UCSD

Lecture : MATH 20B Calculus for Science and Engineering

2020 WI

Lecturer: Yuming Paul Zhang *Scribe:*

Contents

1 Integration

1.1 Review

First I will briefly review how *derivatives* is defined. Let $f(x)$ be a function. If the limit

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists, then we say that $f(x)$ is differentiable at x, and we denote the above limit as its derivative $f'(x)$. If $f(x)$ is defferentiable on $\mathbb{R}$, we can view $f'(x)$ as a function itself. Then we say that $f(x)$ is one antiderivative of $f'(x)$.

Example 1.1.1.

$$(x^n)' = nx^{n-1}, \quad (e^{\alpha x})' = \alpha e^{\alpha x}, \quad (\sin x)' = \cos x, \quad (\cos x)' = -\sin x.$$

Can you write down the derivative, and all the antiderivatives of $1+x+x^2$?

Next, let us review definite integration. Recall that in Math 20A, we start with Riemann sums. *Riemann Sums* is used to approximately find the area enclosed by a function $f(x)$ and the x-axis. For example, when $f(x) = 4x - x^2$ (as shown in the figure), we can compute the areas of the

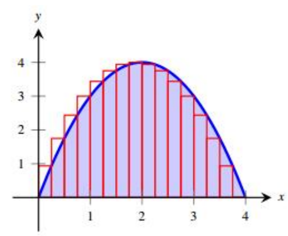

Figure 6.8: Approximating area with the Right Hand Rule and 16 evenly spaced subintervals.

rectangles to approximate the area of the shadow part: for some large n,

$$S_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{4}{n}$ is the length of the short sides of the rectangles, and $x_i = i\Delta x$ (they indicate the position of those rectangles; we have $x_0 = 0, x_n = 4$). These $x_i, 0 \le i \le n$ are called a partition of the interval $[0,4]$, S_n is called one Riemann sum of $f(x)$ on the interval $[0,4]$.

The *definite integral* of $f(x)$ on $[0,4]$ is defined to be the limit of S_n as $n \to \infty$ (assuming that the limit exists).

It turns out that the definite integral of one function is closely related to its antiderivatives. Recall the fundamental theorem of calculus:

Theorem 1.1. *Suppose F is an antiderivative of f on $[a,b]$. Then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

If we are given the antiderivative first, say $F = s(x)$, then f becomes $s'(x)$. The theorem becomes

Theorem 1.2.
$$\int_a^b s'(x)dx = s(b) - s(a).$$

Example 1.1.2.

$$\int_a^b x^n dx = \frac{x^{n+1}}{n+1}\Big|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \quad \textit{where } n \neq -1,$$

$$\int_0^\pi \sin x\, dx = (-\cos x)|_0^\pi = 2,$$

$$\int_1^{10} e^{\alpha x} dx = \frac{e^{\alpha x}}{\alpha}\Big|_1^{10} = \frac{e^{10\alpha}}{\alpha} - \frac{e^\alpha}{\alpha}.$$

1.2 Net Change as the Integral of a Rate of Change

Let us start with the following example.

Example 1.2.1. *Water leaks from a tank at a rate of* $2 + 5t$ *L/hour, where* t *is the number of hours after* 9. *How much water is lost between* 9 *and* 11 *pm?*

Solution. Let $s(t)$ be the quantity of water in the tank at time t. Since $2 + 5t$ represents the rate the water is leaving, the rate of the change of the water is $-(2 + 5t)$. Then

$$s(2) - s(0) = \int_0^2 s'(t) = \int_0^2 -(2 + 5t)dt = -(2t + \frac{5}{2}t^2)\Big|_0^2 = -14.$$

Therefore there is a loss of 14L of water. □

Remark 1.3. We sometimes call $s(2) - s(0)$ the **net change** in $s(t)$ over the interval $[0, 2]$. Since s' is the rate of the change, we know the net change equals the integral of the rate of change.

Theorem 1.4. ***The integral of Velocity.*** *Suppose that there is a particle moving along forward or backward along the real line. Let* $v = v(t)$ *be a function representing velocity. Then*

$$\textit{displacement during } [t_1, t_2] = \int_{t_1}^{t_2} v(t)dt, \tag{1}$$

$$\textit{distance traveled during } [t_1, t_2] = \int_{t_1}^{t_2} |v(t)|dt. \tag{2}$$

Example 1.2.2. *A particle has velocity* $v(t) = t^2 + t - 2$. *Compute the displacement and the total distance traveled over* $[0, 4]$.

Solution. Compute

$$\int_0^4 v(t)dt = \int_0^4 t^2 + t - 2\,dt$$
$$= (\frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t)\Big|_0^4 = \frac{64}{3}.$$

So the displacement is $\frac{64}{3}$. Next,

$$\int_0^4 |v(t)|dt = \int_0^4 |t^2 + t - 2|dt.$$

Since

$$t^2 + t - 2 = (t+2)(t-1)$$

which is positive when $t > 1$ and otherwise when $t \in (0,1)$. So

$$\int_0^4 |v(t)|dt = \int_0^1 -(t^2 + t - 2)dt - \int_4^1 (t^2 + t - 2)dt$$
$$= -(\frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t)\Big|_0^1 + (\frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t)\Big|_1^4$$
$$= \frac{7}{6} + \frac{64}{3} + \frac{7}{6} = \frac{71}{3}.$$

Thus the total distance traveled is $\frac{71}{3}$.

□

1.3 The Substitution Method

The method is about the change of variable. For example, if we change the variable from x to u, then we view $u = u(x)$ and so by chain rule,

$$du = u'(x)dx.$$

Theorem 1.5. *If $F'(x) = f(x)$, and u is a differentiable function, then*

$$\int f(u(x))u'(x)dx = \int f(u(x))du(x) = F(u(x)) + C.$$

Example 1.3.1. *Evaluate $\int x(x^2+9)^5 dx$.*

Solution. Let $u = x^2 + 9$. Then $du = 2xdx$ and so $xdx = \frac{1}{2}du$. We apply substitution

$$\int x(x^2+9)^5 dx = \int (x^2+9)^5 xdx$$
$$= \int u^5 \frac{1}{2}du = \frac{1}{12}u^6 + C = \frac{1}{12}(x^2+9)^6 + C.$$

□

Example 1.3.2. *Evaluate* $\int \cot\theta d\theta$.

Solution. If letting $u = \sin\theta$, then $u = \cos\theta d\theta$. Since $\cot\theta = \frac{\cos\theta}{\sin\theta}$, then

$$\int cot\theta d\theta = \int \frac{\cos\theta}{\sin\theta} d\theta = \int \frac{du}{u} = \ln|u| + C.$$

The answer is $(\ln|\sin\theta| + C)$. □

Example 1.3.3. *Evaluate* $\int \frac{dx}{(1+\sqrt{x})^2}$.

Solution. Let $u = 1 + \sqrt{x}$. Then

$$du = d(1 + \sqrt{x}) = \frac{1}{2\sqrt{x}} dx.$$

Since $\sqrt{x} = u - 1$, we get $du = \frac{1}{2(u-1)} dx$ and so

$$dx = 2(u-1)du.$$

Then

$$\int \frac{dx}{(1+\sqrt{x})^2} = \int \frac{2(u-1)du}{u^2} = \int \frac{2}{u} - \frac{2}{u^2} du$$
$$= 2\ln|u| + \frac{2}{u} + C = 2\ln|1+\sqrt{x}| + \frac{2}{1+\sqrt{x}} + C.$$

□

The change of variables formula can be applied to definite integrals.

Theorem 1.6. *Suppose* $F'(x) = f(x)$.

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du = F(u(b)) - F(u(a)).$$

Example 1.3.4. *Calculate the area under the graph of* $y = \frac{x}{x^2+1}$ *over* $[1, 3]$.

Solution. The area equals $\int_1^3 \frac{x}{x^2+1} dx$. Let $u = x^2$ and then

$$du = 2x\,dx,\ u(1) = 1,\ u(3) = 9.$$

We have

$$\int_1^3 \frac{x}{x^2+1} dx = \frac{1}{2} \int_1^9 \frac{du}{u+1} = \frac{1}{2} \ln|u+1|\Big|_1^9 = \frac{1}{2} \ln 5.$$

□

Example 1.3.5. *A particle has velocity* $v(t) = \sin^2(t)\cos(t)$. *Compute the displacement and the total distance traveled over* $[0, \pi]$.

Solution. Compute

$$\int_0^\pi v(t)dt = \int_0^\pi \sin^2 t \cos t dt = \int_0^\pi \sin^2 t d \sin t = (\frac{1}{3}\sin^3 t)\Big|_0^\pi = 0.$$

So the displacement is 0. Next,

$$\int_0^\pi |v(t)|dt = \int_0^\pi |\sin^2 t \cos t| dt = \int_0^{\pi/2} \sin^2 t \cos t dt - \int_{\pi/2}^\pi \sin^2 t \cos t dt.$$

By symmetry: $\sin^2 t \cos t = -\sin^2(\pi - t)\cos(\pi - t)$. Therefore the above

$$= 2\int_0^{\pi/2} \sin^2 t \cos t dt = 2\int_0^{\pi/2} \sin^2 t \, d \sin t = \frac{2}{3}\sin^3 t\Big|_0^{\pi/2} = \frac{2}{3}.$$

Thus the total distance traveled is $\frac{2}{3}$.

□

2 Applications of the integral

2.1 Area between two curves

Theorem 2.1. *Let* $y = f(x)$ *and* $y = g(x)$ *be two graphs. The signed area from the graph of* $g(x)$ *to the graph of* $f(x)$ *over interval* $[a, b]$ *is*

$$\int_a^b f(x) - g(x)dx.$$

The area between the graphs over interval $[a, b]$ *is*

$$\int_a^b |f(x) - g(x)|dx.$$

Example 2.1.1. *Find the area between the graphs of* $f(x) = x^2 - 5x - 7$ *and* $g(x) = x - 12$ *over* $[-2, 5]$.

Solution. **Step 1. Sketch the region.** (You can skip this step when you are familiar with this type of problems.)

Step 2. Find out the signs of $f(x) - g(x)$**.** Since

$$f(x) - g(x) = (x^2 - 5x - 7) - (x - 12) = x^2 - 6x + 5 = (x - 1)(x - 5),$$

$f(x) - g(x) > 0$ for $x \in (-2, 1)$ and < 0 for $x \in (1, 5)$.

Step 3. Compute the integral. We have

$$\begin{aligned}
\int_{-2}^{5} |f(x) - g(x)| &= \int_{-2}^{1} f(x) - g(x)dx + \int_{1}^{5} g(x) - f(x)dx \\
&= \int_{-2}^{1} x^2 - 6x + 5dx - \int_{1}^{5} x^2 - 6x + 5dx \\
&= (\frac{1}{3}x^3 - 3x^2 + 5x)\Big|_{-2}^{1} - (\frac{1}{3}x^3 - 3x^2 + 5x)\Big|_{1}^{5} \\
&= (\frac{7}{3} - \frac{(-74)}{3}) - (\frac{-7}{3} - \frac{25}{3}) = \frac{113}{3}.
\end{aligned}$$

□

Example 2.1.2. *Find the area of the region bounded by the graphs of $y = \frac{8}{x^2}$ (with positive x), $y = 8x$ and $y = x$.*

Solution. Let us find the intersection of graphs. Suppose $f_1 = \frac{8}{x^2}$, $f_2 = 8x$, $f_3 = x$. The intersection of f_1, f_2 is given by

$$f_1 = f_2 \implies x = 1.$$

The intersection of f_1, f_3 is

$$f_1 = f_3 \implies x = 2.$$

Finally

$$f_2 = f_3 \implies x = 0.$$

When $x \in (0, 1)$, $f_1 \geq f_2 \geq f_3$ and when $x \in (1, 2)$, $f_2 \geq f_1 \geq f_3$. Therefore the area equals

$$\begin{aligned}
&\int_{0}^{1} f_2 - f_3 dx + \int_{1}^{2} f_1 - f_3 \\
&= \int_{0}^{1} 8x - xdx + \int_{1}^{2} \frac{8}{x^2} - xdx \\
&= \int_{0}^{1} 7xdx + \int_{1}^{2} \frac{8}{x^2} - xdx \\
&= (\frac{7}{2}x^2)\Big|_{0}^{1} + (-\frac{8}{x} - \frac{x^2}{2})\Big|_{1}^{2} \\
&= \frac{7}{2} + \frac{5}{2} = 6.
\end{aligned}$$

□

We can also do integration along y-axis. Then we need to rewrite the curves as functions of y.

Example 2.1.3. *Find the area of the region (in $\{x > 0, y > 0\}$) that is bounded by $y = x^2$, $y = (x-2)^2$ and $x = 0$.*

Solution. Let us rewrite the curves as functions of y in the region $y \le 1$ as follows

$$f_1(y) = \sqrt{y}, \quad f_2(y) = 2 - \sqrt{y}.$$

Set $f_1(y) = 0, f_2(y) = 0$ respectively and then we get $y = 0, 4$. We find $y \in (0, 4)$.

