UCSD

Lecture : MATH 130 Differential Equations and Dynamical Systems

2021 WI

Lecturer: Yuming Paul Zhang *Scribe:*

Contents

2 Flows on the line

(The section numbers are the same as those in the textbook.)

We start with reviewing the definitions of differential equations.

- An *Differential Equation (DE)* is any equation that involves derivatives.
- An *Ordinary Differential Equation (ODE)* is a DE that has only one independent variable and all derivatives are with respect to this variable.
- An DE which has derivatives with respect to more than one variables is called a *Partial Differential Equation (PDE)*.

Suppose $x = f(t)$ is a given smooth function. Then the derivative x' describes the rate of change of x with respect to t.

There are several notations for the derivatives:

- $x'(t)$ – Newton notation;
- $\frac{dx}{dt}$ – Leibnitz notation;
- other notations: $x_t, \dot{x}$ etc.

The *Order* of a DE is the number of the highest derivative in the equation.

2.1 Autonomous system

An first order autonomous differential equation is an equation of the form

$$\frac{dx}{dt} = f(x).$$

Let's think of t as indicating time. This equation says that the rate of change $\frac{dx}{dt}$ of the function $x(t)$ is given by some rule. The rule says that if the current value is x, then the rate of change is $f(x)$.

Note that $x' = f(x,t)$ is not an autonomous system.

Let us look at the following example:

Example 2.1.1. $\frac{dx}{dt} = \sin x$.

Solution. This is a separable differential equation which is to say that we are able to separate the variables x and t as follows:

$$\frac{dx}{\sin x} = dt.$$

After integration, we get

$$t = \int \frac{1}{\sin x} dx.$$

Since

$$\int \frac{1}{\sin x}dx = \int \frac{\sin x}{\sin^2 x}dx = \int -\frac{d\cos x}{1-\cos^2 x}$$
$$= -\frac{1}{2}\int \frac{1}{1-\cos x} + \frac{1}{1+\cos x} d\cos x$$
$$= -\frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + C = -\ln\left|\frac{1+\cos x}{\sin x}\right| + C.$$

(Think about why the last equality holds?) □

It is great that we obtain an explicit solution to the problem $x' = \sin x$. However the explicit form is complicated and we can not tell some useful properties of solutions from it immediately.

Some properties of solutions:

- Suppose at $t = 0$, $x(0) = x_0$ for some real number x_0. The solution is increasing or decreasing?
- Suppose at $t = 0$, $x(0) = x_0$ for some real number x_0. What happens to the solution as $t \to \infty$? (Asymptotic behaviour.)

Let us draw the graph of $(x, \sin x)$ which then, by the equation, is the graph of (x, x').

Let us think about the solution $x(t)$ to

$$x'(t) = \sin x, \quad x(0) = x_0$$

as a particle moving on the real line starting at position $x = x_0$ (we call the solution as a *flow along the vector field* $\sin x$). There are several typical cases:

- If $x_0 = \frac{-\pi}{4}$, then from the graph, $x'(0) = \sin x_0 < 0$, and so the solution $x(t)$ is decreasing (moving to the left). The solution moves toward $x = -\pi$ as $t \to \infty$.
- If $x_0 = \frac{\pi}{2}$, think about what happens to $x(t)$ as $t \to \infty$?
- If $x_0 = \pi$, then $x'(0) = 0$ and $x(t) = \pi$ is a constant solution. $x = \pi$ will be called as a *fixed point*. All values z such that $\sin z = 0$ are called fix points.

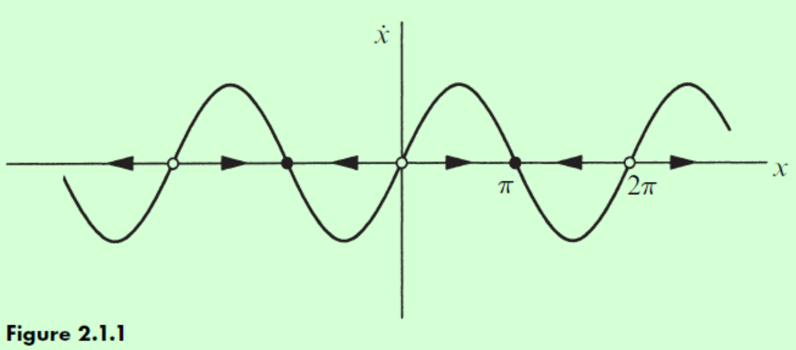


Figure 2.1.1

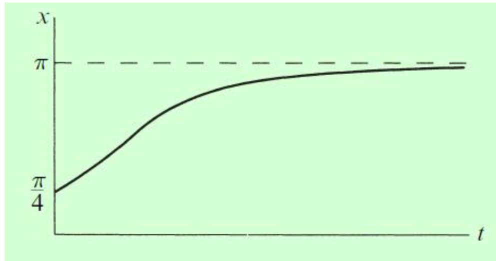


Figure 1: Figure 2.1.2

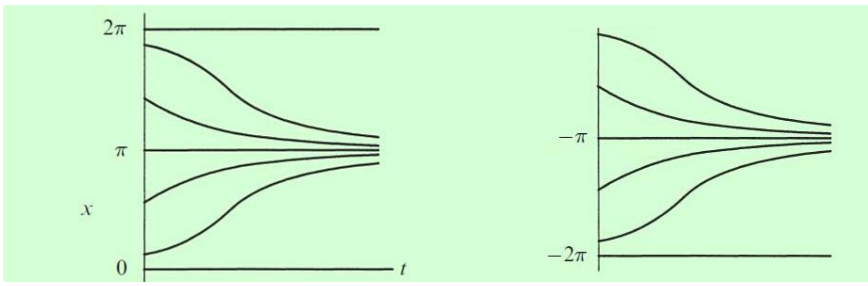


Figure 2: Figure 2.1.3

- $x = (2k+1)\pi$ with $k \in \mathbb{Z}$ are called *stable fixed points* because any solutions started near those points are moving towards those points.

- $x = 2k\pi$ with $k \in \mathbb{Z}$ are called *unstable fixed points* because any solutions started near those points are moving away from those points.

The graph of (x', x) tells us the dynamics of the autonomous system in large time. Using the information, we can roughly draw the picture of solutions starting at different initial positions.

For example, if $x(0) = x_0 = \frac{\pi}{4}$, then the solution with this initial data is increasing as t increases. Also from the graph we know that the rate of $x(t)$ increases is again increases and reaches its maximum when $x(t)$ becomes $\frac{\pi}{2}$. Then then $x(t)$ increases with a smaller and smaller speed. Eventually as $t \to \infty$, $x(t) \to \pi$. Thus we can draw the graph of $x(t)$ as in the Figure 2.1.2 above.

Next we look for the rough shapes of solutions with different initial data and we can gain a big picture of the dynamics, see Figure 2.1.3.

2.2 Fixed points and stability

In general let us consider the one-dimensional autonomous system $x' = f(x)$.

As before, we draw the picture of $f(x)$, and then we put arrows on the real line indicating the sign of x'. The real line with arrows are referred to as the *phase portrait*. Every point on the graph is called a *phase point*. Suppose $(x_0, f(x_0))$ is a phase point. It is called a phase point because one solution to the equation might pass this point i.e. for some $t_0 \geq 0$, $(x(t_0), x'(t_0)) = (x_0, f(x_0))$. Any solution $x(t)$ is called a *trajectory*.

These are just some names. But using these names gives a geometric flavor of solving DE. More than that, the geometric way of thinking is important.

The set

$$\{z \in \mathbb{R} \,|\, f(z) = 0\}$$

are called *equilibruim points* (or fixed points). We know that $x(t) \equiv z$ with z being a equilibruim point, is a constant solution.

Example 2.2.1. *Find all equilibruim points for $x' = x^2 - 1$, and classify their stability.*

Solution. Solving for $x^2 - 1 = 0$ implies that $x = \pm 1$ are the two equilibruim points. From the graph of $x^2 - 1$, we know that $x = -1$ is stable, and $x = 1$ is unstable.

A simple criteria is: suppose $f(x_*) = 0$ and $f'(x_*) > 0$, then x_* is unstable. While if $f'(x_*) < 0$, x_* is stable. □

In the above example, we can only say that $x = -1$ is *locally stable* (in contrast to *globally stable*). Because, solutions starting with initial values < 1 will converge to $x = -1$ as $t \to \infty$. But when the initial data is > 1, the corresponding solution diverges to ∞ as $t \to \infty$.

Can you think of an example where there is a globally stable equilibruim point?

Take a look at Example 2.2.2 in the textbook which is about an electrical circuit system.

Example 2.2.2. *Use the phase portrait to study*

$$x' = -(x-1)^{5/3}(x-2)^2(x-3).$$

Predict the asymptotic behavior as $t \to \infty$ of the solution satisfying $x(0) = 2.5$.

Can you draw the graph of the function in the right-hand side of the above equation?

Sometimes it is hard to portrait the graph for $f(x)$.

Example 2.2.3. *Find the stability of all fixed points for $x' = x - \cos x$.*

Solution. We can use computer to draw the graph for $x - \cos x$ (which is forbidden during the exam..). An simpler way is to draw the graph for $y = x$ and $y = \cos x$ separately. We plot both graphs on the same axes and then their difference corresponds to the graph of $x - \cos x$. Their only one intersection in exactly one point (denoted as x_*) is the fixed point. Notice that $x > \cos x$ when $x > x_*$ and $x < \cos x$ when $x < x_*$. Therefore x_* is unstable. □

2.3 Population growth

Let $N(t)$ be the population of cells, one species or the number of people who get infectious disease etc. Then we believe that the growth rate of the population is proportional to $N(t)$ itself, which implies the simplest model $N' = rN$ for some $r > 0$. After solving the equation, we

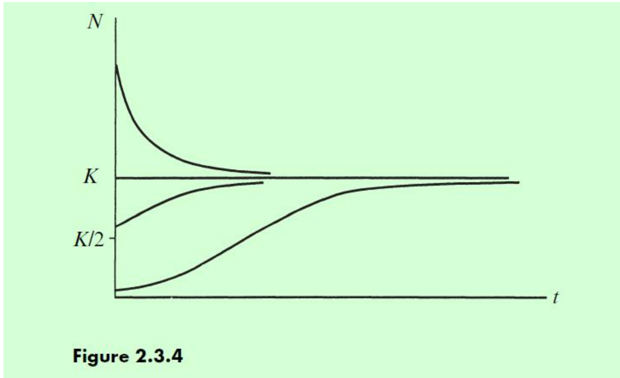


Figure 3: Graph of the population $N(t)$

get $N(t) = N_0 e^{rt}$ where N_0 is the initial population. However due to the limited resources, the interior competition or the predators from outside, the per capita growth rate $\frac{N'}{N}$ decreases (to a negative number) when N becomes sufficiently large. We assume that $\frac{N'}{N}$ becomes negative when N exceeds K which is often called the carrying capacity. We can write the model to be

$$N' = rN(1 - \frac{N}{K})$$

which is firstly used to describe the growth of human population by Verhulst in 1838. It is often called the Logistic model.

