuous. For $x \in [a,b]$, define $F(x) = \int_a^x f(t)dt$. Then F is an antiderivative of f i.e.

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Remark 5.18. Notice that $f(x)$ is one antiderivative of $f'(x)$. Therefore the first part (i) says that taking derivative first and then do definite integration, we get the difference of one antiderivative evaluated at the end points. (ii) says that if we do integration first before differentiation, then we get the function back.

Remark 5.19. (i) can be used to evaluate definite integrals.

Corollary 5.20. *Let $n \in \mathbb{R}$ and $n \neq -1$, and $b > a > 0$. Then*

$$\int_a^b x^n dx = \left[\frac{1}{n+1}x^{n+1}\right]_{x=a}^{x=b} = \frac{1}{n+1}(b^{n+1} - a^{n+1}).$$

What happens if $a = -1, b = 1$ and $n < 0$? (This is not a trivial question.)

Proof. Let $f = \frac{x^{n+1}}{n+1}$. Then the conclusion follows from the first fundamental theorem of calculus (i). □

Example 5.4.1.

$$\int_0^1 xdx = \left[\frac{x^2}{2}\right]_{x=0}^{x=1} = \frac{1}{2} - 0 = \frac{1}{2}.$$

$$\int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{3} - 0 = \frac{1}{3}.$$

$$\int_1^3 (x^4 - x^{-2})dx = \left[\frac{x^5}{5} + x^{-1}\right]_{x=1}^{x=3} = \frac{3^5}{5} + 3^{-1} - \frac{1}{5} - 1 = ...$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos dx = [\sin x]_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin(-\frac{\pi}{2}) = 2.$$

Example 5.4.2.

$$\frac{d}{dx}\int_1^x \cos(t)dt = \cos x.$$

$$\frac{d}{dx}\int_x^{100} \frac{1}{1+t^2}dt = \frac{d}{dx}\left(-\int_{100}^x \frac{1}{1+t^2}dt\right) = -\frac{1}{1+x^2}.$$

$$\frac{d}{dx}\int_1^{x^2} \cos t dt = 2x \cos x^2.$$

$$\frac{d}{dx}\int_{2x}^{x^2} t \ln t dt = 2x \times x^2 \ln x^2 - 2 \times 2x \ln 2x = 4x^3 \ln x - 4x \ln x - 4x \ln 2.$$

Solution. (of the third one.)

Write

$$\frac{d}{dx}\int_1^{x^2} \cos t dt = f(g(x))$$

where

$$f(y) := \int_1^y \cos t dt, \quad g(x) = x^2.$$

By chain rule

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = \cos(g(x))g'(x) = (\cos 2x)2x.$$

□

In general, we have the following formula:

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)dt = f(g_2(x))g_2'(x) - f(g_1(x))g_1'(x).$$

Example 5.4.3.

$$\int_2^8 \frac{dx}{x} = \ln 8 - \ln 2 = \ln 4.$$

$$\int_{-4}^{-2} \frac{dx}{x} = [\ln |x|]_{x=-4}^{x=-2} = \ln 2 - \ln 4 = \ln \frac{1}{2} = -\ln 2.$$

$$\int_0^1 \frac{dx}{x}, \textit{ and } \int_{-1}^2 \frac{dx}{x} \quad \textit{does not exist.}$$

Here this is because the antiderivative of $\frac{1}{x}$ *is* $\ln |x| + C$ *which is not a real number when evaluated at* $x = 0$*.*

Suppose $0 < p < 1$*,*

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{x^{-p+1}}{-p+1}\right]_{x=1}^{x=0} = \frac{1}{1-p}.$$

In the following examples, we will call $y = \ln x$*.*

$$\int_1^2 \frac{\ln x}{x} dx = \int_1^2 \ln x d \ln x = \int_0^{\ln 2} y dy = \frac{(\ln 2)^2}{2}.$$

$$\int_1^2 \frac{1}{x \ln x} dx = \int_1^2 \frac{1}{\ln x} d \ln x = \int_0^{\ln 2} \frac{1}{y} dy \quad \textit{which does not exist.}$$

Proof. (of the Fundamental Theorem of Calculus (i)) Suppose we have a partition on $[a, b]$ that is

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Then

$$\begin{aligned} f(b) - f(a) &= f(x_n) - f(x_0) \\ &= [f(x_n) - f(x_{n-1})] + [f(x_{n-1}) - f(x_{n-2})] + \cdots + [f(x_1) - f(x_0)] \\ &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]. \end{aligned}$$

By the mean value theorem, there exists for each $1 \leq i \leq n$, $c_i \in [x_{i-1}, x_i]$ such that

$$f(x_i) - f(x_{i-1}) = (x_i - x_{i-1}) f'(c_i).$$

We obtain

$$f(b) - f(a) = \sum_{i=1}^{n} (x_i - x_{i-1}) f'(c_i).$$

Note that the right-hand side is one Riemann sum of f'. Since the partition is arbitrary (so we can take $\|P\| \to 0$), by the definition of definite integral, we get

$$f(b) - f(a) = \int_a^b f'(t) dt.$$

$\square$

Proof. (of the Fundamental Theorem of Calculus (ii))
Since F is an antiderivative of f,

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt.$$

Then since $\frac{1}{h} \int_x^{x+h} f(t) dt$ is the average of f on $[x, x+h]$, we have

$$\min_{x \leq t \leq x+h} \{f(t)\} \leq \frac{1}{h} \int_x^{x+h} f(t) dt \leq \max_{x \leq t \leq x+h} \{f(t)\}.$$

Since $f(t)$ is continuous,

$$\lim_{h \to 0} \min_{x \leq t \leq x+h} \{f(t)\} = \lim_{h \to 0} \max_{x \leq t \leq x+h} \{f(t)\} = f(x).$$

Therefore, $\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) dt = f(x)$. We get

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) dt = f(x).$$

$\square$