The intersection of the two curves is given by

$$\sqrt{y} = 2 - \sqrt{y} \quad \Longrightarrow \quad y = 1.$$

For $y \in (0, 1)$, the region is given by those point in between the graphs of $x = 0, x = f_1$. For $y \in (1, 4)$, the region is given by $x = 0, x = f_2$. Thus the area equals

$$\begin{aligned}
&\int_0^1 f_1(y) - 0\,dy + \int_1^4 f_2(y) - 0\,dy \\
&= \int_0^1 \sqrt{y}\,dy + \int_1^4 2 - \sqrt{y}\,dy \\
&= (\frac{2}{3}y^{3/2})\Big|_0^1 + (2y - \frac{2}{3}y^{3/2})\Big|_1^4 \\
&= \frac{2}{3} + (8 - \frac{16}{3} - 2 + \frac{2}{3}) \\
&= 2.
\end{aligned}$$

□

2.2 Volume, Density

Let us compute the volume of a solid body in $\mathbb{R}^3$. Suppose that a solid body extends from height $y = a$ to $y = b$. Let us cut the solid body with a hyperplane $y = y_0$, then we get a **horizontal cross section** and we denote the area of the section as $A(y)$. Then we have the following formula:

Theorem 2.2.

$$\textit{The volume of the solid body (given above)} = \int_a^b A(y)dy.$$

Example 2.2.1. ***Volume of a Pyramid.*** *Calculate the volume V of a pyramid of height $12m$ whise base is a square of side $4m$.*

Solution. **Step 1. Find $A(y)$.** Consider a horizontal cross section at height y. It is a square denoted by S_y and suppose the length of its side is s. Apply the law of similar triangles to the triangle given by one side of S_y and the top point. See Figure 3 on page 367 of the textbook. We find

$$\text{the proportion of the sides } = \frac{s}{4} = \frac{12 - y}{y}.$$

This shows that $s = \frac{1}{3}(12 - y)$ and therefore

$$A(y) = \text{ the area of } S_y = s^2 = \frac{1}{9}(12 - y)^2.$$

Step 2. Compute V.

$$V = \int_0^{12} A(y)dy = \int_0^{12} \frac{1}{9}(12 - y)dy = -\frac{1}{27}(12 - y)^3\Big|_0^{12} = 64.$$

□

Remark 2.3. Suppose a pyramid of base area A and height h, the volume formula is

$$V = \frac{1}{3}Ah.$$

Can you justify this formula?

Example 2.2.2. ***Volume of a Sphere.*** *Compute the volume of a sphere of radius R.*

Solution. Place the sphere centered at the origin. Let $A(y)$ denotes the area of the horizontal cross section. See Figure 5 on page 367. Then y is from $-R$ to R. For each such y, the section is a circle with radius r such that

$$r^2 + y^2 = R^2 \quad \text{and so } r = \sqrt{R^2 - y^2}.$$

Then

$$A(y) = \pi r^2 = \pi(R^2 - y^2).$$

Therefore the volume of the sphere

$$\int_{-R}^{R} \pi(R^2 - y^2)dy = \pi(R^2 y - \frac{y^3}{3})\Big|_{-R}^{R} = \frac{4}{3}\pi R^3.$$

□

Example 2.2.3. *The population of one city and its surrounding suburbs has radial density function $\rho(r) = 15(1 + r^2)^{-1/2}$, where r is the distance from the city center in kilometers and ρ has units of thousands per square kilometer. How many people live in the ring between 10 and 30 km from the city center?*

Solution. Knowing the density of population, say ρ, then the population equals the integration of the density. Suppose A (in dimension 2) is the region and the the population equals

$$\int_A \rho dxdy.$$

We have the following formula where (r, θ) is the polar coordinates in dimension 2:

$$dxdy = rdrd\theta.$$

So the population is

$$\begin{aligned}\int_0^{2\pi}(\int_{10}^{30} 15(1+r^2)^{-1/2}\, rdr)d\theta &= 30\pi \int_{10}^{30} \frac{rdr}{(1+r^2)^{1/2}} \\ &= 15\pi \int_{10}^{30} \frac{dr^2}{(1+r^2)^{1/2}} \\ &= 15\pi \int_{100}^{900} \frac{du}{(1+u)^{1/2}} \qquad (u=r^2) \\ &= 30\pi(1+u)^{1/2}\Big|_{100}^{900} \\ &= 30\pi(\sqrt{901}-\sqrt{101}).\end{aligned}$$

□

2.3 Volumes of Revolution

A **solid of revolution** is a solid obtained by rotating a 2-dimensional region about an axis in 3-dimensional space.

Theorem 2.4. *If $f \geq 0$ on $[a,b]$, then the solid obtained by rotating the region under the graph about the x-axis has volume*

$$V = \pi \int_a^b f(x)^2 dx.$$

Theorem 2.5. *If $f \geq g \geq 0$ on $[a,b]$, then the solid obtained by rotating the region between the graphs f, g about the x-axis has volume*

$$V = \pi \int_a^b f(x)^2 - g(x)^2 dx.$$

Example 2.3.1. *(Region between two curves) Find the volume V obtained by revolving the region between $y = x^2+4$ and $y = 2$ about the x-axis for $1 \leq x \leq 3$.*

Solution. The volume of the solid given by rotating the region under $y = x^2+4$ is

$$V_1 = \pi \int_1^3 (x^2+4)^2 dx.$$

The volume of the solid given by rotating the region under $y = 2$ is

$$V_2 = \pi \int_1^3 2^2 dx.$$

The volume wanted is the difference of the above two volumes:

$$V = V_1 - V_2 = \pi \int_1^3 (x^2+4)^2 - 2^2 dx = \pi \int_1^3 (x^4+8x^2+12)dx = \frac{2126}{15}\pi.$$

□

In the next example, we calculate a volume of revolution about a vertical line that is parallel to the y-axis.

Example 2.3.2. *Find the volume of the solid obtained by rotating the region under the graph of* $f(x) = 9 - x^2$ *for* $0 \le x \le 3$ *about the vertical axis* $x = -2$*.*

Solution. First let us plot the graph of $y = f(x)$. It can be seen from the graph that the solid obtained can be viewed as the difference of two sold bodies coming from rotating two graphs about $x = -2$. Let us write the inner and outer radii as R_1 and R_2 (note that the center is at $x = -2$). Since $y = f(x) = 9 - x^2$, we have $x = \sqrt{9-y}$. Because the center is at $x = -2$, then

$$R_1 = \sqrt{9-y} + 2.$$

Since the region accounts for $0 \le x \le 3$,

$$R_2 = 2.$$

When $0 \le x \le 3$, from the picture of the graphs, y is from 0 to 9. So the volume

$$\begin{aligned} V &= \pi \int_0^9 R_1^2 - R_2^2 dy = \pi \int_0^9 (9 - y + 4\sqrt{9-y}) dy \\ &= \pi(9y - \frac{1}{2}y^2 - \frac{8}{3}(9-y)^{3/2})\Big|_0^9 = \frac{225}{2}\pi. \end{aligned}$$

□

3 Techniques of integration

3.1 Integration by parts

In this section, we give a formula that often allows us to convert an integral to another.

Theorem 3.1. *Let* u, v *be (differentiable) functions of* x*.*

$$\int u dv = uv - \int v du.$$

Proof. Notice by the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

So after integrating both sides,

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx = \int u dv + \int v du.$$

The formula follows by moving $\int v du$ to the left-hand side. □

Example 3.1.1. *Evaluate* $\int \ln x dx$.

Solution. View x as v and $\ln x$ as u. Then

$$\begin{aligned}\int \ln x dx &= (\ln x)x - \int x d \ln x \\ &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - x + C.\end{aligned}$$

□

Sometimes we need to do integration by parts more than once.

Example 3.1.2. *Evaluate* $\int x^2 \sin x \, dx$.

Solution. Since $\sin x \, dx = d(-\cos x)$, view x^2 as u and $-\cos x$ as v. Then

$$\begin{aligned}\int x^2 \sin x \, dx &= \int x^2 d(-\cos x) \\ &= -x^2 \cos x - \int (-\cos x) dx^2 \\ &= -x^2 \cos x + \int 2x \cos x \, dx.\end{aligned}$$

Let us do integration by parts once again for $\int 2x \cos x \, dx$.

$$\begin{aligned}\int 2x \cos x \, dx &= \int 2x \, d \sin x \\ &= 2x \sin x - \int \sin x d(2x) \\ &= 2x \sin x - 2 \int \sin x dx \\ &= 2x \sin x + 2 \cos x + C.\end{aligned}$$

In all we have

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

□

The definite integral version of the integration by parts:

Theorem 3.2.

$$\int_a^b u \, dv = uv \,\Big|_a^b - \int_a^b v \, du.$$

Example 3.1.3. *Evaluate* $\int_0^\pi e^x \cos x\, dx$.

Solution.

$$\begin{aligned}\int_0^\pi e^x \cos x\, dx &= \int_0^\pi e^x\, d \sin x \\ &= e^x \sin x\Big|_0^\pi - \int_0^\pi \sin x\, de^x \\ &= -\int_0^\pi e^x \sin x\, dx.\end{aligned}$$

If we do integration by parts once again, we get the above

$$\begin{aligned}&= \int_0^\pi e^x d \cos x \\ &= e^x \cos x\Big|_0^\pi - \int_0^\pi \cos x\, de^x \\ &= -e^\pi - 1 - \int_0^\pi e^x \sin x\, dx.\end{aligned}$$

In all we obtained

$$\int_0^\pi e^x \sin x\, dx = -e^\pi - 1 - \int_0^\pi e^x \sin x\, dx,$$

which implies

$$\int_0^\pi e^x \sin x\, dx = -\frac{1}{2}e^\pi - \frac{1}{2}.$$

□

3.2 Polar Coordinates

The rectangular coordinates describe the position of one point on the $x - y$ plane. For example if a point P has coordinate (x_1, y_1), it means that x_1 is the projection of P onto the x-axis and y_1 is the projection of P onto the y-axis.

There is another way to describe the position of P. In polar coordinates, we label it by coordinates (r, θ), meaning that the distance from P to the origin is $r(= \overline{\mathrm{OP}})$ and the angle between $\overline{\mathrm{OP}}$ and the x-axis is θ.

Theorem 3.3. *Suppose a point has* (x, y) *rectangular coordinate and* (r, θ) *polar coordinate. Then polar to rectangular coordinates:*

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and rectangular to polar:

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Here I avoided writing $\theta = \arctan\frac{y}{x}$. Because by convention, the range of $\arctan$ is only $(-\frac{\pi}{2}, \frac{\pi}{2}]$, while the angle we want to represent has a larger range. We also remark that (r,θ) and $(r,\theta+2\pi k)$ with $k \in \mathbb{Z}$ label the same point.

We commonly choose $r \geq 0$ and $\theta \in [0, 2\pi)$. When $r > 0$,

$$\begin{aligned}
\theta &= \arctan\frac{y}{x} && \text{if the point lies in the first quadrant,}\\
\theta &= \arctan\frac{y}{x} + \pi && \text{if the point lies in the second or third quadrant,}\\
\theta &= \arctan\frac{y}{x} + 2\pi && \text{if the point lies in the fourth quadrant,}\\
\theta &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2} && \text{if } (x,y) = (0,y).
\end{aligned}$$

Example 3.2.1. *Find the rectangular coordinate of a point Q which has polar coordinate $(3, \frac{5\pi}{6})$. Find the polar coordinate of a point W which has a rectangular coordinate $(3,2)$.*

Solution. Since the polar coordinate of Q is $(r,\theta) = (3, \frac{5\pi}{6})$, so the rectangular coordinate is

$$(x,y) = (r\cos\theta, r\sin\theta) = (-\frac{3\sqrt{3}}{2}, \frac{3}{2}).$$

As for W, since $(x,y) = (3,2)$ lies in the first quadrant, we have

$$r = \sqrt{x^2+y^2} = \sqrt{13} \approx 3.6,$$

$$\theta = \arctan(2/3) \approx 0.588.$$

So the polar coordinate for W is $(\sqrt{13}, \arctan(2/3)) \approx (3.6,\ 0.588)$. □

By convention, we allow negative radial coordinates (though not common!). The definition is

$(-r,\theta)$ is the reflection of (r,θ) through the origin.

Hence $(-r,\theta)$ and $(r,\theta+\pi)$ represent the same point.

Example 3.2.2. *Find two polar representations of $P = (-1,1)$, one with $r > 0$ and one with $r < 0$.*

Solution. Let (r,θ) be one polar coordinate of P. Then

$$r^2 = 2 \quad \text{and} \quad \tan\theta = \frac{y}{x} = -1.$$

First consider $r > 0$, then $r = \sqrt{2}$. Since P is in the second quadrant, the correct angle is

$$\theta = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}.$$

If we wish to use the negative radial coordinate $r = -\sqrt{2}$, then the angle becomes $\frac{3\pi}{4} + \pi = \frac{7\pi}{4}$. Thus

$$P = (\sqrt{2}, \frac{3\pi}{4}) \quad \text{or} \quad (-\sqrt{2}, \frac{7\pi}{4}).$$

□

Example 3.2.3. *Convert the following to an equation in polar coordinates of the form $r = f(\theta)$:*

1. $xy = 1$;
2. *the line whose point closest to the origin is $P_0 = (d, \alpha)$ in polar coordinate.*

Solution. For 1, since $x = r\cos\theta$, $y = r\sin\theta$, we have

$$r^2 \cos\theta \sin\theta = 1.$$

Therefore

$$r = \sqrt{\frac{1}{\cos\theta\sin\theta}}.$$

We also need

$$\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$$

to be positive. Hence

$$\theta \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}).$$

Now consider 2. Let $P = (r, \theta)$ be any point on the line. Then ΔOPP_0 is a right triangle. Therefore

$$\frac{d}{r} = \cos(\theta - \alpha)$$

or

$$r = \frac{d}{\cos(\theta - \alpha)} = d\sec(\theta - \alpha).$$

From the picture, θ belongs to $(-\frac{\pi}{2} + \alpha, \frac{\pi}{2} + \alpha)$. □

Example 3.2.4. *Convert to rectangular coordinates and identify the curve with polar equation $r = 2a\cos\theta$ where a is a positive constant.*

Solution. Use the relation

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad x = r\cos\theta.$$

We get

$$\sqrt{x^2 + y^2} = 2ax/r$$

which gives

$$x^2 + y^2 = 2ax.$$

It can be rewritten into

$$(x - a)^2 + y^2 = a^2.$$

This is the equation of the circle of radius a and center $(a, 0)$. □

3.3 Area in Polar Coordinates

Theorem 3.4. *Let f be a continuous function. The area bounded by a curve in polar form $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ (with $\alpha < \beta$) is equal to*

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 d\theta.$$

Example 3.3.1. *Sketch $r = \sin 3\theta$ and compute the area of one "petal".*

Solution. To sketch the curve, we first graph $r = \sin 3\theta$ in r versus θ rectangular coordinates (see figure 5 on page 647 of the textbook). By periodicity, we only need to look at $\theta \in [0, 2\pi)$. Since $\sin 3\theta$ is periodic with periodicity $\frac{2\pi}{3}$, we only need to look at $\theta \in [0, \frac{2\pi}{3})$. Also since $r(\theta) = -r(\theta + \pi)$, we only need to look at $\theta \in [0, \frac{\pi}{3})$.