This is certainly an autonomous system. Using the phase portrait method, we can find that

1. when $N_0 \in (0, K)$, the population growths and it converges to K;
2. when $N(0) > K$, the population decreases to K;
3. when $N(0)$ is very small, N grows exponentially and the growth rate achieves its maximum at $N = \frac{K}{2}$;
4. when $N(0) < \frac{K}{2}$, $N(t)$ is increasing as $t \to \infty$. Moreover notice that

$$N''(t) = \frac{d}{dt}(f(N(t))) = f'(N(t))N'(t) > 0$$

as long as $N(t) < \frac{K}{2}$. Therefore the rate $N(t)$ increases is also increasing, and the graph of $N(t)$ is convex. While when $N(t)$ becomes greater than $\frac{K}{2}$, we have

$$N''(t) = \frac{d}{dt}(f(N(t))) = f'(N(t))N'(t) < 0$$

and so the graph of $N(t)$ is then concave;

5. when $N(0) > \frac{K}{2}$, we have

$$N''(t) = \frac{d}{dt}(f(N(t))) = f'(N(t))N'(t) < 0$$

for all $t > 0$, and so the graph of $N(t)$, in this situation, is (increasing) concave for all time.

As an application, let us consider fishing. Suppose the number of fishes near the coast can be modeled by the Logistic model. People needs to fish to survive and at the same time overfishing might needs to serious problem like extinction of the fish. The best strategy is keep the population of the fish to be around $\frac{K}{2}$ where the growth rate is the largest.

2.4 Linear stability analysis

Suppose x_* is a fixed point to $x' = f(x)$. The goal of this section is to have a quantitative measure of stability (such as the rate of decay to the stable fixed point x_*) by using *linearization*.

Using Taylor's expansion we have

$$f(x_* + h) = f(x_*) + hf'(x_*) + O(h^2) = hf'(x_*) + O(h^2).$$

From the equation, $f(x_*) = 0$ and $x' = h'$, we obtain

$$h' = hf'(x_*) + O(h^2).$$

If $f'(x_*) \neq 0$, the $O(h^2)$ term is negligible (when h is small) and then the equation becomes

$$h' \approx hf'(x_*)$$

which is a linear equation of h, and this is called the *linearization (of the original equation) about* x_*.

The solution to the linearized equation $H' = Hf'(x_*)$ with initial data $H(0) = \epsilon$ for some $\epsilon \in \mathbb{R}$ is $H(t) = \epsilon e^{f'(x_*)t}$. Therefore when $f'(x_*) > 0$, $H(t) \to \infty$ as $t \to \infty$ and when $f'(x_*) < 0$, $H(t) \to 0$ as $t \to \infty$. We conclude that

1. if $f'(x_*) > 0$, 0 is a unstable fixed point for H equation,
2. if $f'(x_*) < 0$, 0 is a stable fixed point for H equation.

We can assume that h and H are close when both of them are small enough and with the same initial data. Therefore we have

1. if $f'(x_*) > 0$, x_* is a unstable,
2. if $f'(x_*) < 0$, x_* is a stable.

There is a simple rigorous proof for this: Let us consider the case when $f'(x_*) > 0$. Then when h is small enough, $\frac{h'}{h} \geq \frac{f'(x_*)}{2}$ and so h grows exponentially (for a small time as long as $h(t)$ is still "small enough"). This indicates that x_* is unstable.

Example 2.4.1. *Classify the fixed points of the logistic equation.*

Solution. The equation is $N' = rN(1 - \frac{N}{K})$ and $f(N) = rN(1 - \frac{N}{K})$. Set $f(N) = 0$ and we get $N_1 = 0$, $N_2 = K$ are the two fixed points. Then $f'(N) = r - \frac{2rN}{K}$ and so $f'(0) = r$ and $f'(K) = -r$. Hence N_1 is unstable and N_2 is stable. $\square$

Let us consider the linearized equation about 0 for the logistic model:

$$H' = rH.$$

Then $H(t) = H(0)e^{rt}$. As explained before, we expect that $H(t)$ and $N(t)$ are close as long as both of them are close enough to 0. Suppose the initial data $H(0) = N(0)$ are small. In order to have $H(t)$ small, we need rt to be small. Therefore we say that when t is of order $\frac{1}{r}$, the linearized equation approximates the original equation well. Notice that here $r = f'(0)$.

In general cases ($x' = f(x)$ and x_* is a fixed point), we define $\frac{1}{|f'(x_*)|}$, if $f'(x_*)$ is not zero, to be the *characteristic time scale*.

" the new feature is that now we have a measure of how stable a fixed point isthat's determined by the magnitude of $f(x_*)$. This magnitude plays the role of an exponential growth or decay rate. Its reciprocal $1/|f'(x_*)|$ is a characteristic time scale; it determines the time required for $x(t)$ to vary significantly in the neighborhood of x_*. "

2.5 Existence and uniqueness

The mathematical term *well-posed problem* stems from a definition given by 20th-century French mathematician Jacques Hadamard. He believed that mathematical models of physical phenomena should have the properties that:

1. a solution exists,
2. the solution is unique,
3. the solution's behaviour changes continuously with the initial conditions.

Let us start with the following example of an initial value problem where the uniqueness of solutions fails.

Example 2.5.1. *Show that the solution to $x' = x^{1/2}$ starting from $x_0 = 0$ is NOT unique.*

Solution. It is not hard to check that $x(t) \equiv 0$ is a solution. Also $x(t) = \frac{1}{4}t^2$ is a solution.

Moreover "combining" the above two solutions we know that

$$x(t) := \begin{cases} 0, & \text{if } t \in [0, a], \\ \frac{1}{4}(t-a)^2, & \text{if } t \geq a \end{cases}$$

are solutions for any choice of $a > 0$. $\square$

Using the geometric interpretation for this example, what is happening is: if we place a particle (phase point) at the origin at time 0, then the particle can stay there for all times, or it can stay there for time $a \geq 0$ and then move according to $\frac{1}{4}(t-a)^2$.

The following theorem provides sufficient conditions for existence and uniqueness for $x' = f(t,x)$ (in the textbook only the autonomous system is discussed) with some initial data.

Theorem 2.1. *Suppose $f(t,x)$ and $\partial_x f(t,x)$ are continuous in a rectangle centered at (t_0, x_0): $\{(t,x) \,|\, |t-t_0| < R,\ |x-x_0| < R\}$ for some $R > 0$. Then there exists $\tau > 0$ (possible smaller than R) such that the initial value problem*

$$x' = f(t,x), \quad x(t_0) = x_0$$

has a solution $x(t)$ on some time interval $(t_0 - \tau, t_0 + \tau)$ and the solution is unique.

We skip the proof. (Some references for the proof are given in the textbook. You can also google "Picard iteration method" to find some references.)

The next example indicates that even the conditions in Theorem 2.1 hold (actually even f is smooth in the whole domain), there's no guarantee that solutions exist forever. Think about whether this contradicts with Theorem 2.1. If not, why?

Example 2.5.2. *Solve $x' = 1 + x^2$, $x(0) = x_0$ where x_0 is a real number.*

Solution. This is a separable differential equation. By integration, we are able to find out an explicit solution:

$$x = \tan(t + C).$$

Using the initial data we get $C = \tan^{-1} x_0$ and C is a number from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Later we will still denote C as $\tan^{-1} x_0$ below.

Now we claim that this solution only exists for a finite time. Indeed, the function

$$\tan(t + C)$$

is only a continuous function for $t \in (-C - \frac{\pi}{2}, -C + \frac{\pi}{2})$, and as $t \to \pm\frac{\pi}{2}$, the solution $\to \pm\infty$. Thus outside of the time interval $(-C - \frac{\pi}{2}, -C + \frac{\pi}{2})$, there is no solution to the initial value problem for $x(0) = x_0$. $\square$

In the above example the solution reaches infinity in finite time. This phenomenon is called (finite time) *blow-up*. As the name suggests, it is of physical relevance in models of combustion and other runaway processes.

Later we will only deal with smooth vector fields ($f = f(x)$) and we will not worry about issues of existence and uniqueness.

2.6 Impossibility of oscillations

Consider the system $x' = f(x)$. Fixed points dominate the dynamics of first-order systems. In all our examples so far, all trajectories either approached a fixed point, or diverged to $\pm\infty$.

We claim that the trajectories are forced to increase or decrease monotonically, or remain constant. To put it more geometrically, the phase point never reverses direction. To put it in another way, a solution will monotonically converge to a fixed point or to $\pm\infty$. There is no oscillations and no periodic solutions.

This can be proved by the existence and uniqueness theorem.

Proof. (of no oscillations.) Suppose for contradiction that a solution $x(t)$ to $x' = f(x)$ with smooth f is such that $x(t_1) = x(t_2) = a$ and a is not a fixed point. Without loss of generality, suppose $x'(t_1) > 0$ (it cannot be 0 because otherwise a is a fixed point), then $x(t)$ is increasing. If $x(t)$ is always increasing for $t > t_1$, a cannot be reached again. Therefore $x'(t)$ have to be 0 at some $t > t_1$. While this is again impossible, because if $x'(t_3) = 0 = f(x(t_3))$ for some $t_3 > t_1$, $x \equiv x(t_3)$ is then a solution, which contradicts with the uniqueness theorem.

□

Similarly we can show that two trajectories of an autonomous system cannot intersect by using the uniqueness theorem. Can you prove it?

2.7 Potentials

There is another way to visualize the dynamics of the first-order system $x' = f(x)$, based on the physical idea of potential energy.

Suppose a particle has to slog through a think layer of goo. By physics (Newton's law)

$$mx'' + bx' = F(x)$$

where m is the mass, b is the damping coefficient, and f is the force. Suppose the system is *over damped* in the sense that $m << \frac{F(x)}{b} =: f(x)$. In such a case we get $x' = f(x)$. The (re-scaled) force term $f(x)$ is determined by the shape of the goo. If it has a large slope, the force is large, and if it is flat, the force is small. Therefore we have $f(x) = -\frac{dV(x)}{dx}$. The minus sign is due to the pointing direction. This function V will be called the *potential function*.

We claim that, under this interpretation, *along any trajectories (solutions) to the system, the potential decreases*. The proof for this is one line:

$$\frac{dV(x(t))}{dt} = \frac{dV(x)}{dx}\frac{dx}{dt} = -(V'(x(t)))^2 \leq 0.$$

Also $f(x) = 0$ corresponds to $\frac{dV}{dx} = 0$. So fixed points are local extreme points of $V(x)$. Actually we have

- if x_0 is a local minimum point of V, it is a stable fixed point;

- if x_0 is a local maximum of V, it is a unstable fixed point.

Can you prove this?