We know that r varies from 0 to 1 and back to 0 as θ increases from 0 to $\frac{\pi}{3}$. This gives one petal. When θ increase from $\frac{\pi}{3}$ to $\frac{2\pi}{3}$, $r \leq 0$. We shift the angle by π if we use positive radius. This gives that r varies from 0 to 1 and back to 0 as θ increases from $\frac{4\pi}{3}$ to $\frac{5\pi}{3}$. Lastly when θ increases from $\frac{2\pi}{3}$ to π, r varies from 0 to 1 and back to 0.

The area is

$$\frac{1}{2}\int_{0}^{\pi/3} (\sin 3\theta)^2 d\theta = \frac{1}{2}\int_{0}^{\pi/3} \left(\frac{1 - \cos 6\theta}{2}\right) d\theta = \left(\frac{1}{4}\theta - \frac{1}{24}\sin 6\theta\right)\Big|_0^{\pi/3} = \frac{\pi}{12}.$$

□

Example 3.3.2. *Find the area of the region inside the circle $r = 2\cos\theta$ but outside the circle $r = 1$.*

Solution. The two circles intersect at the points where $2\cos\theta = 1$, which gives $\cos\theta = \frac{1}{2}$ and so $\theta = \pm\frac{\pi}{3}$.

From the picture (see figure 7 on page 648 of the textbook) that the target region can be viewed as the difference of two "sectors".

$$\begin{aligned}
\text{the area } &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3} (1)^2 d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4(\cos\theta)^2 - 1\, d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos(2\theta) + 1\, d\theta \\
&= \frac{1}{2}(\sin\theta + \theta)\Big|_{-\pi/3}^{\pi/3} \\
&= \frac{\sqrt{3}}{2} + \frac{\pi}{3}.
\end{aligned}$$

□

Review for Midterm 1

Velocity and Displacement.

Example 3.3.3. *Find the distance travelled and displacement from $t = 1$ to 4 for an object with velocity $8t - t^2$.*

Integration method.

Example 3.3.4.

$$\int \cos x \cos(\sin x) dx, \int x 5^x dx, \int_0^2 \frac{dx}{\sqrt{2x+5}}.$$

Area and Volume.

Example 3.3.5. *Sketch the region enclosed by the curves and compute its area as an integral.*

$$y = x\sqrt{x-2},\ y = -x\sqrt{x-2},\ x = 4.$$

Solution. Note by the given expression, $x \geq 2$. So x is from 2 to 4.

$$\begin{aligned} \text{area } &= \int_2^4 x\sqrt{x-2} - (-x\sqrt{x-2}) dx \\ &= 2\int_2^4 x\sqrt{x-2}\, dx \end{aligned}$$

Set $t := \sqrt{x-2}$ and then $x = t^2 + 2$, $dx = 2tdt$. The integration

$$\begin{aligned} &= 2\int_0^{\sqrt{2}} (t^2+2)t \times 2t\, dt \\ &= 4\int_0^{\sqrt{2}} t^4 + 2t^2\, dt \\ &= 4\left(\frac{t^5}{5} + \frac{2t^3}{3}\right)\Big|_0^{\sqrt{2}} \\ &= \frac{128}{15}\sqrt{2}. \end{aligned}$$

□

Example 3.3.6. *Find the volume V of the solid whose base is the circle $x^2 + y^2 = 16$ and whose cross sections perpendicular to the x-axis are triangles whose height and base are equal.*

Solution. For each hyperplane location $x = a$, it intersects with the solid and the cross section is a triangle with bottom side of length $2y = 2\sqrt{16 - a^2}$. The height of the triangle is the same with the bottom side.

Therefore, the volume of the solid is

$$\begin{aligned}\int_{-4}^{4} \frac{1}{2}(2y)^2 dx &= \int_{-4}^{4} \frac{1}{2} 2(16 - x^2) dx \\ &= 2\int_{-4}^{4} (16 - x^2) dx \\ &= 512/3.\end{aligned}$$

□

Example 3.3.7. *A plane inclined at an angle of* 45 *degree passes through a diameter of the base of a cylinder of radius r. Find the volume of the region with the cylinder and below the plane (see Figure 23 on page 374 of the textbook).*

Solution. Let h be the height. When $h = 0$, the cross section is a half disc. When $h > 0$, the cross section is the smaller part of the intersection of a disc and a straight line which is h away from its center. The area of the cross section is the difference of sector of angle $2\arccos\frac{h}{r}$ and a triangle:

$$\pi r^2 \frac{2\arccos(h/r)}{2\pi} - \frac{1}{2}h(2\sqrt{r^2 - h^2}).$$

Therefore the volume

$$\begin{aligned}&\int_{0}^{r} r^2 \arccos(h/r) - h\sqrt{r^2 - h^2}\, dh \\ &= r^3 \int_{0}^{r} \arccos(h/r) d(h/r) - \frac{1}{2}\int_{0}^{r} \sqrt{r^2 - h^2}\, dh^2 \\ &= r^3 \int_{0}^{1} \arccos x dx - \frac{1}{2}\int_{0}^{r^2} \sqrt{r^2 - y}\, dy \\ &= r^3 (x \arccos x - \sqrt{1 - x^2})\Big|_0^1 + \frac{1}{3}(r^2 - y)^{3/2}\Big|_0^{r^2} \\ &= r^3 - \frac{1}{3}r^3 \\ &= \frac{2}{3}r^3.\end{aligned}$$

□

Example 3.3.8. *The solid* S *(see Figure 24 on page 375 of the textbook) is the intersection of two cylinders of radius* r *whose axes are perpendicular.*

(a) The horizontal cross section of each cylinder at distance y *from the central axis is a rectangular strip. Find the strip's width.*

(b) Find the area of the horizontal cross section of S *at distance* y*.*

(c) Find the volume of S as a function of r.

Solution. The strip's side is given by the intersection of a disc of radius r (the base of the cylinder) and a straight line which is y away from the disc's center. Thus the width is $2\sqrt{r^2-y^2}$.

The horizontal cross section of a cylinder (placed in a way that its bottom is perpendicular to the horizontal plane) is a rectangle. And therefore the horizontal cross section of S is a square of side length $2\sqrt{r^2-y^2}$. Thus the area is $4(r^2-y^2)$.

With the above information, the volume is

$$\int_{-r}^{r} 4(r^2-y^2)dy = 8r^3 - \frac{4}{3}y^3\Big|_{-r}^{r} = \frac{16}{3}r^3.$$

□

Example 3.3.9. *Find the volume of the solid obtained by rotating the region enclosed by the graphs about the given axis.*

$$y = 2\sqrt{x},\ y = x,\ \textit{about } x = -20.$$

Solution. Let us find the intersection of the two curves by setting $2\sqrt{x} = x$. We get the two curves intersect at $x = 0$ and $x = 4$. Rotating the region enclosed by $y = 2\sqrt{x}$ and $y = x$ about $x = -20$ produces a solid whose cross sections are washers with outer radius $R = y - (-20) = y + 20$ and inner radius $r = \frac{1}{4}y^2 - (-20) = \frac{1}{4}y^2 + 20$. The volume of the solid of revolution is

$$\begin{aligned}
V &= \pi \int_0^4 \left((y+20)^2 - (\frac{1}{4}y^2+20)^2\right) dy \\
&= \pi \int_0^4 \left(0 + 40y - 9y^2 - \frac{1}{16}y^4\right) dy \\
&= \pi \left(20y^2 - 3y3 - \frac{1}{80}y^5\right)\Big|_0^4 \\
&= \frac{576}{5}\pi.
\end{aligned}$$

□

Example 3.3.10. *The torus (see Figure 15 on page 384 of the textbook) is obtained by rotating the circle $(x-a)^2 + y^2 = b^2$ around y-axis (assume $a > b > 0$). Show that it has volume $2\pi^2ab^2$.*

Solution. In order to find out the outer and inner radius, we find the distance from the circle to the y-axis. For each y, the inner radius equals $a - \frac{1}{2}s$ where s is the length of the intersection of the circle and a straight line y away from its center. While the outer radius is $a + \frac{1}{2}s$. Thus the volume

$$V = \pi \int_{-b}^{b} (a+\frac{1}{2}s)^2 - (a-\frac{1}{2}s)^2 dy = 2a\pi \int_{-b}^{b} sdy.$$

Notice $\int_{-b}^{b} sdy$ is exactly the area of a circle of radius b. Therefore

$$V = 2a\pi \times \pi b^2 = 2\pi^2ab^2.$$

Alternatively, we can compute $s = 2\sqrt{b^2 - y^2}$ and do the integration by considering the substitution $y = b\sin\theta$. $\square$

Example 3.3.11 (exercise 20 on textbook page 650)**.** *Find the area between* $r = 2 + \sin 2\theta$ *and* $r = \sin 2\theta$.

Solution. [Sketch] Think about why the two curves attach at two points in the picture?

$\theta \in (0, \pi/2)$:

$$S_1 = \frac{1}{2}\int_0^{\pi/2} (2 + \sin 2\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/2} (\sin 2\theta)^2 d\theta.$$

$\theta \in (\pi/2, \pi)$:

$$S_2 = \frac{1}{2}\int_{\pi/2}^{\pi} (2 + \sin 2\theta)^2 d\theta - \frac{1}{2}\int_{\pi/2}^{\pi} (\sin 2(\theta + \pi))^2 d\theta.$$

The area $= 2S_1 + 2S_2$. $\square$

Example 3.3.12. *Set up, but do not evaluate, the area of the region between the inner and outer loop given by* $r = 2\cos\theta - 1$. *See Figure 19 on textbook page 650.*

Solution. What is the graph of $r(\theta)$? Which parts translates to the which parts of the graph? What happens when $r < 0$? This answer is:

$$\int_{\pi/3}^{\pi} (2\cos\theta - 1)^2 d\theta - \int_0^{\pi/3} (2\cos\theta - 1)^2 d\theta.$$

Think about why? $\square$

4 Supplement

4.1 Complex Numbers

Definition 4.1. A complex number is a number of the form $a + bi$, where a and b are real numbers and $i^2 = -1$. Here a is called the real part of the complex number and b is called the imaginary part.

Remark 4.2. A real number is also a complex number (with 0 imaginary part).

We have the following computational laws: if $\alpha = a + bi$ and $\beta = c + di$ are complex numbers, then

$$\begin{aligned}
\alpha + \beta &:= (a + c) + (b + d)i, \\
\alpha - \beta &:= (a - c) + (b - d)i, \\
\alpha\beta &:= (ac - bd) + (ad + bc)i.
\end{aligned}$$

The last law can be deduced from the following computation by using $i^2 = -1$.

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = ac - bd + (ad + bc)i.$$

Example 4.1.1. *Let α, β be as the above. What is the real part and the imaginary part of $\frac{\alpha}{\beta}$?*

Solution. It follows from the direct computation

$$\begin{aligned}\frac{a+bi}{c+di} &= \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right) \\ &= \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2} \\ &= \left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{-ad+bc}{c^2+d^2}\right)i.\end{aligned}$$

Thus the real part is $\frac{ac+bd}{c^2+d^2}$ and the imaginary part is $\frac{-ad+bc}{c^2+d^2}$. □

Definition 4.3. Let $\alpha = a+bi$ be a complex number.

(a) The complex conjugate of α is the complex number $\bar{\alpha} := a - bi$.

(b) The magnitude of α, written $|\alpha|$, is given by $|\alpha| := \sqrt{\alpha\bar{\alpha}} = \sqrt{a^2+b^2}$.

(c) $|\alpha|$ is also called the modulus, length or absolute value of α.

Theorem 4.4. *Let α and β be complex variables. Then,*

$$\overline{(\alpha \pm \beta)} = \bar{\alpha} \pm \bar{\beta}, \quad \overline{(\alpha\beta)} = \bar{\alpha}\bar{\beta}, \quad \overline{\left(\frac{\alpha}{\beta}\right)} = \frac{\bar{\alpha}}{\bar{\beta}}.$$

A complex variable $z = x+iy$ can be represented geometrically by a point (x,y) in the 2-dimensional plane. (Think about the difference between a complex number and a point in $\mathbb{R}^2$?) It is known that any point in the plane can be represented in polar coordinates (r,θ). Thus, we can use polar coordinates to rewrite a complex number as well:

$$z = r[\cos(\theta) + i\sin(\theta)].$$

This is called the polar form of z. Note that $|z| = r$. The angle θ is called the argument of z, written $\theta = arg(z)$. It is important to realize that $\theta = arg(z)$ is not uniquely determined.

Theorem 4.5. *Suppose for $j = 1, 2$,*

$$z_j = r_j[\cos(\theta_j) + i\sin(\theta_j)].$$

Then

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

Example 4.1.2. *Let $\alpha = 1+i$. By what angle will multiplication by $\beta = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ rotate α? Compute $\alpha\beta$.*

Solution. Since
$$\beta = -\frac{\sqrt{3}}{2} + \frac{1}{2}i = \cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6}),$$
thus multiplication by β produces a rotation by $\frac{5\pi}{6}$. Also since
$$\alpha = \sqrt{2}[\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})],$$
we obtain
$$\alpha\beta = |\alpha||\beta|[\cos(\frac{\pi}{4} + \frac{5\pi}{6}) + i\sin(\frac{\pi}{4} + \frac{5\pi}{6})] = \sqrt{2}[\cos(\frac{13\pi}{12}) + i\sin(\frac{13\pi}{12})].$$
□

Theorem 4.6 (de Moivre's Theorem)**.** *Let $z = r[\cos(\theta) + i\sin(\theta)]$ be a complex number in polar form and let n be an integer. Then,*
$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Example 4.1.3. *Compute $(1+\sqrt{3}i)^8$ and write the result in the standard $(a+bi)$ form.*

Solution. Notice
$$1 + \sqrt{3}i = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 2[\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})].$$
Therefore
$$\begin{aligned}(1+\sqrt{3}i)^8 &= 2^8[\cos(\frac{8\pi}{3}) + i\sin(\frac{8\pi}{3})] \\ &= 256(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\ &= -128 + 128\sqrt{3}i.\end{aligned}$$
□

Example 4.1.4. *Find all the sixth roots of -64.*

Solution. Since
$$-64 = 2^6[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)],$$
the 6^{th} roots are given by
$$w_k = 2\left[\cos\left(\frac{\pi + 2\pi k}{6}\right) + i\sin\left(\frac{\pi + 2\pi k}{6}\right)\right], \quad k = 0, 1, 2, 3, 4, 5.$$
Then are
$$w_0 = \sqrt{3} + i,\ w_1 = 2i,\ w_2 = -\sqrt{3} + i,\ w_3 = -\sqrt{3} - i,\ w_4 = -2i,\ w_5 = \sqrt{3} - i.$$
The roots appear in pairs. For example, w_1 and w_6 are conjugate to each other, and w_2 and w_5 are conjugate to each other. Think about why? □

Why there are six sixth roots? In general, for any complex number, there n different n'th roots.

4.2 Complex Exponential

Definition 4.7. Complex Exponential.

$$e^{a+bi} = e^a[\cos b + i \sin b].$$

Why this definition? We are going to provide one answer after learning Taylor series. In particular, we have the well-known Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{i\pi} = 1.$$

Theorem 4.8. *Let z_1, z_2 be complex numbers. Then*

$$e^{z_1}e^{z_2} = e^{z_1+z_2}.$$

Example 4.2.1. *Show*

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}.$$

Proof. Notice

$$e^{ix} = \cos x + i\sin x, \quad e^{-ix} = \cos x - i \sin x.$$

Adding and dividing by 2 gives us $\cos(x)$ whereas subtracting and dividing by $2i$ gives us $\sin(x)$. □

This expression can be used to define $\cos z$, $\sin z$ for complex z. Can you do it?

Example 4.2.2. *Use Theorem 4.8 to show*

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

Proof. Consider $e^{i\alpha}, e^{i\beta}$. Then

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta},$$

which implies

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos(\alpha)+i\sin(\alpha)) \times (\cos(\beta)+i\sin(\beta)).$$

The RHS of the above equals

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).$$

To have the RHS = the LHS, we need the corresponding real parts equal and imaginary parts equal. Thus

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

□

Remark 4.9. The proof of example 4.2.2 also implies

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Set $\alpha = \beta$, and then we get

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha,\\ \cos 2\alpha &= (\cos\alpha)^2 - (\sin\alpha)^2 = 2\cos^2\alpha - 1.\end{aligned}$$

4.3 Trig integration

Complex exponential method.

We know that for an exponential function e^{ax} we have

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C.$$

Actually the same formula holds if we replace a by a complex number z (and in this case the constant C can be either a real number or a complex number depending on what anti-derivative you want to find).

Example 4.3.1. *Let us integrate*

$$8\cos(3x)\sin(x).$$

Solution. We have

$$\begin{aligned} 8\cos(3x)\sin(x) &= 8\left(\frac{e^{3ix}+e^{-3ix}}{2}\right)\left(\frac{e^{ix}-e^{-ix}}{2i}\right) \\ &= \frac{2}{i}\left(e^{4ix}+e^{-2ix}-e^{2ix}-e^{-4ix}\right). \end{aligned}$$

After integration,

$$\begin{aligned} \int 8\cos(3x)\sin(x)dx &= \int \frac{2}{i}\left(e^{4ix}+e^{-2ix}-e^{2ix}-e^{-4ix}\right)dx \\ &= \frac{2}{i}\left(\frac{e^{4ix}}{4i}-\frac{e^{-2ix}}{2i}-\frac{e^{2ix}}{2i}+\frac{e^{-4ix}}{4i}\right)+C \\ &= -\left(\frac{e^{4ix}}{2}+\frac{e^{-4ix}}{2}\right)+e^{-2ix}+e^{2ix}+C \\ &= -\cos(4x)+2\cos(2x)+C. \end{aligned}$$

□

For a complex number $z = x + yi$, we write the real part of it as $\text{Re}(z)$, and so $\text{Re}(z) = x$. It can be easily checked that $z + \bar{z} = 2\text{Re}(z)$.

Example 4.3.2. *Let us integrate* $e^{2x}\sin(x)$.

Solution.

$$\begin{aligned} \int e^{2x}\sin(x)dx &= \frac{1}{2i}\int e^{2x}(e^{ix}-e^{-ix})dx \\ &= \frac{1}{2i}\int e^{(2+i)x}-e^{(2-i)x})dx \\ &= \frac{1}{2i}\left(\frac{e^{(2+i)x}}{2+i}-\frac{e^{(2-i)x)}}{2-i}\right)+C \\ &= -\frac{e^{2x}}{2}\left(\frac{e^{ix}}{1-2i}+\frac{e^{-ix}}{1+2i}\right)+C. \end{aligned}$$

Since the two expressions are conjugate to each other, the above equals,

$$
\begin{aligned}
&= -e^{2x}\,\mathrm{Re}\left(\frac{e^{ix}}{1-2i}\right) + C \\
&= -e^{2x}\,\mathrm{Re}\left(\frac{(1+2i)e^{ix}}{5}\right) + C \\
&= -\frac{e^{2x}}{5}\left(\cos(x) - 2\sin(x)\right) + C.
\end{aligned}
$$

□

Substitution.

Substitution can also help solving an integration problem. The following substritution are frequently used:

$$u = \sin x, \quad u = \cos x, \quad u = \tan x \quad u = \tan\frac{x}{2}, \quad \text{etc.}$$

Example 4.3.3. *Evaluate*

$$\int \tan^3 x \sec^5 x \, dx.$$

Solution. Notice

$$\tan^3 x \sec^5 x = \frac{\sin^3 x}{\cos^8 x} = \frac{1-\cos^2 x}{\cos^8 x}\sin x.$$

Let $u = \cos x$ and then $du = -\sin x dx$. Therefore

$$
\begin{aligned}
\int \tan^3 x \sec^5 x dx &= \int \frac{1-\cos^2 x}{\cos^8 x}\sin x \, dx \\
&= -\int \frac{1-u^2}{u^8} du \\
&= -\int u^{-8} - u^{-6} du \\
&= \frac{u^{-7}}{7} - \frac{u^{-5}}{5} + C \\
&= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{6} + C.
\end{aligned}
$$

□

Example 4.3.4. *Evaluate*

$$\int \tan^2 x \sec^4 x \, dx.$$

Solution. Notice

$$\tan^2 x \sec^4 x = \frac{\sin^2 x}{\cos^6 x}$$

and doing substitution for as in the previous example won't work. Instead, let us consider

$$u = \tan x \quad \text{and then } du = \frac{1}{\cos^2 x} dx.$$

Hence

$$\begin{aligned}
\int \tan^2 x \sec^4 x \, dx &= \int \frac{\sin^2 x}{\cos^4 x} d\tan x \\
&= \int \frac{\sin^2 x (\sin^2 x + \cos^2 x)}{\cos^4 x} d\tan x \\
&= \int u^4 + u^2 \, du \\
&= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C.
\end{aligned}$$

□

Example 4.3.5. *Evaluate*

$$(1). \int \tan^3 x \, dx, \quad (2). \int \tan^4 x \, dx.$$

Solution. For (1). consider $u = \sin x$. Then

$$\begin{aligned}
\int \tan^3 x \, dx &= \int \frac{\sin^3 x}{\cos^3 x} \frac{1}{\cos x} d\sin x \\
&= \int \frac{\sin^3 x}{(1 - \sin^2 x)^2} d\sin x \\
&= \int \frac{u^3}{(1 - u^2)^2} du.
\end{aligned}$$

Then we apply the Partial fraction method (in Section 5.4) to get

$$\frac{u^3}{(1-u^2)^2} = \frac{u^3}{(u-1)^2(u+1)^2} = \frac{1}{2\,(u+1)} - \frac{1}{4\,(u+1)^2} + \frac{1}{2\,(u-1)} + \frac{1}{4\,(u-1)^2}.$$

After integration, we get

$$\begin{aligned}
&\frac{1}{2} \ln |(u-1)(u+1)| + \frac{1}{4} \frac{1}{u+1} - \frac{1}{4} \frac{1}{u-1} + C \\
&= \frac{1}{2} \ln |1 - u^2| + \frac{1}{2} \frac{1}{1 - u^2} \\
&= -\ln |\sec x| + \frac{1}{2} \sec^2 x + C.
\end{aligned}$$

As for (2), we let $v = \tan x$. Then

$$\begin{aligned}\int \tan^4 x\, dx &= \int \frac{\sin^4 x}{\cos^2 x}\, d\tan x \\ &= \int \frac{\sin^4 x}{\cos^2 x(\sin^2 x + \cos^2 x)} d\tan x \\ &= \int \frac{v^4}{v^2+1} dv = \int v^2 - 1 + \frac{1}{v^2+1} dv \\ &= \frac{v^3}{3} - v + \arctan v + C.\end{aligned}$$

The integral becomes:

$$\frac{1}{3}\tan^3(x) + x - \tan(x) + C.$$

□

Remark 4.10. We can also apply the following formula to Example 4.3.5:

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx.$$

Please do it as an exercise.

Example 4.3.6 (Integral of Secant)**.** *Derive the formula*

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C.$$

Solution. Let $u = \tan \frac{x}{2}$. Then

$$du = \frac{1}{2\cos^2(x/2)} dx = \frac{1}{2}(1+u^2) dx,$$

$$\sec x = \frac{1}{\cos^2(x/2) - \sin^2(x/2)} = \frac{1+u^2}{1-u^2}.$$

The integral becomes

$$\begin{aligned}\int \sec x\, dx &= \int \frac{1+u^2}{1-u^2} \frac{2}{1+u^2} du \\ &= \int \frac{1}{1-u} + \frac{1}{1+u} du \\ &= \ln\left|\frac{1+u}{1-u}\right| + C.\end{aligned}$$

After transferring from $\tan(x/2)$ to u, we can show the formula. I will leave it to you. :) □

4.4 Partial fractions

Theorem 4.11. ***[Fundamental Theorem of algebra]*** *A polynomial P of degree n is a function of the form*

$$P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x_n$$

where a_i are complex numbers and $a_n \neq 0$. Then $P(x)$ has n complex roots (counting multiplicity) and we can factor $P(x)$ as

$$P(x) = a_n (x - z_1)^{n_1} (x - z_2)^{n_2} ... (x - z_k)^{n_k},$$

where z_k are k distinct complex numbers and $\sum_i n_i = n$.

Now we consider a polynomial differentiate another polynomial and we study partial fractions.

Theorem 4.12. *Suppose that the n numbers $\alpha_1, ..., \alpha_n$ are pairwise distinct and that $P(x)$ is a polynomial with degree less than n. Then, there are constants $C_1, ..., C_n$ such that*

$$\frac{P(x)}{(x - \alpha_1)...(x - \alpha_n)} = \frac{C_1}{x - \alpha_1} + ... + \frac{C_n}{x - \alpha_n}. \tag{3}$$

To determine the constants $C_1, ..., C_n$, we carry out the following steps:

- Multiply both sides of (3) by $x - \alpha_j$ and then set $x = \alpha_j$. The left side will evaluate to a number Z_j.

- The right side evaluates to C_j, since the other terms have a factor of $x - \alpha_j$ which is 0 when $x = \alpha_j$. We conclude that $Z_j = C_j$.

Now for some illustrations, we consider the following example.

Example 4.4.1. *Let us expand $f(x) = \frac{x^2+2}{(x-1)(x+2)(x+3)}$ by partial fractions.*

Solution. By the theorem,

$$f(x) = \frac{x^2 + 2}{(x - 1)(x + 2)(x + 3)} = \frac{C_1}{x - 1} + \frac{C_2}{x + 2} + \frac{C_3}{x + 3}.$$

Multiply by $x - 1$ to eliminate the pole at $x = 1$ and get

$$\frac{x^2 + 2}{(x + 2)(x + 3)} = C_1 + \frac{C_2(x - 1)}{x + 2} + \frac{C_3(x - 1)}{x + 3}.$$

Set $x = 1$ and obtain

$$C_1 = \frac{1 + 2}{(1 + 2)(1 + 3)} = \frac{1}{4}.$$

Similarly,

$$C_2 = \frac{x^2 + 2}{(x - 1)(x + 3)}|_{x=-2} = -2.$$

and

$$C_3 = \frac{x^2+2}{(x-1)(x+2)}|_{x=-3} = \frac{11}{4}.$$

We conclude that

$$f(x) = \frac{1}{4(x-1)} - \frac{2}{(x+2)} + \frac{11}{4(x+3)}.$$

□

A cultural aside is that the numbers C_1, C_2, C_3 are often called the **residues** of the poles at $1, -2, -3$, many of you will see them later in your career under that name. If we wish to find the antiderivatives of f from this, we immediately get

$$\int f(x)dx = \frac{1}{4}\ln|x-1| + 2\ln|x+2| + \frac{11}{4}\ln|x+3| + C.$$

Repeated roots.

When we have repeated root, each factor $(x-a)^n$ contributes the following sum of terms to the partial fraction decomposition

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + ... + \frac{A_n}{(x-a)^n}.$$

Let us apply the method to

$$f(x) = \frac{1}{(x-1)^2(x-3)}.$$

Solution. The partial fraction expansion is of the form

$$f(x) = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{x-3}.$$

We can find C quickly from

$$C = (x-3)f(x)|_{x=3} = \frac{1}{(3-1)^2} = \frac{1}{4}$$

and A from

$$A = (x-1)^2 f(x)|_{x=1} = \frac{1}{1-3} = -\frac{1}{2}.$$

We get

$$f(x) = -\frac{1}{2(x-1)^2} + \frac{B}{(x-1)} + \frac{1}{4(x-3)}.$$

Let us plug in a convenient value of x, say $x = 0$ and obtain

$$f(0) = \frac{1}{(-1)^2(-3)} = -\frac{1}{2} - B - (\frac{1}{3})(\frac{1}{4})$$

We get $B = -\frac{1}{4}$, and then

$$f(x) = -\frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x-3)}.$$

□

Quadratic factor.