Example 2.7.1. *For the system $x' = -x$, the potential $V(x) = \frac{1}{2}x^2$ (any functions $\frac{1}{2}x^2 + C$ for $C \in \mathbb{R}$ are correct potential functions). The equilibrium point is $x = 0$, and it is stable.*

For the system $x' = x - x^3$, the potential can be $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$. $x = \pm 1$ are the two stable equilibrium points, $x = 0$ is the only unstable equilibrium point.

3 Bifurcations

Bifurcation is the phenomena of qualitative changes in the structure of a given family of solutions to a continuous family of differential equations.

3.1 Saddle-node bifurcation

Let us consider the following continuous family of first-order autonomous system with parameter $r \in \mathbb{R}$:

$$x' = r + x^2. \tag{1}$$

When r is negative, there are two fixed points, one stable and one unstable. As r approaches 0 from below, the parabola moves up and the two fixed points move toward each other. When $r = 0$, the fixed points coalesce into a half-stable fixed point at $x_* = 0$. As soon as $r \to 0$, the fixed point vanishes and now there are no fixed points at all.

In this example, we say that a *bifurcation* occurred at $r = 0$, since the solutions to the equation exhibit significantly different behaviour for $r < 0$ and $r > 0$ (different number of fixed points and different stability).

Now we try to depict the bifurcation by plotting the fixed points as a function of r (this makes sense because we know that fixed points alone capture the main property of the vector fields or the equation).

To do so we first depict the phase portraits for different r. And then we think of r as an independent variable, and for each r, we plot the fixed points and we get a curve of fixed points (as a function of r). To distinguish between stable and unstable fixed points, we use a solid line for stable points and a broken line for unstable ones. This picture is called the *bifurcation diagram* of (1). Bifurcation diagram of this shape will be called the *saddle-node bifurcation* or *fold bifurcation*. They are used to describe the situation that two fixed points of a dynamical system collide and annihilate each other.

Another example is

$$x' = r - x^2.$$

When $r > 0$ there are two fixed points (one is stable and the other is unstable, check this!); when $r = 0$, we have only one half-stable fixed point; and when r becomes negative, there is no fixed point.

Example 3.1.1. *Show that the first-order system $x' = r - x - e^{-x}$ undergoes a saddle-node bifurcation as r is varied, and find the value of r at the bifurcation point.*

Solution. Let $f(x, r) = r - x - e^{-x}$. The problem is to find the root(s) of $f(x, r) = 0$ for various r.

We adopt a geometric approach. We plot $r - x$ and e^{-x} on the same picture. The intersections of the two curves represents the root(s) to $f(\cdot, r) = 0$ for fixed r. This picture also allows us to read off the direction of flow on the x-axis: the flow is to the right where the line lies above the curve, since $r - x > e^{-x}$ and therefore $x' > 0$. Hence, the fixed point on the right is stable, and the one on the left is unstable.

Then we start decreasing the parameter r. The line $r - x$ slides down and the fixed points approach each other. At some critical value $r = r_c$ (we know r_c exists and we will find out its value below), the line becomes tangent to the curve and the fixed points coalesce in a saddle-node bifurcation. For r below this critical value, the line lies below the curve and there are no fixed points.

Now we solve for the value of r_c. Since $r_c - x$ and e^{-x} attaches each other at one point. Then at the point, the slope has to be the same. We get two equations from this: at the attaching point (say x),

$$r_c - x = e^{-x}, \quad \frac{d}{dx}(r_c - x) = \frac{d}{dx}(e^{-x}).$$

Solving the two equations, we find that the bifurcation point is $r_c = 1$. $\square$

It can be deduced easily from the above that

$$f(x, r) = (r - r_c) - (x - z)^2 \quad \text{and} \quad f(x, r) = (r - r_c) + (x - z)^2$$

exhibit Saddle-node bifurcation (occurred at $r = r_c, x = z$), and they will be called *the normal forms* for saddle-node bifurcation. More generally, as before, we can use Taylor's expansion characterize the functions and fixed points that have saddle-node bifurcation.

Taylor's expansion for multi-variables ($\mathbf{x}, \mathbf{a} \in \mathbb{R}^2$):

$$f(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})\mathbf{H}(f)(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + O(\|\mathbf{x} - \mathbf{a}\|^3).$$

Here $Df(\mathbf{a}) := (f_{x_1}(\mathbf{a}), f_{x_2}(\mathbf{a}))$ is the gradient of f at $\mathbf{x} = \mathbf{a}$, and

$$\mathbf{H}(f)(\mathbf{a}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{a}) & f_{x_1x_2}(\mathbf{a}) \\ f_{x_1x_2}(\mathbf{a}) & f_{x_2x_2}(\mathbf{a}) \end{pmatrix}.$$

This matrix involving second derivatives of f is called the *Hessian* of f evaluated at $\mathbf{a}$.

We regard f as a function of both x and r, and examine the behavior of $x' = f(x, r)$ near the bifurcation at $x = z$ and $r = r_c$. Applying the Taylor's expansion to $f(x, r)$ at point $(x, r) = (z, r_c)$, we get

$$f(x, r) = f(z, r_c) + f_x(z, r_c)(x - z) + f_r(z, r_c)(r - r_c) + \frac{1}{2}f_{xx}(z, r_c)(x - z)^2$$
$$+ O(\|x - z\|\|r - r_c\| + \|r - r_c\|^2 + \|(x - z, r - r_c)\|^3).$$

Now if $f(z, r_c) = 0$ (z is a fixed point for $f(\cdot, r_c)$) and $f_x(z, r_c) = 0$ (r_c is critical), we find that

$$f(x, r) = a(r - r_c) + b(x - z)^2 + ...$$

where $a = f_r(z, r_c)$ and $b = \frac{1}{2} f_{xx}(z, r_c)$. Assuming $a, b \neq 0$, we get the same expressions as the prototypical examples (at least for the first several terms of the Taylor's expansion of f, and the first several terms approximate the original f well near $(x, r) = (z, r_c)$).

Therefore we draw **the conclusion**: If $f = f(x, r)$ satisfies for some (z, r_c),

1. $f(z, r_c) = 0$, and $f_x(z, r_c) = 0$,
2. $f_r(z, r_c) \neq 0$, and $f_{xx}(z, r_c) \neq 0$,

then $x' = f(x, r)$ exhibits Saddle-node bifurcation at $(x, r) = (z, r_c)$.

3.2 Transcritical bifurcation

The normal form for a transcritical bifurcation is

$$x' = rx - x^2.$$

Let us figure out how the fixed points changes as we change r from a negative number to a positive number.

By using the phase portrait method, when $r < 0$, we have two fixed points: $x = 0$ which is stable and $x = -r$ which is unstable.

When $r = 0$, there is only one half-stable fixed point $x = 0$.

And when $r > 0$, there are two fixed points again: $x = 0$ which is unstable and $x = r$ which is stable.

We can also draw the bifurcation diagram: figure 3.2.2 in the textbook. We see that 0 is always a fixed point. But as r varies, $x = 0$ interchanges its stability with the other fixed point.

Example 3.2.1. *A typical example could be the consumer-producer problem where the consumption is proportional to the (quantity of) resource.*

$$x' = rx(1 - x) - px$$

where $rx(1 - x)$ is the logistic equation of resource growth; and px is the consumption, proportional to the resource.

The next example is from the textbook:

Example 3.2.2. *Show that the first-order system $x' = x(1 - x^2) - a(1 - e^{-bx})$ undergoes a transcritical bifurcation at $x = 0$ when the parameters a, b satisfy $ab = 1$.*

Proof. Let us only consider the case $a, b > 0$. Let us call the right-hand side of the equation $f(x)$. By the Taylor's expansion,

$$f(x) = (1 - ab)x + \frac{1}{2}ab^2 x^2 + x^3 g(x)$$

where $g(x)$ is a continuous function depending also continuously on a, b. We see that $x = 0$ is always a fixed point.

When $ab = 1$, $x = 0$ is a half-stable fixed point.

When $ab > 1$, notice that

$$f(x) = \frac{1}{2}ab^2 x(x - \frac{2(ab-1)}{ab^2} + \frac{2x^2 g(x)}{ab^2}).$$

Hence for $|ab - 1|$ small enough, $x \approx \frac{2(ab-1)}{ab^2}$ is a fixed point and (from the graph of the vector filed of $f(x)$ near 0) it is unstable. Also we can see that $x = 0$ is stable.

Similarly it is not hard to derive that when $ab < 1$, for $|ab - 1|$ small enough, $x \approx \frac{2(ab-1)}{ab^2}$ is a stable fixed point and $x = 0$ is unstable.

Hence we can conclude that the equation undergoes a transcritical bifurcation. □

3.3 Laser threshold

In this section we look at one scientific example: a simplified model for a laser.

It is known that when an external energy source is used to pump the atoms out of their ground states, laser is created from a process that photons stimulate excited atoms to emit additional photons. Let us use $n(t)$ to denote the number of photons in the laser field and $N(t)$ to denote the number of excited atoms. Since photons and atoms meet at random, the gaining rate of $n(t)$ is expected to be proportional to $n(t)N(t)$. We also have a lose term which models the escape of photons through the endfaces of the laser. Therefore we have

$$n' = GnN - kn$$

where $G, k > 0$ are parameters. We also need an equation to relate N, n. Assume for simplicity that the excited atoms is reduced by the laser process in a linear way:

$$N(t) = N(0) - \alpha n(t)$$

where $N(0) > 0$ comes from the pump, and $\alpha > 0$ is some rate. Putting the equations together gives

$$n' = Gn(N(0) - \alpha n) - kn = (GN(0) - k)n - (\alpha G)n^2.$$

Now let us analyze the equation by classifying the fixed points. When $N(0) < k/G$, the fixed point at $n = 0$ is stable. This means that there is no stimulated emission, and eventually we cannot see the laser. As the pump strength $N(0)$ is increased, the system undergoes a transcritical bifurcation when $N(0) = k/G$, and $n = 0$ is a half-stable fixed point. When $N(0)$ further increases, a stable fixed point occurs. Thus $N(0) = k/G$ can be interpreted as the laser threshold in this model.

Next let us look at a improved model of a laser:

$$n' = GnN - kn,$$
$$N' = -GnN - fN + p.$$

The physical background is given in Exercise 3.3.1 in the textbook.

Suppose that N relaxes much more rapidly than n and assume that $N' = 0$. Then the two-dimensional system becomes a one-dimensional equation:

$$n' = \frac{Gnp}{Gn + f} - kn =: F(n).$$

Sending $F(n) = 0$, we find that $n = 0$ and $n = \frac{p}{k} - \frac{f}{G}$ are two fixed points.

If $p > \frac{fk}{G}$, there is only one fixed point $n = 0$ in the non-negative region. In this case since $F(n) < 0$ for $n > 0$, $n = 0$ is an unstable fixed point. When $p = \frac{fk}{G}$, $n = 0$ is half-stable and when $p < \frac{fk}{G}$, the other fixed point $n = \frac{p}{k} - \frac{f}{G} > 0$ and it is unstable. In this case $n = 0$ becomes stable. We can therefore conclude that the equation undergoes a transcritical bifurcation.

3.4 Pitchfork Bifurcation

In this section we discuss a third kind of bifurcation, the so-called pitchfork bifurcation.

Again we discuss it by looking at a prototypical type problem:

$$x' = rx - x^3.$$

Before analyzing the phase portrait, let us mention a property of the equation: *invariant* under the change of variables $x \to -x$. This means that if $x(t)$ is a solution, then $y(t) := -x(t)$ is also a solution, because

$$y' = -x' = -rx + x^3 = ry - y^3.$$

We used that the vector field is an odd function here. This invariant property tells that if $x = a$ is a fixed point and so is $x = -a$. Moreover, if $x = a$ is a stable fixed point and so is $x = -a$ – the stability are the same as well.

When $r < 0$, the origin is the only fixed point, and it is stable. The linearized equation is $x' = rx$ with solutions of the form $x(t) = x_0 e^{rt}$.

When $r = 0$, the origin is still the only fixed point which is stable. But now solutions no longer decay exponentially fast, instead the decay is a much slower algebraic function of time. Actually, we can compute the explicit form of solutions: since $x' = -x^3$, we get

$$x(t) = (\frac{1}{2}t + C)^{-\frac{1}{2}}.$$

This lethargic decay is called *critical slowing down* in the physics literature.

Finally, when $r > 0$ the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x = \pm\sqrt{r}$.

The corresponding bifurcation diagram is given in Figure 3.4.2. This bifurcation is called a *Supercritical Pitchfork Bifurcation*.

Next we introduce the *Subcritical Pitchfork Bifurcation*. The prototypical equation is:

$$x' = rx + x^3.$$

Comparing to the supercritical case discussed in the above the cubic term is negative which then has a stabilizing effect: it acts as a restoring force that pulls $x(t)$ back toward x = 0. While here the effect is opposite.

Now we study the phase portrait. When $r > 0$, the origin is the only fixed point, and it is unstable. When $r = 0$, the origin is still the only fixed point which is unstable. When $r > 0$, there are three fixed points: the origin is stable, $x = \pm\sqrt{-r}$ are two unstable fixed points. Figure 3.4.6 in the textbook shows the bifurcation diagram.

Example 3.4.1. *Let us consider*

$$x' = rx + x^3 - x^5.$$

We can imagine before any computations that since near $x = 0$, x^5 is of very low order, the bifurcation occurs at $x = 0, r = 0$ is a subcritical pitchfork bifurcation. The phase portrait and bifurcation diagram near these values are the same as those of $x' = rx + x^3$.

Now let us consider for x that are away from 0 (and we will see that the x^5 term plays an important role). Let us compute the fixed points for different r below.

Notice that $rx + x^3 - x^5 = 0$ is a polynomial of degree 5 and so there are five roots: $x = 0, \pm\sqrt{\frac{1\pm\sqrt{1+4r}}{2}}$. Not all roots are fixed points because some of them can be of complex value. When $r > 0$, there are three fixed points (three roots are real): $x = 0, \pm\sqrt{\frac{1+\sqrt{1+4r}}{2}}$. $x = 0$ is unstable and the other two are stable. When $r \in (-\frac{1}{4}, 0)$, we have five fixed points. When $r < -\frac{1}{4}$, 0 is the only fixed point. They are

$$x_1 = \sqrt{\frac{1+\sqrt{1+4r}}{2}},\ x_2 = \pm\sqrt{\frac{1-\sqrt{1+4r}}{2}},\ x_3 = -x_1,\ x_4 = -x_1.$$

We can also see that $0, x_1, x_4$ are stable fixed points and x_2, x_3 are unstable by the graph of f. The bifurcation at $r = -\frac{1}{4}$ is a saddle-node bifurcation, in which stable and unstable fixed points are born as r is increased.

With the knowledge of these fixed points and their stability, we can certainly analyze the long time behaviour of solutions as we did in section 2. Please take a look at the textbook page 60 – page 61.

3.5 Overdamped bead on a rotating hoop

Let us use this section to analyze a classic problem from first-year physics, the motion of bead on a rotating hoop. The story is: A bead of mass m slides along a wire hoop of radius r. The hoop is constrained to rotate at a constant angular velocity ω about its vertical axis. Furthermore, suppose that there's a frictional force on the bead that opposes its motion. The friction is due to viscous damping.

Let me write down the equation below and then I will briefly explain how it comes from. The equation is

$$mr\phi'' = -b\phi' - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi,$$

where ϕ is the angle between the bead and the downward vertical direction; $r\phi''$ is the tangential acceleration; mg represents the gravitational force; and $b\phi'$ is a tangential damping force. The term $mg\sin\phi$ comes from the gravity force projecting to the tangential direction. The term $mr\omega^2\sin\phi\cos\phi$ is the projection of the sideways centrifugal force $m\rho\omega^2$ with $\rho = r\sin\phi$ into the tangential direction.

Let us neglect the second order term. (Cautious! When doing research, it is OK to neglect terms that are minor, but it is not OK to neglect a term that makes things hard. We will explain in the end of this section that when the system is overdamped, such approximation makes sense.) Then the equation becomes a first order differential equation:

$$b\phi' = mg\sin\phi(\frac{r\omega^2}{g}\cos\phi - 1).$$

We need to keep in mind that, for this equation, the domain for the independent variable ϕ is $(-\pi, \pi]$ (and if we think of $-\pi$ the same as π, the domain is a circle). From this, we see that all the ϕ's satisfying $\sin\phi = 0$ are fixed points. Then we get $\phi = 0$ (the bottom of the hoop), and $\phi = \pi$ (the top). The more interesting result is that if

$$\gamma := \frac{r\omega^2}{g} > 1$$

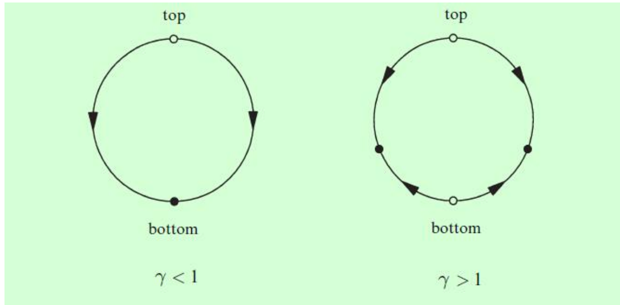
there are more fixed points i.e. ϕ's satisfying $\cos\phi = \frac{g}{r\omega^2} = \gamma^{-1}$.

Think about what does these fixed points correspond to in the physical model? What does the assumption $\gamma > 1$ correspond to in the physical model?

Overall when $\gamma \leq 1$, there are two fixed points (think about what happened when $\gamma = 1$?). Thus when $\gamma > 1$, we have four fixed points: $\phi = 0, \pi, \pm\arccos(\gamma^{-1})$. Since the domain for ϕ is a circle, we have the following phase portrait (the direction of arrows are obtained from the sign of the vector field as before). Also it is worth mentioning that as $\gamma \to \infty$, $\pm\arccos(\gamma^{-1})$ converge to $\pm\pi/2$.

We can also plot the fixed points as a graph of the parameter γ to obtain the bifurcation diagram (see Figure 3.5.6 in the textbook). We now see that a supercritical pitchfork bifurcation occurs at $\gamma = 1$.

In terms of the bead's motion, we see that when $\gamma < 1$ (the hoop is rotating slowly), the bead slides down to the bottom and stays there. But if $\gamma > 1$ (the hoop is spinning fast), the bead is therefore pushed up the hoop until gravity balances the centrifugal force; this balance occurs at one of the two fixed points. Which of these two fixed points is actually selected depends on the initial disturbance. Even though the two fixed points are entirely symmetrical, an asymmetry in



the initial conditions will lead to one of them being chosen – physicists sometimes refer to these as symmetry-broken solutions.

Now we need to address the question: When is it valid to neglect the second order term. To answer the question, let us try to rewrite the equation into a simpler form (try to combine the "unimportant" constants). Let us change the variable: $\tau = \frac{t}{T}$ where $T > 0$ is a *characteristic time scale* to be chosen later. We have

$$\phi' = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$$

and

$$\phi'' = \frac{d}{d\tau}(\frac{1}{T}\frac{d\phi}{d\tau})\frac{d\tau}{dt} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}.$$

Using the τ variable, the equation becomes

$$\frac{r}{gT^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT}\frac{d\phi}{d\tau} - \sin\phi + \frac{r\omega^2}{g}\sin\phi\cos\phi.$$

Therefore in order to neglect the second order term and preserve the first order term, we need to pick $T = \frac{b}{mg}$, and require the coefficient of the second order term to be small:

$$\frac{r}{g}(\frac{mg}{b})^2 << 1 \Longleftrightarrow b^2 >> m^2gr.$$

Since b is the damping coefficient and m is the mass, this can be interpreted as saying that the damping is very strong, or that the mass is very small. Setting $\epsilon = \frac{r}{g}(\frac{mg}{b})^2$, the equation

$$\epsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \frac{r\omega^2}{g}\sin\phi\cos\phi$$

is said to be in the *dimensionless form*. Generally, mathematician are more willing to deal with an equation in its dimensionless form, instead of the original one from physics.

3.6 Imperfect Bifurcations

In this section we discuss imperfect bifurcations. Here by "imperfect" we mean that, compared to pitchfork bifurcations where we have a symmetry (the vector field is an odd function), the symmetry is lost.

We look at the following example:

$$x' = h + rx - x^3,$$

where h, r are two parameters. We refer to h as an *imperfection parameter* (think about why?). If $h = 0$, this is the normal for a supercritical pitchfork bifurcation. However here we are more interested for $h \neq 0$.

One obstacle of analyzing this equation is that there are two parameters. One idea is to plot the graphs of $y_1 = rx - x^3$ and $y_2 = -h$ on the same axes, and look for intersections. Let us fix r first and vary h.

When $r \leq 0$, the graph of y_1 is monotonically decreasing, and so it intersects the horizontal line y_2 in exactly one point. When $r > 0$, the number of intersection points depend on the parameters. As we sliding down the horizontal line, the number of fixed points changes from 1 to 2, to 3 and then to 2, and to 1 again. The critical cases are when y_1, y_2 attach tangentially at one point, and it happens at the place where y_1 reaches its local maximum and local minimum (see Figure 3.6.1 b in the textbook). Thus for the fixed r, saddle-node bifurcation happens twice in the system.

It is known that to find out the local extreme points, we solve for $\frac{d}{dx}(rx - x^3) = 0$. We get

$$x = \pm\sqrt{r/3}.$$

Let us call the local maximum $h_c(r)$ and, by symmetry, the local minimum is $-h_c(r)$. Then $h_c(r) = \frac{2r}{3}\sqrt{r/3}$. So for $h = \pm h_c(r)$, there are two fixed points and bifurcations occur. Therefore, for possibly different positive r, the curves $h = \pm h_c(r)$ are called the *bifurcation curves*. We can plot the bifurcation curves in an $r - h$ coordinates (see Figure 3.6.2 for the *stability diagram*) (think about how we obtain the stability of fixed points). In the picture we see a cusp point at $(r, h) = (0, 0)$.