Irreducible quadratic factors $(x^2 + ax + b)^N$ contributes the following sum of terms to the partial fraction decomposition

$$\frac{A_1 x + B_1}{(x^2 + ax + b)} + \frac{A_2 x + B_2}{(x^2 + ax + b)^2} + \ldots + \frac{A_N x + B_N}{(x^2 + ax + b)^N}.$$

Example 4.4.2. *Evaluate*

$$\int \frac{4 - x}{x(x^2 + 2)^2} dx.$$

Solution. The partial fraction decomposition has the form

$$\frac{4 - x}{x(x^2 + 2)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2}.$$

Multiplying both side by x and then set $x = 0$, we get $A = 1$.

Then multiplying both side by $x(x^2 + 2)^2$, we get

$$\begin{aligned} 4 - x &= (x^2 + 2)^2 + (Bx + C)x(x^2 + 2) + (Dx + E)x \\ &= (1 + B)x^4 + Cx^3 + (4 + 2B + D)x^2 + (2C + E)x + 4. \end{aligned}$$

Now equate the coefficients on the two sides gives

$$B = -1, \quad C = 0, \quad D = -2, \quad E = -1.$$

Thus

$$\begin{aligned} \int \frac{4 - x}{x(x^2 + 2)^2} dx &= \int \frac{dx}{x} - \int \frac{x dx}{x^2 + 2} - \int \frac{2x + 1}{(x^2 + 2)^2} dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2 + 2) - \int \frac{2x + 1}{(x^2 + 2)^2} dx. \end{aligned}$$

Finally using the result from example 4.5.2, we have

$$\int \frac{4 - x}{x(x^2 + 2)^2} dx = \ln|x| - \frac{1}{2}\ln(x^2 + 2) + \frac{1}{4}\frac{4 - x}{x^2 + 2} - \frac{1}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C.$$

Here by $\tan^{-1} x$, I mean $\arctan x$ which is the inverse function of $\tan$ function.

□

Since we often write $\sin^2 x$ to denote $(\sin x)^2$. It is easy to confuse $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$ with $\frac{1}{\sin x}, \frac{1}{\cos x}, \frac{1}{\tan x}$. So I strongly recommend you to use, instead, the notations of $\arcsin, \arccos, \arctan$ to denote the inverse functions of trig functions.

4.5 Trig Substitution

In this section, let us use substitution with trigonometric function to integrate functions. We will see that the substitutions help simplify some complicated terms like $(\pm 1 \pm x^2)^\alpha$ where α can be any real number.

Example 4.5.1. *Evaluate*

$$\int \frac{1}{\sqrt{1-x^2}}dx.$$

Solution. Set $x = \sin\theta$ and then

$$dx = \cos\theta d\theta, \quad \sqrt{1-x^2} = \cos\theta.$$

So

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{1}{\cos\theta}\cos\theta d\theta = \theta + C = \arcsin x + C.$$

□

Example 4.5.2. *Solve*

$$\int \frac{2x+1}{(x^2+2)^2}dx.$$

Solution. Note

$$\int \frac{2x+1}{(x^2+2)^2}dx = \int \frac{2x}{(x^2+2)^2}dx + \int \frac{1}{(x^2+2)^2}dx =: A + B.$$

For A, we have

$$A = \int \frac{d(x^2+2)}{(x^2+2)^2} = -\frac{1}{(x^2+2)}.$$

As for B, we use the trigonometric substitution

$$x = \sqrt{2}\tan\theta.$$

And then

$$dx = \frac{\sqrt{2}}{\cos^2\theta}d\theta, \quad x^2+2 = 2\tan^2\theta + 2 = \frac{2}{\cos^2\theta}.$$

$$\begin{aligned}
B &= \int \frac{\cos^4\theta}{4}\frac{\sqrt{2}}{\cos^2\theta}d\theta = \frac{\sqrt{2}}{4}\int \cos^2\theta\, d\theta \\
&= \frac{\sqrt{2}}{8}\int (1 + 2\cos 2\theta)\, d\theta \\
&= \frac{\sqrt{2}}{8}(\theta + \sin\theta\cos\theta) + C \\
&= \frac{1}{4\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + \frac{1}{4}\frac{x}{x^2+2} + C.
\end{aligned}$$

Therefore

$$\int \frac{4-x}{x(x^2+2)^2}\,dx = \frac{1}{4}\frac{x-4}{x^2+2} + \frac{1}{4\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C.$$

□

Summary of trig substitution.

$\sqrt{a^2-x^2}$, try $x = a\sin\theta$ and then $dx = a\cos\theta d\theta$, $\sqrt{a^2-x^2} = a\cos\theta$;
$\sqrt{a^2+x^2}$, try $x = a\tan\theta$ and then $dx = a(\cos\theta)^{-2}d\theta$, $\sqrt{a^2+x^2} = a(\cos\theta)^{-1}$;
$\sqrt{x^2-a^2}$, try $x = a(\cos\theta)^{-1}$ and then $dx = a\sin\theta\cos^{-2}\theta d\theta$, $\sqrt{a^2-x^2} = a\tan\theta$.

Sometimes you might need to do the substitution for more than once.

Example 4.5.3. *Evaluate*

$$\int \frac{dx}{(x^2+2x+3)^{3/2}}.$$

Solution. Since

$$x^2+2x+3 = (x+1)^2+2,$$

let $u = x+1$ and we get

$$\int \frac{dx}{(x^2+2x+3)^{3/2}} = \int \frac{du}{(u^2+2)^{3/2}}.$$

Now we set

$$u = \sqrt{2}\tan\theta,$$

and then

$$du = \sqrt{2}\cos^{-2}\theta d\theta, \quad (x^2+2x+3)^{3/2} = (2\cos^{-2}\theta)^{3/2}.$$

The integration becomes

$$\int \frac{du}{(u^2+2)^{3/2}} = \int \frac{\sqrt{2}\cos^{-2}\theta}{(2\cos^{-2}\theta)^{3/2}} = \frac{1}{2}\int \cos\theta\,d\theta$$
$$= \frac{1}{2}\sin\theta + C.$$

Use the fact that

$$\sin\theta = \sqrt{\frac{1}{1+1/\tan^2\theta}}.$$

We get the above

$$= \frac{u}{2\sqrt{u^2+2}} + C.$$

Convert to the original x variable, we obtain

$$\frac{x+1}{2\sqrt{x^2+2x+3}} + C.$$

□

4.6 Improper Integrals

We know that the integrals represent signed areas of bounded regions. One question is can we compute the area of an unbounded regions? Is it possible that the area is bounded? For example, we can compute the area of regions below $f(x) = e^{-x}$ for $x \in [0, \infty)$. Then we need to compute the following:

$$\int_0^\infty e^{-x} dx.$$

We can do the following:

$$\int_0^\infty e^{-x} dx = \lim_{R\to\infty} \int_0^R e^{-x} dx = \lim_{R\to\infty} (-e^{-x})|_0^R = \lim_{R\to\infty} (-e^{-R} + 1) = 1.$$

The area is bounded!

This is an integration of a function in a unbounded domain, which belongs to **improper integrals**. We give a "proper" definition below.

Definition 4.13. Let a be a real number. The improper integral of f over $[a, \infty)$ is defined as the following limit (if it exists):

$$\int_a^\infty f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx.$$

We say that the improper integral **converges** if the limit exists and it **diverges** if the limit does not exist (including $= \pm\infty$).

Similarly we can define

$$\int_{-\infty}^a f(x)dx = \lim_{R\to-\infty} \int_R^a f(x)dx.$$

Example 4.6.1. *Show that* $\int_2^\infty \frac{dx}{x^{1.1}}$ *converges and compute its value. Show that* $\int_{100}^\infty \frac{dx}{x}$ *diverges.*

In the following theorem, we consider the integration with integrand $= x^{-p}$ in a unbounded region away from 0.

Theorem 4.14. *For* $a > 0$*,*

$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{p-1} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \le 1. \end{cases}$$

Example 4.6.2. *Determine if* $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$ *converges and, if so, compute its value.*

Notice that the anti-derivative of $\frac{1}{1+x^2}$ is $\arctan x$.

Example 4.6.3. *Use L'Hopital's rule to calculate* $\int_0^\infty xe^{-x} dx$.

Example 4.6.4. ***Escape Velocity.*** *The earth exerts a gravitational force of magnitude* $F(r) = GM_e m/r^2$ *on an object of mass* m *at distance* r *from the center of the earth.*

(a) Find the work required to move the object infinitely far from the earth.

(b) Calculate the escape velocity v_{esc} *on the earth's surface.*

Solution. To find out the work, recalling that $W = F \times d$, we do integration. Suppose the object start at the distance r_e to the center of the earth. To move it to R, we need force:

$$\int_{r_e}^{R} \frac{GM_e m}{r^2} dr = GM_e m(\frac{1}{r_e} - \frac{1}{R}).$$

Sending $R \to \infty$, we obtain $\frac{GM_e m}{r_e}$.

For the second question, we need some more knowledge of the physics. We need the velocity to be large enough such that the object has enough energy which should be at least equal to the "work" we found in part (a) (this is the so called the principle of conservation of energy). From physics we know that the kinetic energy equals $\frac{1}{2}mv^2$. So solving for

$$\frac{1}{2}mv_{\text{esc}}^2 = \frac{GM_e m}{r_e},$$

we get $\frac{GM_e m}{r_e} = \sqrt{2GM_e/r_e}$.

□

4.6.1 Unbounded Functions

An integral over a finite interval for a unbounded integrand is also **improper**.

Definition 4.15. If f is continuous on $[a, b)$ and $\lim_{x \to b^-} f(x) = \pm\infty$, we define (assuming the limit on the right-hand side exists)

$$\int_a^b f(x)dx = \lim_{R \to b^-} \int_a^R f(x)dx.$$

We say that the improper integral **converges** if the limit exists and that it **diverges** otherwise.

Example 4.6.5. *Calculate*

$$(a). \int_0^9 \frac{dx}{\sqrt{x}}, \quad (b). \int_0^{1/2} \frac{dx}{x}.$$

Example 4.6.6. *Calculate* $\int_0^9 \frac{dx}{(x-1)^{2/3}}$.

In the following theorem, we study the integration with integrand $= x^{-p}$ in a region containing 0.

Theorem 4.16. *For* $a > 0$,

$$\int_0^a \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{1-p} & \text{if } p < 1, \\ \text{diverges} & \text{if } p \geq 1. \end{cases}$$

4.6.2 Comparing Integrals

Sometimes we need to determining whether an improper integral converges or not, without finding its exact value. Then we can do comparison.

Theorem 4.17. *Assume for $x > a$, $f(x) \geq g(x) \geq 0$. We have*

if $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ also converges;

if $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty g(x)dx$ also diverges.

The comparison test is also valid for improper integrals of unbounded functions.

Example 4.6.7. *Show that $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converges.*

Proof. Let us use the comparison test. To show convergence, we need to construct a simpler and larger function. Looking at the denominator, x^3 is the main ingredient comparing to 1 when x is large. So consider

$$\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}} = x^{-3/2}.$$

Notice

$$\int_1^\infty x^{-3/2}dx \quad \text{converges}$$

and therefore $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ converges by the comparison test. □

Example 4.6.8. *Determine whether the following integral converges or not:*

$$\int_1^\infty \frac{dx}{\sqrt{x}+e^{3x}}, \quad \int_0^2 \frac{dx}{x^8+x^2}.$$

Solution. When x is large $\frac{1}{\sqrt{x}+e^{3x}}$ behaves like $\frac{1}{e^{3x}}$ and the integration from 1 to ∞ for the latter function converges:

$$\int_1^\infty \frac{dx}{e^{3x}} = \frac{1}{3}e^{-3x}.$$

Since

$$\frac{1}{\sqrt{x}+e^{3x}} \leq \frac{1}{e^{3x}},$$

comparison test yields the convergence of the first integral.

For the second one, the integral is improper near $x = 0$. When $x = 0$,

$$x^8 + x^2 \quad \text{is comparable to} \quad x^2.$$

Notice

$$\int_0^1 \frac{dx}{x^2} \quad \text{diverges.}$$

To do comparison, we observe that when $x \leq 1$

$$x^8 + x^2 \leq 2x^2.$$

Since $\int_0^1 1/(2x^2)dx$ diverges,

$$\int_0^2 \frac{dx}{x^8 + x^2} \geq \int_0^1 \frac{dx}{x^8 + x^2} \geq \int_0^1 \frac{dx}{2x^2} \quad \text{diverges.}$$

□

5 Infinite Series

5.1 Sequences

By sequence we just mean a sequence of numbers, denoted as $\{a_n\} = \{a_1, a_2, ...\}$ (sometimes people also start with $n = 0$: $\{a_0, a_1, a_2, ...\}$).

First let me introduce to you the well-known **Fibonacci Sequence**:

- We can define the sequence iteratively (recursive sequence) by taking $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.

Given the first two terms, we can easily find out the first 10 terms one by one:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

If we continue this process, we are able to find as many terms as we want. But this is not convenient. For example, one may ask can we find F_{1000} without computing all 999 terms ahead? Actually we can! There exists a function f such that $F_n = f(n)$.

The Fibonacci sequence appears in a surprisingly wide variety of situations, particularly in nature. For instance, the number of spiral arms in a sunflower almost always turns out to be a number from the Fibonacci sequence.

Another Recursive Sequence. Compute the two terms a_2, a_3 for the sequence defined recursively by

$$a_1 = 1, a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right).$$

Solution.

$$a_2 = \frac{1}{2}(1 + 2/1) = \frac{3}{2},$$
$$a_3 = \frac{1}{2}(\frac{3}{2} + \frac{2}{3/2}) = \frac{17}{12}.$$

□

One special class of sequence is the so-called converging sequence. We say that a sequence $\{a_n\}$ converges to a limit L, if $|a_n - L|$ becomes arbitrarily small when n is sufficiently large, and if so we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

Definition 5.1. (not required) We say $\{a_n\}$ converges to a limit L and write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L$$

if, for every $\epsilon > 0$, there is a number M such that $|a_n - L| < \epsilon$ for all $n > M$.