Next, for any fixed $r \leq 0$ and $r > 0$, we can plot the bifurcation diagrams for the parameter h. When $r \leq 0$, there is only one fixed point: x_* such that $h + rx_* - x_*^3 = 0$. When $r > 0$, we plot curves (h, x_*) subjected to $h = -rx_* + x_*^3$ (and so the graph is the same as the graph of the inverse function of $-rx_* + x_*^3$).

Another way to think of the problem is to fix h first and vary the r. We can again draw some bifurcation diagrams. When $h = 0$ we have the usual pitchfork diagram. Let us only discuss what happens when $h > 0$. In this case when $r < 0$, we have one fixed points for sure. While when $h > 0, r > 0$, we plot (r, x_*) such that $r = x_*^2 - \frac{h}{x_*}$. One interesting observation is that the graph in the bifurcation diagram $(h - x)$ is smooth when $h \neq 0$ and it is a pitchfork when $h = 0$!

Of course we can also try to plot the fixed points in an $h - r - x$ coordinates (in $\mathbb{R}^3$). But this is hard to plot and visualize.

3.7 Insect outbreak

We are going to look at a biological model for the sudden outbreak of an insect called the spruce budworm. This insect is a serious pest in eastern Canada, where it attacks the leaves of the balsam fir tree. When an outbreak occurs, the budworms can defoliate and kill most of the fir trees in the forest in about four years.

Let us skip the modeling part and present the equation: let N be the population of the budworm,

$$N' = RN(1-\frac{N}{K}) - \frac{bN^2}{a^2+N^2}$$

where (the first term in the RHS is discussed) the second term represents the death rate due to predation, chiefly by birds.

First let us try to find out the dimensionless formulation (which we had done once before for a different equation). In order to simplify the predation term, let $N = ax$ and diving the equation by b:

$$\frac{a}{b}\frac{dx}{dt} = \frac{Ra}{b}x(1-\frac{ax}{K}) - \frac{x^2}{1+x^2}.$$

Then to remove the coefficient in front of the differential term, we let $\tau = \frac{b}{a}t$. Then

$$\frac{dx}{d\tau} = \frac{Ra}{b}x(1-\frac{ax}{K}) - \frac{x^2}{1+x^2} = rx(1-\frac{x}{k}) - \frac{x^2}{1+x^2}.$$

where we set

$$r := \frac{Ra}{b}, \quad k := \frac{K}{a}.$$

Then $\frac{dx}{d\tau} = rx(1-\frac{x}{k}) - \frac{x^2}{1+x^2}$ with some $r, k > 0$ is the desired dimensionless form equation.

We look for the fixed points by solving $r(1-\frac{x}{k}) = \frac{x^2}{1+x^2}$. We plot the functions on the left hand side and on the right hand side of the qualities on the same axes. From the picture (Figure 3.7.2), when k is sufficiently small, there are always only one intersection (for all $k > 0$ small and $r > 0$). Let us call this case 1. Next when k is large there can be one, two, three fixed points, and we call this case 2. Actually we can determine a critical value of k_0 such that if $k < k_0$ corresponds to case 1 and $k > k_0$ corresponds to case 2. Think about how to find k_0?

Now suppose we are in the region of $k > k_0$ and we fix one such k. Then as r increases from 0 there are mainly three regions (if we do not count the critical case): 1 intersection with $y = \frac{x^2}{1+x^2}$, 3 intersections, and no intersection. So we have saddle-node bifurcations when the line intersects the curve tangentially (the critical case).

To find the place where tangential intersections occur, we solve for the following two equations:

$$r(1-\frac{x}{k}) = \frac{x}{1+x^2}$$

and

$$\frac{d}{dx}\left(r(1-\frac{x}{k})\right) = \frac{d}{dx}\left(\frac{x}{1+x^2}\right).$$

Theoretically two equations involving three variables (r, k, x), after eliminating x we get a relation between r and k which then define the stability diagram (figure 3.7.5).

Finally let us take a look at $k > k_0$. In this region we have three fixed points and suppose that they are x_1, x_2, x_3 such that $x_3 > x_2 > x_1 > 0$. And we can plot the phase portrait. We see that x_1, x_3 are stable and x_2 is unstable. In biology, x_1 is called the *refuge* level of the budworm population, while x_3 is called the *outbreak* level (the level where there are too many budworms). We view x_2 as a *threshold* since if the population is below x_2 at a time, the population converges to x_1; and if the population is above x_2, the population converges to x_3.

4 Flows on the circle

In this section we still consider first order autonomous differential equations, while the novelty is that the range of the unknown variable is no longer a subset of $\mathbb{R}$ but a circle. This will be a short section.

The equation is

$$\theta' = f(\theta)$$

where we can interpret θ as angles. We will discuss the dynamics of some simple oscillators, and then show that these equations arise in a wide variety of applications.

4.1 Examples and definitions

Example 4.1.1. *Sketch the vector field on the circle corresponding to $\theta' = \sin\theta$.*

Example 4.1.2. *Explain why $\theta' = \theta$ cannot be regarded as a vector field on the circle, for θ in the range $-\infty < \theta < \infty$.*

Solution. We cannot regard $f(\theta) = \theta$ with range $(-\infty, \infty)$ as a function on a circle because the value is not uniquely defined. On a circle, $\theta = 0, 2\pi, 4\pi...$ are all denoting the same point, and so we need

$$f(0) = f(2\pi) = f(4\pi) = ...$$

If we try to avoid this non-uniqueness by restricting the range of $f(\theta)$ to the range $(-\pi, \pi]$, then the velocity vector jumps discontinuously at the point corresponding to $\theta = \pi$. Try as we might, there is no way to consider $\theta' = \theta$ a smooth vector field on the entire circle.

In practice, $\theta' = f(\theta)$ for some real-valued, 2π-periodic function f is a well-defined differential equation on the circle. So through tout the section, we will mainly consider smooth $2\pi-$periodic functions as the vector fields. □

4.2 Uniform oscillator

Recall that when we consider flows of a given vector field on the real line or an open subset of the real line, periodic solutions is not possible. However for solutions on a circle, it is common. The simplest example can be:

$$\theta' = \omega$$

where $\omega > 0$ is a constant. The solution is $\theta(t) = \omega t + \theta_0$ where θ_0 is the initial data. This solution is periodic, in the sense that $\theta(t)$ changes by 2π, and therefore returns to the same point on the circle, after a time $T = \frac{2\pi}{\omega}$. We call T the period of the oscillation.

Let us take a look at Example 4.2.1.

Example 4.2.1. *Two joggers, Ada and Bob, are running at a steady pace around a circular track. It takes Ada T_1 seconds to run once around the track, whereas it takes Bob $T_2 > T_1$ seconds. Of course, Ada will periodically overtake Bob; how long does it take for Ada to lap Bob once, assuming that they start together?*

Solution. Let $\theta_1(t)$ be Ada's position on the track, and let $\theta_2(t)$ be Bob's. Then we have for $i = 1, 2$,

$$\theta_i' = \omega_i := \frac{2\pi}{T_i}.$$

Ada laps Bob once, assuming that they start together, is the same as $\theta_1(0) = \theta_2(0)$ and T is the smallest positive time such that

$$\theta_1(T) - \theta_2(T) = 2\pi.$$

($\theta_1 - \theta_2$ is often called the *phase difference*.) From the equation we get

$$\theta_1(t) - \theta_2(t) = (\omega_1 - \omega_2)t$$

and so

$$T = \frac{2\pi}{\omega_1 - \omega_2} = (\frac{1}{T_1} - \frac{1}{T_2})^{-1}.$$

□

4.3 Nonuniform oscillator

$\theta' = \omega$ is called uniform oscillator. In this section we consider the following nonuniform oscillator ($a \neq 0$)

$$\theta' = \omega - a \sin \theta.$$

The equation arises in many different branches of science and engineering like: Electronics (phase-locked loops), Biology (oscillating neurons, firefly flashing rhythm, human sleep-wake cycle), and Mechanics (Overdamped pendulum driven by a constant torque).

Let us assume that $\omega \geq 0, a > 0$. We can plot the vector field on $(-\pi, \pi]$ for different values of a (see Figure 4.3.2). When a is less than ω, since the vector field is strictly positive, we obtain an oscillator. As can be seen from the picture, $\theta = \frac{\pi}{2}$ is a bottleneck: $\theta(t)$ increases slowly around the point. And we can imagine that when $a < \omega$ is very close to ω, it takes a very long time for the solution to go over the bottleneck and the periodicity of solutions is very large.

Next when $a = \omega$, we see a half-stable fixed point has been born in a saddle-node bifurcation at $\pi/2$, and so the system stops oscillating. When $a > \omega$, the half-stable fixed point splits into a stable and unstable fixed point. All trajectories are attracted to the stable fixed point as $t \to \infty$.

For $a < \omega$, we know that solutions are periodic. One natural question is what is the periodicity? Suppose the periodicity of θ is T. Then, assuming that $\theta(0) = 0$, we have $2\pi = \theta(T)$ which implies that

$$T = \theta^{-1}(2\pi) = \int_0^{2\pi} \frac{d\theta^{-1}(s)}{ds} ds.$$

(Here θ^{-1} is well-defined because θ is a strictly increasing function when viewed as a function on a real line.) Since $\theta^{-1}(s) = t$ with $s = \theta(t)$, chain rule yields

$$\frac{d\theta^{-1}(s)}{ds} = \frac{d\theta^{-1}(\theta(t))}{dt} \frac{dt}{ds} = 1 \times \left(\frac{ds}{dt}\right)^{-1} = \left(\frac{d\theta(t)}{dt}\right)^{-1} = \frac{1}{\omega - a \sin \theta}.$$

Therefore

$$T = \int_0^{2\pi} \frac{1}{\omega - a\sin\theta} d\theta = \frac{2\pi}{\sqrt{\omega^2 - a^2}}.$$

Let me briefly recall for you how to do this integration problem in the last equality in the above (this is basically a problem of Math 20). Let $u = \tan\frac{\theta}{2}$, and then

$$\frac{du}{d\theta} = \frac{1}{2\cos^2(\theta/2)}, \quad \sin\theta = 2\sin(\theta/2)\cos(\theta/2).$$

Therefore the integration can be rewritten as

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{\omega - a\sin\theta} d\theta &= \int_{-\pi}^{\pi} \frac{1}{\omega - a\sin\theta} d\theta \\
&= \int_{\mathbb{R}} \frac{2\cos^2(\theta/2)}{\omega - 2a\sin(\theta/2)\cos(\theta/2)} du \\
&= \int_{\mathbb{R}} \frac{2\cos^2(\theta/2)}{\omega(\sin^2(\theta/2) + \cos^2(\theta/2)) - 2a\sin(\theta/2)\cos(\theta/2)} du \\
&= \int_{\mathbb{R}} \frac{2}{\omega(1+u^2) - 2au} du \\
&= \int_{\mathbb{R}} \frac{2}{(\sqrt{\omega}u - a/\sqrt{\omega})^2 + (\omega^2 - a^2)/\omega} du.
\end{aligned}$$

Recall the formula: $\int \frac{1}{a^2+x^2} dx = a^{-1}\tan^{-1}(x/a) + C$. We obtain

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{\omega - a\sin\theta} d\theta &= 2(\sqrt{\omega})^{-1} \int_{\mathbb{R}} \frac{1}{v^2 + (\omega^2 - a^2)/\omega} dv \\
&= 2(\sqrt{\omega})^{-1}((\omega^2 - a^2)/\omega)^{-1/2}\pi = \frac{2\pi}{\sqrt{\omega^2 - a^2}}.
\end{aligned}$$

Example 4.3.1. *Estimate the period of $\theta' = \omega - a\sin\theta$ in the limit $a \to \omega^-$.*

Solution. From the exact formula of T, we can estimate the order of $T \to \infty$ as $a \to \omega^-$:

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}} = \frac{2\pi}{\sqrt{\omega + a}}(\omega - a)^{-\frac{1}{2}} \approx \pi\sqrt{2/\omega}(\omega - a)^{-\frac{1}{2}}$$

when a is close to ω.