- If no limit exists, we say that $\{a_n\}$ diverges.
- If the terms increase without a bound (or decrease without a bound), we say that $\{a_n\}$ diverges to infinity (or $-\infty$).

Example 5.1.1. *The Fibonacci sequence increases without a bound, and so the sequence diverges.*
The sequence $a_n := \cos(\frac{n\pi}{2})$ has no limit because

$$\{a_n\} = \{1, 0, -1, 0, 1, ...\}.$$

and $1, -1, ...$ do not converge.

5.1.1 Prove convergence

Example 5.1.2. *Let $a_n = \frac{n+4}{n+3}$. Prove that $\lim_{n\to\infty} a_n = 1$.*

Proof. By definition, we need to find, for every $\epsilon > 0$, a number M (which depends on ϵ) such that

$$|a_n - 1| \le \epsilon \quad \text{for } n \ge M.$$

We have

$$a_n - 1 = \frac{1}{n+3}$$

which can be arbitrarily small when n is large ($\lim_{n\to\infty}(a_n - 1) = 0$)). Indeed for every $\epsilon > 0$, when $n \ge \frac{1}{\epsilon}$,

$$|a_n - 1| = \frac{1}{n+3} \le \epsilon.$$

This proves the convergence. □

For some sequence $\{a_n\}$, we can understand it as a sequence defined by a function:

$$a_n = f(n).$$

Then we have the following criteria about convergence of sequences.

Theorem 5.2. *If* $\lim_{x\to\infty} f(x)$ *exists and equals* L*. Then the sequence* $a_n = f(n)$ *converges to* L*.*

Can you apply this theorem to Example 5.1.2 with $f(x) = \frac{x+4}{x+3}$? Do you have $\lim_{x\to\infty} f(x) = 1$?

Example 5.1.3. *Calculate the limit of the sequence* $a_n = \frac{n+\ln n}{n^2}$*.*

Solution. Consider the following function

$$f(x) = \frac{x + \ln x}{x^2}.$$

By L'Hopital's rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x + \ln x}{x^2} = \lim_{x\to\infty} \frac{1 + 1/x}{2x} = 0.$$

Therefore the sequence $a_n \to 0$ as $n \to \infty$.

□

5.1.2 Geometric sequence

A **geometric sequence** is a sequence of the form

$$a_n = cr^n.$$

The number r is called the **common ratio**.

Example 5.1.4. *Prove that for* $r \geq 0, c > 0$*,*

$$\lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if } r \in [0,1), \\ c & \text{if } r = 1, \\ \infty & \text{if } r \in (1,\infty). \end{cases}$$

The proof follows by considering the function $f(x) = cr^x$.

Theorem 5.3. ***[Squeeze Theorem].*** *Let* $\{a_n\}, \{b_n\}, \{c_n\}$ *be sequences such that for some M,*

$$a_n \leq b_n \leq c_n \quad \text{for } n > M$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then

$$\lim_{n\to\infty} b_n = L.$$

Example 5.1.5. *Suppose $c \neq 0$. Prove*

$$\lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if } r \in (-1,0), \\ \text{diverges} & \text{if } r \leq -1. \end{cases}$$

Proof. Notice

$$-c|r|^n \leq cr^n \leq c|r|^n.$$

It follows from example 5.1.4, when $r \in (-1,0)$, which implies $|r| < 1$, we have both $\{-c|r|^n\}$ and $\{c|r|^n\}$ converge to 0. Hence $cr^n \to 0$.

When $r = -1$, the sequence is an alternating sequence with values $\{c, -c, c, -c...\}$ which is diverging.

When $r < -1$, the sequence also diverges since $cr^{2n} \to \infty$ and $cr^{2n+1} \to -\infty$. □

Example 5.1.6. *Prove that $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for any R.*

Proof. I will only consider the case when $R > 0$. (Think about what you can say if $R \leq 0$.) Let M be an integer such that

$$M \leq R < M + 1.$$

We have

$$\frac{R^n}{n!} = \left(\frac{R}{1}\frac{R}{2}\cdots\frac{R}{M}\right)\left(\frac{R}{M+1}\frac{R}{M+2}\cdots\frac{R}{n}\right).$$

Let us call the value in the first bracket C and then we know $C \leq R^M$ which is finite, and it is independent of n. Then we obtain that

$$0 \leq \frac{R^n}{n!} \leq C\left(\frac{R}{M+1}\frac{R}{M+2}\cdots\frac{R}{n}\right) \leq C\frac{R}{n}$$

and the latter $\to 0$ as $n \to \infty$. By Squeeze theorem, $\frac{R^n}{n!} \to 0$. □

Given a converging sequence $\{a_n\}$ and a function f, we can form the new sequence $\{f(a_n)\}$. When f is continuous, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n).$$

Example 5.1.7. *For example given a sequence $a_n = \frac{3n}{n+e^{-n}}$ which converges to 3. Then for any continuous function f, we have $f(a_n) \to f(3)$. For example, if $f = x^2$, then $f(a_n) = (\frac{3n}{n+e^{-n}})^2$ converges to 9; if $f = \ln x$, then $f(a_n) \to \ln 3$.*

5.1.3 Bounded sequence and Monotonic sequence (not required)

Next, we define the concepts of a bounded sequence and a monotonic sequence, concepts of great importance for understanding convergence.

Definition 5.4. A sequence $\{a_n\}$ is

- Bounded from above if there is a number M such that $a_n \leq M$ for all n. The number M is called an upper bound.
- Bounded from below if there is a number m such that $a_n \geq m$ for all n. The number m is called a lower bound.

The sequence $\{a_n\}$ is called bounded if it is bounded from above and below. A sequence that is not bounded is called an unbounded sequence.

Theorem 5.5. *Convergent sequences are bounded sequences.*

Divergent sequences can be both bounded or unbounded. Can you give both the examples?

Definition 5.6. $\{a_n\}$ is monotonic if either $a_n \leq a_{n+1}$ for all n or $a_n \geq a_{n+1}$. We call

- $\{a_n\}$ is (strictly) increasing if $a_n < a_{n+1}$ for all n,
- $\{a_n\}$ is (strictly) decreasing if $a_n > a_{n+1}$ for all n,
- $\{a_n\}$ is non-decreasing if $a_n \leq a_{n+1}$ for all n,
- $\{a_n\}$ is non-increasing if $a_n \geq a_{n+1}$ for all n.

Theorem 5.7. *Bounded Monotonic Sequences Converge.*

If $\{a_n\}$ is increasing and $a_n \leq M$, then $\{a_n\}$ converges and the limit $\leq M$.

If $\{a_n\}$ is decreasing and $a_n \geq m$, then $\{a_n\}$ converges and the limit $\geq m$.

Example 5.1.8. *Verify that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded from below. What is the limit of a_n?*

Solution. Let us consider the function $f(x) = \sqrt{x+1} - \sqrt{x}$. The function is decreasing because $f'(x) < 0$ for all $x > 0$. So $a_n = f(n)$ is a decreasing sequence. It is not hard to see that the sequence is bounded below by 0 i.e. $a_n \geq 0$ for all n. Therefore by the theorem, the sequence has a limit.

Notice

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Therefore $a_n \to 0$. □

Example 5.1.9. *Show that the following sequence is bounded and increasing:*

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, ...$$

Prove the limit $\lim_{n\to\infty} a_n$ exits and find out the value.

Solution. **Step 1. bounded from above.** We certainly have that $a_1 < 2$. Suppose $a_k < 2$. Notice

$$a_{k+1} = \sqrt{2\sqrt{a_k}},$$

then

$$a_{k=1} < \sqrt{2\sqrt{2}} \leq 2.$$

Thus the sequence is bounded from above by 2.

Step 2. Increasing.
Since a_n is positive and $a_n \leq 2$, then

$$a_{n+1} = \sqrt{2a_n} < \sqrt{a_n \times a_n} = a_n$$

which implies that the sequence is increasing. From the monotone convergence theorem, a_n converges and let us suppose the limit equals L.

Step 3. Find L.
Since $a_{n+1} = \sqrt{2a_n}$, by passing $n \to \infty$, we get

$$\lim_{n\to\infty} a_{n+1} = \sqrt{2 \lim_{n\to\infty} a_n}$$

which implies that

$$L = \sqrt{2L}.$$

Thus L can only be 0 or 2. We eliminate $L = 0$, because the terms a_n are positive and they all $\geq a_1 = \sqrt{2} > 0$. We must have $L = 2$. □

Example 5.1.10. *(Very hard!) The Fibonacci sequence $\{F_n\}$ diverges since it is unbounded. Please show the sequence defined by the ratios $a_n = \frac{F_{n+1}}{F_n}$ converges. The limit is known as the golden ratio.*

Proof. (Hint) Since $F_{n+2} = F_{n+1} + F_n$, dividing by F_{n+1}, we get

$$a_{n+1} = 1 + \frac{1}{a_n}. \tag{4}$$

Known

$$a_2 = F_3/F_2 = 2 \geq \frac{1+\sqrt{5}}{2} =: c.$$

It can be shown that for all $n \geq 2$, $a_n \geq c$. Why?

Next it can be shown that a_n is decreasing. Why? Therefore a_n converges. Suppose the limit is L.

Passing $n \to \infty$ on both sides of (4), we get

$$L = 1 + \frac{1}{L}$$

this tells that $L = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$. We pick $L = \frac{1+\sqrt{5}}{2} = c$, because $a_n \geq c$ for all n. □

5.2 Infinite Series

Given a sequence $\{a_n\}$, in this section we study the sum of a_n:

$$a_1 + a_2 + a_3 + ... + a_n + ..$$

This infinite sums is also called **infinite series**. Clearly if $\{a_n\}$ is a converging sequence, then the infinite sum equals to $\pm\infty$ as long as the limit of a_n is not 0.

In order to compute the infinite sum, we try to make it a "finite sum". Therefore we define the **partial sums**:

$$\begin{aligned} &S_1 = a_1, \\ &S_2 = a_1 + a_2, \quad ... \\ &S_n = a_1 + a_2 + ... + a_n, \quad ... \end{aligned}$$

We use the following notations:

$$\sum_{n=1}^{N} a_n = a_1 + a_2 + ... + a_N, \quad \sum_{n=1}^{\infty} a_n = a_1 + a_2 + ...$$

Definition 5.8. An infinite series $\sum_{n=1}^{\infty} a_n$ converges to S if the sequence of its partial sums $\{S_n\}$ converges to S.

If the limit does not exist or it is $\pm\infty$, we say that the infinite series diverges.

Example 5.2.1. *Compute*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Solution. Notice

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{(n+1)} \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

which converges to 1 as $N \to \infty$. So the series equals 1. □

Theorem 5.9. ***[Geometric series]*** *For the geometric series $\sum_{n=0}^{\infty} cr^n$ with $r \neq 1$,*

$$S_N = c + cr + ... + cr^n = \frac{c(1 - r^{N+1})}{1 - r}.$$

If $|r| < 1$, then

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1 - r}.$$

Example 5.2.2. *Evaluate $\sum_{n=0}^{\infty} \frac{2+3^n}{5^n}$.*

Solution.

$$\sum_{n=0}^{\infty} \frac{2 + 3^n}{5^n} = \sum_{n=0}^{\infty} \frac{2}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^n}$$
$$= 2\frac{1}{1 - (1/5)} + \frac{1}{1 - (3/5)} = 5.$$

□

Theorem 5.10. ***[nth Term divergence test]*** *If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=k}^{\infty} a_n$ diverges for any k.*

While if $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=k}^{\infty} a_n$ can be both divergent or convergent.

Example 5.2.3. *Prove the divergent of $\sum_{n=1}^{\infty}(-1)^n \frac{n}{4n+1}$.*

Solution. Notice that $a_n = (-1)^n \frac{n}{4n+1}$ does not approach a limit. The series diverges by the test. □

Example 5.2.4. *Prove the divergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.*

Solution. Notice the partial sums

$$S_N = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{N}}.$$

We have N terms in S_N and each one $\geq \frac{1}{\sqrt{N}}$. Therefore

$$S_N \geq N \times \frac{1}{\sqrt{N}} = \sqrt{N}$$

which $\to \infty$ as $N \to \infty$. The series diverges.

□

5.3 Convergence of Series

In this section we consider non-negative series $\sum_{n=1}^{\infty} a_n$ where each $a_n \geq 0$. For such a series, we immediately know that the corresponding partial sums S_n has to be non-decreasing in n.

Theorem 5.11. *Given a positive series $\sum_{n=1}^{\infty} a_n$, then either*

(i) The partial sums S_N are bounded from above and then $\sum_{n=1}^{\infty} a_n$ converges. Or

(ii) The partial sums S_N are unbounded and then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 5.12. ***[Integral Test]*** *Suppose $a_n = f(n) \geq 0$ for some non-increasing, continuous function $f(x), x \geq 1$.*

(i) If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Think about if given an alternating series, can we have such an integral test? Why or why not?

Example 5.3.1. *Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Solution. Consider $f(x) = \frac{1}{x}$ and then $a_n = f(n)$. Since the integral of $\int_1^{\infty} f(x)dx$ diverges, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges. □

Example 5.3.2. *The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.*

Example 5.3.3. *Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converge?*

Solution. Let us use the integral test for $f(x) = \frac{1}{x(\ln x)^2}$. The substitution $u = \ln x$ yields that

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln x)^2}dx &= \int_{\ln 2}^{\infty} \frac{du}{u} \quad (\text{ here we do the integration from } 2 \text{ to } \infty\ ) \\ &= \lim_{R\to\infty} \left(\frac{1}{\ln 2} - \frac{1}{R}\right) \\ &= \frac{1}{\ln 2} < \infty.\end{aligned}$$

The integral test shows that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges. □

Next we introduce the comparison test. This test is, in the spirit, very similar to the comparing techniques we used to determine an improper integral is convergent or not, or the squeeze theorem we used to compute the limit of a sequence.