Next we use the normal form method instead of the exact result to estimate the period. We use a Taylor expansion about $\theta = \pi/2$, where the bottleneck occurs. Let $\phi = \theta - \pi/2$ and so ϕ is small (dealing with ϕ is more convenient). Then we have

$$\phi' = \omega - \sin(\phi + \pi/2) = \omega - a\cos\phi = \omega - a + \frac{1}{2}a\phi^2 ...$$

Then we might solve the following equation to approximate the original equation:

$$x' = (\omega - a) + \frac{1}{2}ax^2.$$

Separating the variables we get $\frac{dx}{(\omega-a)+ax^2/2} = dt$. Thus the periodicity

$$T \approx \int_{-\infty}^{\infty} \frac{dx}{(\omega - a) + \frac{1}{2}ax^2} = (\frac{1}{2}a(\omega - a))^{-\frac{1}{2}}\pi \approx (\pi\sqrt{2/\omega})(\omega - a)^{-\frac{1}{2}}.$$

(We used the limits of the integration $-\infty$, ∞ here. It is not too different from -1 to 1 since the integrand is large when x is large. And here we only need an order of $T \to \infty$ as $\phi \to 0$.) $\square$

4.4 Overdamped pendulum

We now consider a simple mechanical example of a nonuniform oscillator: an overdamped pendulum driven by a constant torque. Let θ denote the angle between the pendulum and the downward vertical direction, and suppose that θ increases counterclockwise. Then Newton's law yields

$$mL^2\theta'' + b\theta' + mgL\sin\theta = \Gamma$$

where $m > 0$ is the mass and $L > 0$ is the length of the pendulum, $b > 0$ is a viscous damping constant, $g > 0$ is the acceleration due to gravity, and $\Gamma > 0$ is a constant applied torque. Similarly as before, when considering the case that b is extremely large (overdamped), the equation can be approximated by the following first order equation

$$b\theta' + mgL\sin\theta = \Gamma.$$

As done several times before, we first simplify the equation to the dimensionless form (and I skip the details):

$$\theta' = \gamma - \sin\theta$$

where $\theta' := \frac{d\theta}{d\tau}$, $\tau := \frac{mgL}{b}t$, and $\gamma = \frac{\Gamma}{mgL}$.

For this equation, if $\gamma > 1$, then the vector field is always positive which implies that the applied torque is strong enough to overturn the pendulum continually.

When $\gamma < 1$, we have two fixed points. It is clear that the lower of the two equilibrium positions is the stable, and the other is unstable.

When $\theta = 1$, there is a fixed point. Unlike in the case of flows on the real-line, where we call it half-stable, here a better name is *asymptotically stable fixed point* because even though a flow might not converges to it instantly but it will converge to it eventually.

Finally, when $\gamma = 0$, the applied torque vanishes and there is an unstable equilibrium at the top (inverted pendulum) and a stable equilibrium at the bottom.

4.5 Fireflies

In this section we discuss a very interesting example of Fireflies' synchronization phenomena in nature. In some parts of southeast Asia, thousands of male fireflies gather in trees at night and flash on and off in unison.

Suppose that $\theta(t)$ is the phase of one firefly's flashing rhythm, where $\theta = 0$ corresponds to the instant when a flash is emitted. When the firefly is alone, θ goes around the circle cycle at a constant frequency ω, and so $\theta' = \omega$. It is known that for different fireflies, the ω is different. So what is spectacular is that when thousands of fireflies are together, they are synchronized in the sense that they are flashing with the same frequency! The key is that the fireflies influence each other, and one interesting question is how the influence happens?

Hanson (1978) studied this effect experimentally, by periodically flashing a light at a firefly and watching it try to synchronize. For a range of periods close to the firefly's natural period, the firefly was able to match its frequency to the periodic stimulus (the range of Entrainment). However, if the stimulus was too fast or too slow, the firefly could not keep up –then a kind of beat phenomenon occurred. But Hanson found that the phase difference increased slowly during part of the beat cycle, as the firefly struggled in vain to synchronize, and then it increased rapidly through 2π after which the firefly tried again on the next beat cycle. This process is called *phase drift*.

Now suppose that the phase Θ of periodic stimulus satisfies

$$\Theta' = \Omega.$$

We model the firefly's response to this stimulus as follows: If the stimulus is ahead in the cycle, then we assume that the firefly speeds up in an attempt to synchronize. Conversely, the firefly slows down if it's flashing too early. A simple model that incorporates these assumptions is

$$\theta' = \omega + A\sin(\Theta - \theta)$$

where $A > 0$ is the resetting strength measuring the firefly's ability to modify its instantaneous frequency. For example, if Θ is ahead of θ (i.e. $0 < \Theta - \theta < \pi$) the firefly speeds up ($\theta' > \omega$).

In order to analyze the dynamics of θ, we look at the phase difference $\phi = \Theta - \theta$, which then satisfies

$$\phi' = \Omega - \omega - A\sin\phi.$$

Here $A > 0$ is called the resetting strength.

By doing the following substitution

$$\tau := At, \quad \mu = (\Omega - \omega)/A,$$

we get a dimensionless equation

$$\phi' = \mu - \sin\phi,$$

where $\phi' = \frac{d\phi}{d\tau}$. The dimensionless parameter μ is a measure of the frequency difference, relative to the resetting strength. We have the nonuniform oscillator equation.

When $\mu = 0$, 0 is a stable fixed point. All trajectories flow toward $\phi = 0$. Thus the firefly and the stimulus flash simultaneously eventually. And when $0 < \mu < 1$, there are two positive fixed points. All trajectories are still attracted to the stable one which is now positive. Since the phase difference approaches a constant, one says that the firefly's rhythm is phase-locked to the stimulus. Similarly when $-1 < \mu < 0$, the phase difference approaches a constant (which is negative now), and the firefly's rhythm is phase-locked to the stimulus.

For $\mu > 1$ or $\mu < -1$ there is no fixed point. When $\mu > 1$, the phase difference ϕ increases indefinitely and periodically. Notice that the phases don't separate at a uniform rate, in qualitative agreement with the experiments of Hanson (1978): ϕ increases most slowly when it passes under the minimum of the sine wave at $\phi = \pi/2$, and most rapidly when it passes under the maximum at $-\pi/2$.

The model makes testable predictions: range of entrainment is the same as $|\mu| \leq 1$ which is equivalent to

$$\omega - A \leq \Omega - \omega \leq \omega + A.$$

By measuring the range of entrainment experimentally, one can nail down the value of the parameter A.

4.6 Superconducting Josephson Junctions

Let us roughly introduce the physical background of Superconducting Josephson Junctions (the textbook provides a much more vivid story).

A Josephson junction consists of two superconductors separated by a weak connection. Brian Josephson (1962) (at his age of 22) suggested that it should be possible for a current to pass between the two superconductors, even if there were no voltage difference. This means that there is no resistance. (Josephson won the Nobel Prize in 1973.)

1. It is known that when the current I is smaller than a critical current I_c, no voltage will be developed across the junction; that is the junction acts as if it has zero resistance. However the phases of the two superconductors will be driven apart to a constant phase difference!

2. When $I > I_c$, constant phase difference disappears, and there is a voltage develops across the junction.

Using the Josephson relations and Kirchhoff's voltage and current laws, we arrive at the following differential equation:

$$\frac{hC}{2e}\phi'' + \frac{h}{2eR}\phi' + I_c \sin\phi = I$$

where the constants are all positive. ϕ denotes the phase difference of the two superconductors. The physical meanings of them can be found in the textbook.

Next, as before, we perform substitutions to simplify the equation, and we consider the "overdamped limit". We obtain

$$\phi' = \frac{I}{I_c} - \sin\phi.$$

As we know from Section 4.3 that the solutions of the equation tend to a stable fixed point when $I \leq I_c$ and vary periodically when $I > I_c$. These agree with the above two listed observations.

One more interesting application of the DE is that we can find the *current-voltage curve*. According to the textbook

$$V = \frac{h}{2e}\phi'.$$

Hence the average of V is proportional to the average of ϕ'.

When $I \leq I_c$, all solutions ϕ converge to a fixed point. Therefore $\phi' = 0$. When $I > I_c$, ϕ is periodic with period

$$T = \frac{2\pi}{\sqrt{(I/I_c)^2 - 1}}.$$

And so the average of ϕ' is

$$\frac{1}{T}\int_0^T \phi' d\tau = \frac{1}{T}\int_0^{2\pi} d\phi = \frac{2\pi}{T} = \sqrt{(I/I_c)^2 - 1}.$$

Overall, we get a relation between the average of voltage and the current. Clearly it is not a smooth function (see Figure 4.6.4 in the textbook).

5 Two-dimentional flows: linear systems

5.1 Definitions and examples

A *two-dimensional linear system* with constant coefficients is a system of the form

$$\begin{aligned} x' &= ax + by, \\ y' &= cx + dy \end{aligned}$$

where a, b, c, d are some constants. Using the matrix notation, the equation can be written compactly into the following form:

$$\mathbf{x}' = A\,\mathbf{x}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The system is linear in the sense that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then so is the linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ for any $c_1, c_2 \in \mathbb{R}$.

One way to visualize the solution is to plot the graphs of $x(t), y(t)$ in one plane. However to emphasize the relation between $x(t)$ and $y(t)$, we plot them as moving trajectories in the (x, y) plane, in this context called the phase plane.