Theorem 5.13 (Comparison test). *Assume for some $N > 0$, $b_n \geq a_n \geq 0$ holds for all $n \geq N$. Then*

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Theorem 5.14 (Limit Comparison test). *Let a_n, b_n be non-negative sequences. Assume that there exist m, M such that*

$$m \leq \liminf_{n\to\infty} \frac{a_n}{b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n} \leq M.$$

Here if the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then

$$\liminf_{n\to\infty} \frac{a_n}{b_n} = \limsup_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

(i) If $\infty > M > m > 0$, then $\sum_{n=1}^{\infty} a_n$ converges, if and only if, $\sum_{n=1}^{\infty} b_n$ converges.

(ii) If $m = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

(iii) If $M = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then in the above we can select

$$m = M = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Example 5.3.4. *Show that $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ converges.*

Solution. Let us set

$$a_n = \frac{n^2}{n^4 - n - 1} \qquad \text{and } b_n = \frac{1}{n^2}.$$

Notice

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4}{n^4 - n - 1} = 1.$$

Since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, therefore by the Limit comparison test, $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ also converges. □

Example 5.3.5. *Does $\sum_{n=2}^{\infty} \frac{1}{(n^2-3)^{1/3}}$ converge?*

Solution. Notice

$$\lim_{n\to\infty} \left(\frac{1}{(n^2-3)^{1/3}}\right) \Big/ \left(\frac{1}{n^{2/3}}\right) = \lim_{n\to\infty} \left(\frac{1}{(1-3n^{-2})^{1/3}}\right) = 1.$$

By the limit comparison test, the divergence of $\sum_{n=2}^{\infty} \frac{1}{n^{2/3}}$ yields the divergence of $\sum_{n=2}^{\infty} \frac{1}{(n^2-3)^{1/3}}$. □

Review for Midterm 2

Complex numbers and trig integration

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$$e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2}(1+i), \quad e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i...$$

We can use this formula to compute trig integration: Example 5.3.1, Example 5.3.2. The idea is to use

$$\cos(cx) = \frac{e^{cix} + e^{-cix}}{2}, \quad \sin(cx) = \frac{e^{cix} - e^{-cix}}{2i}.$$

Substitution is another frequently used method. Recall Example 5.3.5: Evaluate $\int \tan^3 x\, dx$.
We are going to use both substitution method and partial fractions.

Solution. For (1). consider $u = \sin x$. Then

$$\begin{aligned}\int \tan^3 x\, dx &= \int \frac{\sin^3 x}{\cos^3 x}\frac{1}{\cos x} d\sin x \\ &= \int \frac{\sin^3 x}{(1-\sin^2 x)^2} d\sin x \\ &= \int \frac{u^3}{(1-u^2)^2} du.\end{aligned}$$

Then we apply the Partial fraction method (in Section 5.4) to get

$$\frac{u^3}{(1-u^2)^2} = \frac{u^3}{(u-1)^2(u+1)^2} = \frac{1}{2\,(u+1)} - \frac{1}{4\,(u+1)^2} + \frac{1}{2\,(u-1)} + \frac{1}{4\,(u-1)^2}.$$

After integration, we get

$$\begin{aligned}&\frac{1}{2}\ln|(u-1)(u+1)| + \frac{1}{4}\frac{1}{u+1} - \frac{1}{4}\frac{1}{u-1} + C \\ =& \frac{1}{2}\ln|1-u^2| + \frac{1}{2}\frac{1}{1-u^2} + C \\ =& -\ln|\sec x| + \frac{1}{2}\sec^2 x + C.\end{aligned}$$

□

The last technique in this part is the trig substitution: Recall

$\sqrt{a^2 - x^2}$, try $x = a\sin\theta$ and then $dx = a\cos\theta d\theta$, $\sqrt{a^2 - x^2} = a\cos\theta$;
$\sqrt{a^2 + x^2}$, try $x = a\tan\theta$ and then $dx = a(\cos\theta)^{-2} d\theta$, $\sqrt{a^2 + x^2} = a(\cos\theta)^{-1}$;
$\sqrt{x^2 - a^2}$, try $x = a(\cos\theta)^{-1}$ and then $dx = a\sin\theta(\cos\theta)^{-2} d\theta$, $\sqrt{a^2 - x^2} = a\tan\theta$.

Sometimes the integrand is not exactly of the above forms, we might need to do substitution.

Example 5.3.6. *Evaluate*

$$\int \frac{x^2dx}{x^2+2x+3}.$$

Solution. Since

$$x^2+2x+3=(x+1)^2+2,$$

let $u=x+1$ and we get

$$\begin{aligned}\int \frac{x^2dx}{x^2+2x+3} &= \int \frac{(u-1)^2}{u^2+2}du \\ &= \int \frac{u^2-2u+1}{u^2+2}du \\ &= \int \frac{u^2+2}{u^2+2}du - \int \frac{2u}{u^2+2}du - \int \frac{1}{u^2+2}du \\ &=: A_1+A_2+A_3.\end{aligned}$$

Then $A_1=u+C$. For A_2, notice $2udu=du^2$ and thus we use substitution $w:=u^2$. For A_3, we use trig substitution

$$u=\sqrt{2}\tan\theta.$$

Please finish the computations. :) □

Improper Integration

Example 5.3.7. *Compute* $\int_1^\infty \frac{1}{2+x^2}dx$.

Solution. Since

$$\frac{1}{2+x^2}\le\frac{1}{x^2}$$

and

$$\int_1^\infty \frac{1}{x^2}<\infty,$$

So the original improper integral converges.

Actually we can use trig substitution to find its value: let $x=\sqrt{2}\tan\theta$ and we have

$$\begin{aligned}\int_1^\infty \frac{1}{2+x^2}dx &= \int_{\arctan(1/\sqrt{2})}^{\pi/2} \frac{\cos^2\theta\, d\tan\theta}{2} \\ &= \int_{\arctan(1/\sqrt{2})}^{\pi/2} \frac{d\theta}{2} \\ &= \frac{\pi}{4}-\frac{1}{2}\arctan(1/\sqrt{2})\end{aligned}$$

□

Let us solve Example 5.6.3: calculate $\int_0^\infty xe^{-x}dx$.

Solution. Use integration by parts we get

$$\int_0^R xe^{-x}dx = (-xe^{-x})|_0^R - \int_0^R e^{-x}dx = 1 - \frac{R+1}{e^R}.$$

Take $R \to \infty$ and apply the L'Hopital's rule,

$$\int_0^\infty xe^{-x}dx = 1 - \lim_{R\to\infty} \frac{R+1}{e^R} = 1.$$

□

Sequence and series

Example 5.3.8. *Determine the limit*

$$a_n = \ln\left(\frac{12n+2}{-9+4n}\right).$$

The answer is $\ln 3$.

Recall Series: $\sum_1^\infty a_n$.
Power series: for $|r| < 1$,

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + ... = \frac{c}{1-r}$$

We can use functions to determine whether a positive series is convergent. f needs to be positive, decreasing and continuous. See Example 6.3.1- Example 6.3.3

Another method is the comparison test and limit comparison test.

Example 5.3.9. *Does* $\sum_{n=1}^\infty \frac{1}{\sqrt{n}3^n}$ *converge?*

Solution. For $n \geq 1$,

$$\frac{1}{\sqrt{n}3^n} \leq \frac{1}{3^n}.$$

The larger series $\sum_{n=1}^\infty \frac{1}{3^n}$ converges and therefore the original one converges. □

5.4 Absolute and Conditional Convergence

In the previous section, we mainly considered positive series. Now let us study the general series.

Definition 5.15. *[Absolute Convergence]* The series $\sum_{n=1}^\infty a_n$ converges absolutely if $\sum_{n=1}^\infty |a_n|$ converges.

Theorem 5.16. ***[Absolute Convergence Implies Convergence]*** *If* $\sum_{n=1}^\infty |a_n|$ *converges,* $\sum_{n=1}^\infty a_n$ *also converges.*

Example 5.4.1. *Verify that* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ *converges.*

Solution. Since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is a convergent series, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent. By the theorem, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is convergent. □

Definition 5.17. *[Conditional Convergence]* An infinite series $\sum_{n=1}^{\infty} a_n$ converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example 5.4.2. *Show* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges conditionally.*

Proof. It is not hard to see that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. To prove the claim, we only need to show the partial sum of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. Note

$$\begin{aligned} S_{2N} &= \sum_{n=1}^{2N} \frac{(-1)^n}{n} \\ &= -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots - \frac{1}{2N-1} + \frac{1}{2N} \\ &= -(\frac{1}{1} - \frac{1}{2}) - (\frac{1}{3} - \frac{1}{4}) - \dots - (\frac{1}{2N-1} - \frac{1}{2N}) \\ &= -\sum_{i=1}^{N} (\frac{1}{2i-1} - \frac{1}{2i}). \end{aligned}$$

So $\{S_{2N}\}$ (as a sequence) is a decreasing sequence. Since

$$\left| \frac{1}{2i-1} - \frac{1}{2i} \right| = \left| \frac{1}{(2i-1)(2i)} \right| \le \frac{1}{(2i-1)^2},$$

triangle inequality implies that

$$|S_{2N}| \le \sum_{i=1}^{N} \frac{1}{(2i-1)^2} \le \frac{\pi^2}{6}.$$

It follows from Theorem 5.7 that $\{S_{2N}\}$ has a limit. Finally since the difference of S_{2N+1} and S_{2N} converges to 0, S_n has the same limit as $\{S_{2N}\}$.

The convergence of the partial sum implies that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. □

In this example, we see that the numbers in the series is aligned as one positive number followed by a negative one, and then positive again. We will call it an alternating series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

where b_n are positive. For such series, we have the following theorem.

Theorem 5.18. ***[Alternating Series Test]*** *Assume that $\{b_n\}$ is a positive sequence that is decreasing and converges to 0. Then*

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n \quad \textit{converges.}$$

The proof is given by estimating the difference of partial sums: $|S_{n_2} - S_{n_1}|$ for some $n_2 > n_1 >> 1$. If $|S_{n_2} - S_{n_1}|$ can be arbitrarily small as $(n_2 >)\ n_1$ becomes large, then the partial sums $\{S_n\}$ converges which implies the convergence of the alternating series. (The proof is not required. You are required to be able to apply the theorem which includes checking the assumptions, and supply a complete justification.)

Example 5.4.3. *Show that*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$$

converges conditionally. Furthermore if S is the sum of the series, then $0 < S < 1$.

Proof. It is direct to see that $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{\sqrt{n}}\right|$ diverges.

Next the terms $b_n = \frac{(-1)^{n-1}}{\sqrt{n}}$ are positive and decreasing, and $\lim_{n\to\infty} b_n = 0$. By the Alternating series test, the series converges conditionally. Since

$$S = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}} = b_1 - \sum_{n=1}^{\infty}(b_{2n} - b_{2n+1}) < b_1,$$

we get $S < 1$. Also

$$S = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}} = \sum_{n=1}^{\infty}(b_{2n-1} - b_{2n}) > 0.$$

□

Remark 5.19. In general for alternating series converging to S with partial sums S_N, we have

$$S_p < S < S_q$$

for any even p and odd q.

Also if $b_n > 0$ for all n, we have

$$|S - S_N| < b_{N+1}.$$

Can you prove these two claims?

Example 5.4.4. *Consider a conditionally convergent sequence* $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ *(why?). If* S *represent the sum, show that*

$$|S-S_6|<\frac{1}{7}.$$

Next find an N *such that* S_N *approximates* S *with error less than* 10^{-2}.

Solution. By the remark

$$|S-S_6|<b_7=\frac{1}{7}.$$

Also we have

$$|S-S_N|<b_{N+1}=\frac{1}{N+1}.$$

To have error smaller than $\frac{1}{100}$, we need $N\geq 99$. □

Review for Midterm 2

Complex numbers and trig integration

$$e^{i\theta}=\cos\theta+i\sin\theta.$$

$$e^{i\frac{\pi}{4}}=\frac{\sqrt{2}}{2}(1+i),\quad e^{i\frac{\pi}{3}}=\frac{1}{2}+\frac{\sqrt{3}}{2}i,\quad e^{i\frac{\pi}{6}}=\frac{\sqrt{3}}{2}+\frac{1}{2}i...$$

We can use this formula to compute trig integration: Example 5.3.1, Example 5.3.2. The idea is to use

$$\cos(cx)=\frac{e^{cix}+e^{-cix}}{2},\quad \sin(cx)=\frac{e^{cix}-e^{-cix}}{2i}.$$

Substitution is another frequently used method. Recall Example 5.3.5: Evaluate $\int\tan^3 x\,dx$. We are going to use both substitution method and partial fractions.

Solution. For (1). consider $u=\sin x$. Then

$$\begin{aligned}\int\tan^3 x\,dx&=\int\frac{\sin^3 x}{\cos^3 x}\frac{1}{\cos x}d\sin x\\&=\int\frac{\sin^3 x}{(1-\sin^2 x)^2}d\sin x\\&=\int\frac{u^3}{(1-u^2)^2}du.\end{aligned}$$

Then we apply the Partial fraction method (in Section 5.4) to get

$$\frac{u^3}{(1-u^2)^2}=\frac{u^3}{(u-1)^2(u+1)^2}=\frac{1}{2\,(u+1)}-\frac{1}{4\,(u+1)^2}+\frac{1}{2\,(u-1)}+\frac{1}{4\,(u-1)^2}.$$

After integration, we get

$$\begin{aligned}
&\frac{1}{2}\ln|(u-1)(u+1)| + \frac{1}{4}\frac{1}{u+1} - \frac{1}{4}\frac{1}{u-1} + C \\
=& \frac{1}{2}\ln|1-u^2| + \frac{1}{2}\frac{1}{1-u^2} + C \\
=& -\ln|\sec x| + \frac{1}{2}\sec^2 x + C.
\end{aligned}$$

□

The last technique in this part is the trig substitution: Recall

$\sqrt{a^2-x^2}$, try $x = a\sin\theta$ and then $dx = a\cos\theta d\theta$, $\sqrt{a^2-x^2} = a\cos\theta$;
$\sqrt{a^2+x^2}$, try $x = a\tan\theta$ and then $dx = a(\cos\theta)^{-2}d\theta$, $\sqrt{a^2+x^2} = a(\cos\theta)^{-1}$;
$\sqrt{x^2-a^2}$, try $x = a(\cos\theta)^{-1}$ and then $dx = a\sin\theta(\cos\theta)^{-2}d\theta$, $\sqrt{a^2-x^2} = a\tan\theta$.