Example 5.1.1. *Solve the linear system $\mathbf{x}' = A\mathbf{x}$, where A is a diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$. Graph the phase portrait as a varies from $-\infty$ to ∞, showing the qualitatively different cases.*

Proof. The system is the same as

$$\begin{aligned} x' &= ax, \\ y' &= -y \end{aligned}$$

which shows that the two equations are uncoupled; there's no x in the y-equation and vice versa. Thus we get the solutions:

$$x(t) = x_0 e^{at}, \quad y(t) = y_0 e^{-t}.$$

In order to plot the trajectories in an $x - y$ coordinates for various values of initial data x_0, y_0 (called the *phase portrait of the system*), let us use the two equalities to get rid of t:

$$y^a x = y_0^a e^{-at} x_0 e^{at} = y_0^a x_0.$$

Then

$$y = cx^{-\frac{1}{a}} \quad \text{where } c = y_0 x_0^{-\frac{1}{a}}.$$

Then we plot the graphs of $y = cx^{-\frac{1}{a}}$ for various of c and separately for $a < -1$, $a = -1$, $a \in (-1, 0)$, $a = 0$ (the equation is $x = x_0$), and $a > 0$ (see Figure 5.1.5). When $c > 0$ and $a < -1$, the curve is concave and is connecting to the origin; when $c > 0$ and $a \in (-1, 0)$, the curve is convex and is connecting to the origin; when $a > 0$, general curves are away from the origin.

Next we also need to draw arrows on the curves indicating the flows' direction (how the solutions move as time increases). To do this we need to use the original expressions for $x = x(t)$ or $y = y(t)$. Now let us revisit the first case of $c > 0, a < -1$. We have $x(t)$ decays more rapidly than $y(t)$ as $t \to \infty$. On the other hand, if we look backwards along a trajectory ($t \to -\infty$) then the trajectories all become parallel to the faster decaying direction (here, the x-direction). This is because

$$\frac{dy}{dx} = \frac{y'}{x'} = -\frac{y}{ax} = -\frac{y_0}{ax_0} e^{-t-at}$$

which $\to 0$ as $t \to -\infty$. When $a = 0$, solutions are $x(t) \equiv x_0$ which is called *a line of fixed points*.

Stability of the origin. Notice that $x(t) \equiv 0$ and $y(t) \equiv 0$ is a solution, and so we can call the original point a fixed point. Now we discuss the stability of this fixed point for the different values of a. When $a < 0$, all trajectories converge to the origin as $t \to \infty$. Therefore, $\mathbf{x}_* = 0$ is called a stable node for $a < 0$. When $a > 0$, $\mathbf{x}_* = 0$ becomes unstable. Except the case when $x_0 = 0$, all trajectories move away from the origin eventually as $t \to \infty$. If we look at backwards in time, then except the case when $y_0 = 0$, all trajectories move away from the origin eventually as $t \to -\infty$. We call $\mathbf{x}_* = 0$ is called a saddle point.

□

Some definitions of stable fixed points. We say that $\mathbf{x}_* = 0$ is an *attracting* fixed point if all trajectories that start near $\mathbf{x}_*$ approach it as $t \to \infty$. It could be called *globally attracting* if it attracts all trajectories in the phase plane. So we can call $\mathbf{x}_* = 0$ globally attracting in the above example when $a < 0$.

Another stability we introduce is the *Lyapunov stability* (note that the spelling in the textbook is wrong!). We say that a fixed point $\mathbf{x}_*$ is Lyapunov stable if all trajectories that start sufficiently close to $\mathbf{x}_*$ remain close to it for all time. More precisely (the former definition), for every $\epsilon > 0$, there exists a $\delta > 0$ (probably $<< \epsilon$) such that, if $\|\mathbf{x}(0) - \mathbf{x}_*\| < \delta$, then $\|\mathbf{x}(t) - \mathbf{x}_*\| < \epsilon$ for all $t \geq 0$. Then in the case $a = 0$ in the example, $\mathbf{x}_* = 0$ is Lyapunov stable, but not attracting. When a fixed point is Lyapunov stable but not attracting, it is called *neutrally stable*.

Conversely, it's possible for a fixed point to be attracting but not Lyapunov stable (see Figure 5.1.6) (we had studied examples with phase portrait like Figure 5.1.6. Can you recall it?). Here $\theta = 0$ attracts all trajectories as $t \to \infty$, but it is not Lyapunov stable; because there are trajectories that start infinitesimally close to $\theta = 0$ but go on a very large excursion before returning to $\theta = 0$.

Finally if a fixed point is both Lyapunov stable and attracting, we'll call it *stable* or *aymptotically stable*. if a fixed point is neither Lyapunov stable nor attracting, we call it *unstable*.

5.2 Classification of linear system

Let me recall that we learned the characteristic equation method to find general solutions to

$$x'' + bx' + cx = 0.$$

If you call $y := x'$. Then the equation becomes the 2-dimensional linear system:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now let us consider general matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It was discussed in Math 20B that if we plug in the following function valued vector in the the equation:

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

where λ is a constant to be determined, and $\mathbf{v}$ is a constant vector to be determined, we get $\lambda\mathbf{v} = A\mathbf{v}$. This equality holds as long as λ is an eigenvalue and $\mathbf{v}$ is an associated eigenvector.

Example 5.2.1. *Solve the initial value problem*

$$x' = x + y, y' = 4x - 2y,$$

subject to the initial condition $(x_0, y_0) = (2, -3)$.

Solution. In this example, the matrix $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$. First let us find out the eigenvalues: we compute

$$\det(\lambda - A) = 0.$$

This leads to $\lambda^2 + \lambda - 6 = 0$ which is often called *the characteristic equation*. Solving the equation gives two eigenvalues

$$\lambda_1 = 2, \quad \lambda_2 = -3.$$

Since the two eigenvalues are different, we need to find one eigenvector associated to each eigenvalue. Namely we find $\mathbf{v}_i$ such that

$$(\lambda_i I - A)\mathbf{v} = 0.$$

We get

$$\mathbf{v}_1 = (1,1)^T, \quad \mathbf{v}_2 = (1,-4)^T \quad (\text{ here the up-script } T \text{ denotes transpose}).$$

Then the general solutions are linear combinations of $e^{\lambda_i t}\mathbf{v}_i$:

$$c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Finally we use the initial data to solve for the constants c_1, c_2 to get

$$x(t) = e^{2t} + e^{-3t}, \quad y(t) = e^{2t} - 4e^{-3t}.$$

□

Example 5.2.2. *Draw the phase portrait for the above system, and classify the stability of the fixed points* 0 *(the origin).*

Solution. The system has eigenvalues $\lambda_1 = 2, \lambda_2 = -3$. Hence the first eigensolution grows exponentially, and the second eigensolution decays. This means the origin is a saddle point. To find out the shape more precisely, we plot the trajectories of (on $x - y$ plane)

$$c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

with either $c_1 = 0$ or $c_2 = 0$. We get two lines: one is spanned by v_1 and the other is spanned by v_2. The former is often called the *unstable manifold* (since $\lambda_1 > 0$), and the latter is called the *stable manifold*. Figure 5.2.2 in the textbook shows the phase portrait

From the definition, since most trajectories diverge to ∞ as $t \to \infty$, the origin is unstable. □

Example 5.2.3. *Sketch the phase portrait for the case* $\lambda_2 < \lambda_1 < 0$.

Solution. See Figure 5.2.3. As can be seen from the figure, the origin is globally attracting (it is also lyapunov stable and so it is stable). □

Example 5.2.4. *What happens if the eigenvalues are complex?*

Solution. The solutions are linear combinations of the following two linearly independent solutions:

$$e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b},$$

$$e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b}$$

where $\alpha \pm i\beta$ are the eigenvalues of the matrix, and $\mathbf{a} \pm i\mathbf{b}$ are the corresponding eigenvectors.

The fixed point is either a *center* or a *spiral* (Figure 5.2.4). When $\alpha = 0$, we have centers. Note that centers are neutrally stable, since nearby trajectories are neither attracted to nor repelled from the fixed point. When $\alpha > 0$ we have growing oscillations, that is a unstable spiral (all trajectories, except the fixed point itself, converge to ∞ as $t \to \infty$). When $\alpha < 0$, we have decaying oscillations, that is a stable spiral.

For both centers and spirals, it's not hard to determine whether the rotation is clockwise or counterclockwise; just compute a few vectors in the vector field and the sense of rotation should be obvious. □

Example 5.2.5. *What happens if the eigenvalues are equal?*

Solution. As discussed before, there are two cases. If there are two independent eigenvectors, then the general solutions are

$$c_1 e^{\lambda t}\mathbf{e}_1 + c_2 e^{\lambda t}\mathbf{e}_2$$

where $\mathbf{e}_1, \mathbf{e}_2$ denote x, y directions respectively. They span the plane and actually every vector is an eigenvector with this same eigenvalue λ. Then if $\lambda \neq 0$, all trajectories are straight lines through the origin and the fixed point is a star node (Figure 5.2.5).

The other possibility is that there's only one eigenvector (more accurately, the eigenspace is one-dimensional). The general solutions are

$$c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)$$

where $\mathbf{v}_1$ is the eigenvector, and $\mathbf{v}_2$ is one particular vector that is perpendicular to $\mathbf{v}_1$. For example,

$$\mathbf{x}' = A\mathbf{x} \text{ with } A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

We can use the undetermined method to obtain the general solutions

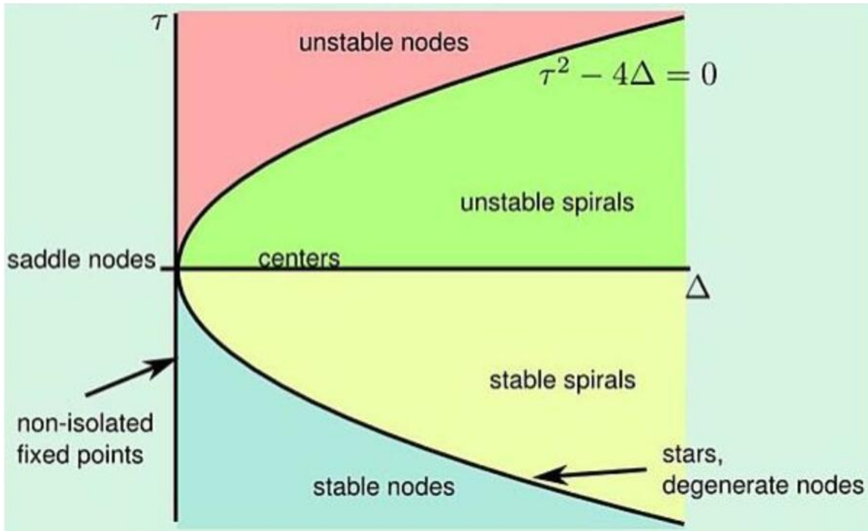
$$c_1 e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda t}\begin{pmatrix} t \\ 1 \end{pmatrix}.$$

In this case, the fixed point is a degenerate node. A typical phase portrait is shown in Figure 5.2.6. In the figure is for $\lambda > 0$, and we see that all trajectories are parallel to the eigendirection $\mathbf{v}_1$ as $t \to \infty$. This is clear from the explicit formula.