Sometimes the integrand is not exactly of the above forms, we might need to do substitution.

Example 5.4.5. *Evaluate*

$$\int \frac{x^2dx}{x^2+2x+3}.$$

Solution. Since

$$x^2+2x+3 = (x+1)^2+2,$$

let $u = x+1$ and we get

$$\begin{aligned}
\int \frac{x^2dx}{x^2+2x+3} &= \int \frac{(u-1)^2}{u^2+2}du \\
&= \int \frac{u^2-2u+1}{u^2+2}du \\
&= \int \frac{u^2+2}{u^2+2}du - \int \frac{2u}{u^2+2}du - \int \frac{1}{u^2+2}du \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

Then $A_1 = u + C$. For A_2, notice $2udu = du^2$ and thus we use substitution $w := u^2$. For A_3, we use trig substitution

$$u = \sqrt{2}\tan\theta.$$

Please finish the computations. :) □

Improper Integration

Example 5.4.6. *Compute* $\int_1^\infty \frac{1}{2+x^2}dx$.

Solution. Since

$$\frac{1}{2+x^2} \leq \frac{1}{x^2}$$

and

$$\int_1^\infty \frac{1}{x^2} < \infty,$$

So the original improper integral converges.

Actually we can use trig substitution to find its value: let $x = \sqrt{2}\tan\theta$ and we have

$$\begin{aligned}\int_1^\infty \frac{1}{2+x^2} dx &= \int_{\arctan(1/\sqrt{2})}^{\pi/2} \frac{\cos^2\theta \, d\tan\theta}{2} \\ &= \int_{\arctan(1/\sqrt{2})}^{\pi/2} \frac{d\theta}{2} \\ &= \frac{\pi}{4} - \frac{1}{2}\arctan(1/\sqrt{2})\end{aligned}$$

□

Let us solve Example 5.6.3: calculate $\int_0^\infty xe^{-x} dx$.

Solution. Use integration by parts we get

$$\int_0^R xe^{-x} dx = (-xe^{-x})|_0^R - \int_0^R e^{-x} dx = 1 - \frac{R+1}{e^R}.$$

Take $R \to \infty$ and apply the L'Hopital's rule,

$$\int_0^\infty xe^{-x} dx = 1 - \lim_{R\to\infty} \frac{R+1}{e^R} = 1.$$

□

Sequence and series

Example 5.4.7. *Determine the limit*

$$a_n = \ln\left(\frac{12n+2}{-9+4n}\right).$$

The answer is $\ln 3$.

Recall Series: $\sum_1^\infty a_n$.

Power series: for $|r| < 1$,

$$\sum_{n=0}^\infty cr^n = c + cr + cr^2 + cr^3 + \ldots = \frac{c}{1-r}$$

We can use functions to determine whether a positive series is convergent. f needs to be positive, decreasing and continuous. See Example 6.3.1- Example 6.3.3

Another method is the comparison test and limit comparison test.

Example 5.4.8. *Does* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ *converge?*

Solution. For $n \geq 1$,

$$\frac{1}{\sqrt{n}3^n} \leq \frac{1}{3^n}.$$

The larger series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges and therefore the original one converges. □

5.5 Ratio and Root Tests

Let us present two tests concerning the problem whether a series converges or diverges.

Theorem 5.20. ***[Ratio Test]*** *Assume that the following limit exists:*

$$\rho := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $\rho < 1$*, the the series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely. If* $\rho > 1$*, then* $\sum_{k=1}^{\infty} a_k$ *diverges. If* $\rho = 1$*, the test is inconclusive.*

Remark 5.21. Let us compare the test with the geometric series:

$$a + ar + ar^2 + ar^3 + ...$$

which converges if $|r| < 1$ and diverges if $|r| > 1$.

Example 5.5.1. *Prove that* $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}$ *converges.*

Example 5.5.2. *Prove that* $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ *converges.*

Example 5.5.3. *Show* $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{10^n}$ *diverges.*

Theorem 5.22. ***[Root Test]*** *Assume that the following limit exists:* $L := \lim_{n\to\infty} \sqrt[n]{|a_n|}$*. If* $L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely. If* $L > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Example 5.5.4. *Show that* $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ *converges.*

Example 5.5.5. *Show* $\sum_{n=1}^{\infty} \frac{10^n}{n^{\sqrt{n}}}$ *diverges.*

Solution. Let us use the root test:

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{10^n}{n^{\sqrt{n}}}\right)^{1/n} = \lim_{n\to\infty} \frac{10}{n^{\frac{1}{\sqrt{n}}}}.$$

Notice by L'Hopital's rule

$$\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}} = e^{\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}} = e^{\lim_{n\to\infty} \frac{1/n}{1/(2\sqrt{n})}} = e^{\lim_{n\to\infty} \frac{2}{\sqrt{n}}} = 1.$$

Therefore

$$\lim_{n\to\infty} |a_n|^{1/n} = 10$$

which implies that the series diverges. □

5.6 Power Series

By power series we mean

$$a_0 + a_1 r + a_2 r^2 + ...$$

If we vary the "r", we get a polynomial of infinite degree:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ...$$

More generally, we consider a power series near point x_0:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ...$$

These are functions of x.

Shifting the Summation index.

Example 5.6.1. *Express the series*

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

as a series where the generic term is x^k.

Solution. Set $k = n - 2$. Then

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k.$$

(This is like doing substitution in the summation index.) □

We say the series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ...$$

converges at $x = c$ if $\sum_{n=0}^{\infty} a_n (c - x_0)^n$ converges. If the limit does not exist, we say the series **diverges** at $x = c$. Moreover if

$$\sum_{n=0}^{\infty} |a_n (c - x_0)^n|$$

converges, we say the series **converges absolutely** at point $x = c$.

As before, for series we are interested in the its convergence property. So for power series, we want to find the points at which the series converges. There is a surprisingly simple characterization of the structure of those points.

Theorem 5.23. ***[Radius of convergence]*** *The radius of convergence r is a nonnegative real number or ∞ such that the series converges if $|x - x_0| < r$, and diverges if $|x - x_0| > r$.*

Let us generalize the previous "root test" and "ratio test" (for series) to the ones for power series. The radius of convergence r can be derived through the root test or the ratio test:

Root test. $r = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{a_n}}$,

Ratio test. when the following limit exists, it satisfies, $r = \lim\limits_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right|$.

Example 5.6.2. *Determine the converge set of*

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n+1} x^n.$$

Solution. By the ratio test, $r = \frac{1}{2}$. Let us check the endpoints. When $x = \frac{1}{2}$, we get an alternating series, which converges. When $x = -\frac{1}{2}$, the series diverges. Thus the series converges in $(-\frac{1}{2}, \frac{1}{2}]$. □

Sum of two Power Series.
Given two power series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

Product of two Power Series.

$$\begin{aligned} f(x)g(x) &= (\sum_{n=0}^{\infty} a_n x^n) \times (\sum_{n=0}^{\infty} b_n x^n) \\ &= (a_0 b_0) + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + ... \end{aligned}$$

The general formula is

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{with } c_n := \sum_{k=0}^{n} a_k b_{n-k}. \tag{5}$$

This is called the **Cauchy Product**.

Theorem 5.24 (Differentiation and Integration)**.** *If*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

has a positive radius of convergence r*, then* f *is differentiable in the interval* $|x| < r$*:*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Also f *has antiderivatives in* $|x| < r$*:*

$$\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Remark 5.25. We can replace x in the above theorem by $(x - x_0)$.

If we inductively apply the first part of the theorem, we know that f is nth differentiable for all $n \geq 1$. In particular, this implies that functions that can be represented by power series are smooth.

Example 5.6.3. *Find the power series for* $\frac{1}{1-x}$*.*

Solution.

$$\frac{1}{1-x} = 1 + x + x^2 + ... = \sum_{0}^{\infty} x^n. \tag{6}$$

The radius of convergence is 1. □

Example 5.6.4. *Find a power series for each of the following functions:*

$$(a) \quad \frac{1}{1+x^2}, \quad (b) \quad \frac{1}{(x-1)^2}, \quad (c) \quad \arctan x.$$

Solution. Replacing x by $-x^2$ in (6), we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - ... = \sum_{0}^{\infty} (-1)^n x^{2n}. \tag{7}$$

For (b), since $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$, by differentiating (6) twice, we get

$$(\frac{1}{1-x})' = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... = \sum_{1}^{n} n x^{n-1}.$$

For (c), notice

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt.$$

Therefore we can integrate the series (7) to get

$$\begin{aligned}\arctan x &= \int_0^x \frac{1}{1+t^2}dt \\ &= \sum_0^\infty \int_0^x (-1)^n t^{2n} dt \\ &= \sum_0^\infty \frac{(-1)^n x^{2n+1}}{2n+1}.\end{aligned}$$

Can you figure out the radius of convergence?

□

5.7 Taylor Series

The Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point.

Definition 5.26. If f is infinitely differentiable at $x = c$, then the Taylor series for $f(x)$ centered at c is the power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + ... = \sum_{n=0}^\infty \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

If you look at the partial sums of the Taylor series, then the polynomials give a good approximation of the function at $x = c$. For example, the first two terms of the Taylor series is just $L(x) := f(c) + f'(c)(x-c)$. This is a linear approximation of f at $x = c$ which satisfies $L(c) = f(c)$ and $L'(c) = f'(c)$.

Example 5.7.1. *Find the Taylor series of a polynomial* $f(x) = 1 + x + x^2 + x^3$ *at* $x = 0$.

What is the Taylor series at $x = c$ for general $c \in \mathbb{R}$?

Example 5.7.2. *Show that the Taylor series for* e^x *at* $x = 0$ *is*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

and the radius of convergence is ∞.

Example 5.7.3. *Find the Taylor series for* $f(x) = x^{-3}$ *centered at* $c = 1$.

Solution. Let us compute $f^n(x)$.

$$f'(x) = -3x^{-4}, \quad f''(x) = (-3)(-4)x^{-5}...$$

in general we have

$$f^n(x) = (-1)^n 3 \times 4 \times ... \times (n+2) x^{-3-n}.$$

Therefore

$$f^n(1) = (-1)^n \frac{(n+2)!}{2}.$$

The Taylor series is

$$\sum_{n=0}^{\infty} (-1)^n \frac{(n+2)(n+1)}{2} (x-1)^n.$$

The radius of convergence equals 1.

□

Theorem 5.27. *1. If f can be represented as a power series centered at c in an interval I containing c, then the power series is the Taylor series.*

2. If f is only known to be smooth on I, then Taylor series and f may not equal on the whole interval I.

3. If f is smooth on I and there exists a constant $C > 0$ such that

$$|f^{(k)}(x)| \leq C^{k+1} k! \quad \textit{for all } k \geq 0 \textit{ and } x \in I.$$

Then f equals its Taylor series on I.

Let me show you an example of 2. Actually, the function

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is infinitely differentiable at $x = 0$, and has all derivatives zero there. Consequently, the Taylor series of $f(x)$ about $x = 0$ is identically zero. However, $f(x)$ is not the zero function.

Example 5.7.4. *Show that for all $x \in \mathbb{R}$,*

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Solution. Step 1. Compute derivatives of f. Step 2. Find the Taylor series. Step 3. Check the conditions of the Theorem 5.27 for $I = (-R, R)$ for any $R \to \infty$. Step 4. Passing $R \to \infty$. □

Definition 5.28. The Taylor series at $x = 0$ is also called the Maclaurin series.

Example 5.7.5. *Find the Maclaurin series for*

$$x^2 e^x, \quad e^{-x^2}, \quad \frac{1}{(x-1)^2}, \quad \ln(1+x), \quad \arctan x.$$

Example 5.7.6. *Write out the terms up to degree* 5 *in the Maclaurin series for* $f(x) = e^x \cos x$.

Solution. We multiply the fifth-order Maclaurin polynomials of e^x and $\cos x$ together:

$$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!})(1 - \frac{x^2}{2!} + \frac{x^4}{4!}).$$

Distribute the term and drop the terms of degree greater than 5. We obtain

$$1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30}.$$

Thus

$$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + ...$$

□

Euler's Formula. Use $i^2 = 1$, we find

$$e^{ix} = \cos x + i \sin x.$$

Example 5.7.7. *Express* $\int_0^1 \sin(x^2)dx$ *as an infinite series centered at point* 0.

Solution. Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1},$$

we get

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}.$$

Then

$$\int_0^1 \sin(x^2)dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n}{(2n+1)!} x^{4n+2} = \sum_{0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{4n+3}.$$

□

Using series to represent a function can be helpful in determine the limit. For example we can get

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x^2} = 0.$$

Moreover using the Taylor series of $\cos x$, we find

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{1 - (1 - \frac{x^2}{2!} + ...)}{x^2} = \frac{1}{2}.$$

Example 5.7.8. *Show* $\lim_{x \to 0} \frac{x - \sin x}{x^3 \cos x} = \frac{1}{6}$.

Binomial Series. We define the **binomial coefficient**: if n, k are non-negative integers and $k \leq n$,

$$\binom{n}{k} := \frac{n(n-1)(n-2)...(n-k+1)}{k!}, \quad \binom{n}{0} := 1.$$

If n, k are non-negative integers, and $k > n$, then $\binom{n}{k} := 0$. We have the binomial formula (n, k are non-negative integers),

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$

More generally, if k is a positive integer, and n is any number but not an integer, then we define

$$\binom{n}{k} := \frac{n(n-1)(n-2)...(n-k+1)}{k!}.$$

Example 5.7.9.

$$\binom{4}{2} = 6, \quad \binom{4/3}{3} = -\frac{4}{81}.$$

Theorem 5.29 (Binomial series)**.** *For any exponent $a \in \mathbb{R}$ and for $|x| < 1$, we have*

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k.$$

Example 5.7.10. *Find the Maclaurin series for $f(x) = \frac{1}{\sqrt{1-x^2}}$.*

Solution. Let us first find out the Maclaurin series for $(1+x)^{-1/2}$: by the binomial series,

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} x^k.$$

Therefore

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{2k} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + ...$$

$\square$