The textbook also explained how the name "degenerate" comes from. You can take a look at that part if you are interested. □

In the following figure we summarize the classification of fixed points (of 2-dimensional linear system): The axes are the trace τ and the determinant Δ of the matrix A. They are real numbers. All of the information in the diagram is implied by the following formulas:

$$\lambda_{1,2} := \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}, \quad \Delta = \lambda_1\lambda_2, \quad \tau = \lambda_1 + \lambda_2.$$



5.3 Love affairs

This section is devoted to an example of the 2-dimensional linear system.

Let us consider the following "strange" love story: Romeo is in love with Juliet. Suppose the more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

Let $R(t)$ denotes Romeo's love / hate for Juliet at time t, and $J(t)$ denotes Juliet's love / hate for Romeo at time t. Positive values of R, J signify love, negative values signify hate. This story yields the equation

$$R' = aJ, \quad J' = -bR$$

where a, b are positive, to be consistent with the story. The sad outcome of their affair is a never-ending cycle of love and hate; the governing system has a center at $(R, J) = (0, 0)$. So they continually embrace the unhealthy relation: loving and hating each other periodically and they can never be far apart or be too close. Probably you can find this sad story in movies and even in the real life..

Let us think about what happens if we have the following system instead?

$$R' = aR + bJ, \quad J' = bR + aJ$$

with $a < 0, b > 0$. Here a is a measure of cautiousness (they each try to avoid throwing themselves at the other) and b is a measure of responsiveness (they both get excited by the other's advances).

To solve the equation, we need to find out the eigenvalues and the eigenvectors. Then are

$$\lambda_1 = a + b, \quad \mathbf{v}_1 = (1, 1)^T;$$

$$\lambda_2 = a - b, \quad \mathbf{v}_1 = (1, -1)^T.$$

Clearly $\lambda_2 < 0$, if $\lambda_1 = a + b > 0$ we have a saddle point, and if $\lambda_1 < 0$ we have a stable node.

6 Phase plane

In this section, we begin to study nonlinear systems. Again analyzing the fixed points will be the key to understanding the systems. (Fixed points are always important to a dynamic system.) There are various different backgrounds for the problem that we will study, ranging from biology (competition between two species), to physics (conservative systems, reversible systems, and the pendulum).

6.1 Phase portrait

The general 2-dimensional autonomous first-order system is of the form

$$x_1' = f_1(x_1, x_2), \quad x_2' = f_2(x_1, x_2).$$

We can also view (f_1, f_2) as a given *vector field* on the space $\mathbb{R}^2$, and then we call the corresponding solutions the *flows* along the vector field. In some content, people can also call the corresponding solutions as *integral curves* along the vector field or the *streamlines* of the vector field.

This system can be written more compactly in vector notation as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \quad \text{where } \mathbf{x} = (x_1, x_2),\ \mathbf{f} = (f_1, f_2).$$

The domain of $\mathbf{x}$ is called the phase plane (which is often the whole domain $\mathbb{R}^2$ in the course), and $\mathbf{f}(\mathbf{x})$ (or $\mathbf{x}'$) is the vector field at $\mathbf{x}$.

For nonlinear systems, there's typically no hope of finding the trajectories analytically (this is also the case for most general ODE and PDE). Even when explicit formulas are available, they are often too complicated to provide much insight. Instead we will try to determine the qualitative behavior of the solutions. Our goal is to find the system's phase portrait directly from the properties of $\mathbf{f}(\mathbf{x})$.

Example 6.1.1. *Consider the system $x' = x + e^{-y}, y' = -y$. Use qualitative arguments to obtain information about the phase portrait.*

Solution. First we find the fixed points by solving $x' = y' = 0$. Then there is only one fixed point: $(x_*, y_*) = (-1, 0)$. To study the stability of the fixed points, we look at the linearized equation near the point $(-1, 0)$. Let us call $z = x + 1$, in which way we made the origin $(0, 0)$ the fixed point for the system of (z, y). Since

$$e^{-y} = 1 - y + O(y^2),$$

the linearized system is

$$z' = z - y, \quad y' = -y.$$

The associated eigenvalues of the linearized system are $1, -1$. According to the linearization theorem (Hartman-Grobman theorem):

if the matrix has no eigenvalue with real part equal to zero, and $\tau^2 - 4\Delta \neq 0$, then the classification of the fixed point of the original system is (locally) the same the one of the fixed point of the linearized system.

This yields that $(-1, 0)$ is (locally) a saddle point.

However this is not enough for us to gain the full picture of the system. To sketch the phase portrait, it is helpful to plot the *nullclines*, defined as the curves where either $x' = 0$ or $y' = 0$. The nullclines indicate where the flow is purely horizontal or vertical. Note that $y' = 0$ yields $y = 0$. Thus along this line, the flow is to the right where $x' = x + 1 > 0$, that is where $x > -1$, and the flow is to the left where $x < -1$. Similarly, the flow is vertical where $x' = x + e^{-y} = 0$, which occurs on the curve shown in Figure 6.1.3. On the upper part of the curve where $y > 0$, the flow is downward, and it is upward where $y < 0$.

The nullclines also partition the plane into four regions. We know the signs of x', y' in the four regions and these give us a rough picture of how the vector field looks like. After plotting out the vector field, we see that the fixed point is indeed a saddle point (see Figure 6.1.4). $\square$

6.2 Existence, and uniqueness

The following theorem grantees that general nonlinear system that we are looking at has at least a solution near its initial data.

Theorem 6.1. *Consider the initial value problem* $\mathbf{x}' = \mathbf{f}(x), \mathbf{x}(0) = \mathbf{x}_0$. *Suppose that* $\mathbf{f}$ *is continuous and that all its partial derivatives are continuous for* $\mathbf{x}$ *in some open connected set* $D \subseteq \mathbb{R}^n$. *Then for* $\mathbf{x}_0 \in D$, *the initial value problem has a solution* $\mathbf{x}(t)$ *on some time interval* $(-\tau, \tau)$ *about* $t = 0$, *and the solution is unique.*

The uniqueness part of the theorem has an important corollary: different trajectories never intersect. If two trajectories did intersect, then there would be two solutions starting from the same point (the crossing point), and this would violate the uniqueness part of the theorem. In more intuitive language, a trajectory can't move in two directions at once.

In two-dimensional phase spaces, the result that trajectories can't intersect has a strong geometric consequences. For example, suppose there is a closed orbit C in the phase plane. Then any trajectory starting inside C is trapped in there forever (Figure 6.2.2).

What is the fate of such a bounded trajectory? If there are fixed points inside C, then the trajectory might eventually approach one of them, or the trajectory is a closed orbit itself. If there aren't any fixed points, the *Poincaré-Bendixson theorem* states that if a trajectory is confined to a closed, bounded region and there are no fixed points in the region, then the trajectory must eventually approach a closed orbit.

6.3 Fixed points and linearization

In this section we elaborate the *linearization* technique that we have already used. The technique is used to determine the phase portrait near a fixed point by that of a corresponding linear system in some robust cases.

Consider

$$x' = f(x, y), \quad y' = g(x, y),$$

and suppose that (x_*, y_*) is a fixed point i.e. $f(x_*, y_*) = g(x_*, y_*) = 0$. Then we shift the fixed point to the origin by calling

$$u = x - x_*, \quad v = y - y_*.$$

Then the original equation can be transformed to

$$\begin{aligned} u' = x' &= f(x_* + u, y_* + v) \\ &= f_x(x_*, y_*)u + f_y(x_*, y_*)v + O(u^2, v^2, uv) \end{aligned}$$

where we used Taylor's expansion and $f(x_*, y_*) = 0$, and similarly

$$v' = g_x(x_*, y_*)u + g_y(x_*, y_*)v + O(u^2, v^2, uv).$$

Here

$$f_x(x_*, y_*) := \frac{\partial f}{\partial x}(x_*, y_*)$$

(sometimes we also use the notation $\partial_x f(x_*, y_*)$, and sometimes we drop (x_*, y_*) for simplicity), and similarly for f_y, g_x, g_y. Here $O(u^2, v^2, uv)$ denotes some quadratic terms (lower order terms). Since the quadratic terms are comparably much smaller than the first order terms when u, v are small, we get the linearized system

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x_*, y_*)} \begin{pmatrix} u \\ v \end{pmatrix}.$$

The matrix is call the *Jacobian matrix* of (f, g) at point (x_*, y_*).

We need to address the question that does the linearized system give a qualitatively correct picture of the phase portrait near (x_*, y_*). The answer is yes, as long as the fixed point for the linearized system is not one of the borderline cases discussed in Section 5.2. In other words, if the linearized system predicts a saddle, node, or a spiral, then the fixed point really is a saddle, node, or spiral for the original nonlinear system with the same stability. This is the linearization theorem we mentioned before.

Example 6.3.1. *Find all the fixed points of the system $x' = -x + x^3, y' = -2y$, and use linearization to classify them.*

Solution. To find the fixed points, we solve for the following two simultaneously:

$$-x + x^3 = 0, \quad -2y = 0.$$

We get three fixed points: $(0, 0)$, $(\pm 1, 0)$. The Jacobian matrix at a general point (x, y) is

$$A = \begin{pmatrix} \partial_x x' & \partial_y x' \\ \partial_x y' & \partial_y y' \end{pmatrix} = \begin{pmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then we can compute the Jacobian matrix at those fixed points. The Jacobian matrices are all diagonalize matrices which is not surprising since the two equations in the system are uncoupled. After computing the eigenvalues, we get that $(0, 0)$ is a stable node, and $(\pm 1, 0)$ are two saddle points.

Because stable nodes and saddle points are not borderline cases, we can be certain that the fixed points for the nonlinear system have been predicted correctly.

The phase portrait is given in Figure 6.3.1 in the textbook which can be obtained by studying the nullclines of the system. □

The borderline cases (centers, degenerate nodes, stars, or non-isolated fixed points) are much more delicate. They can be altered by small nonlinear terms. The next example shows that small nonlinear terms can change a center into a spiral.

Example 6.3.2. *Consider*

$$x' = -y + ax(x^2 + y^2), \quad y' = x + ay(x^2 + y^2)$$

where $a > 0$ is a parameter. Draw the phase portrait at the origin.

Solution. Clearly the origin is the only fixed point. The corresponding linearized system is

$$x' = -y, \quad y' = x.$$

Thus in view of the linearized system, the origin is a center.

Now we show that it is actually like a unstable spiral for the original nonlinear system. Use polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Now we compute r', θ'. Since $x^2 + y^2 = r^2$,

$$rr' = xx' + yy' = a(x^2 + y^2)^2 = ar^4$$

which implies that $r' = ar^3$. In the second equality we used the nonlinear system. Next since $\theta = \arctan\frac{y}{x}$,

$$\theta' = \frac{xy' - yx'}{x^2 + y^2} = 1,$$

by the system again.

Thus in polar coordinates the original system becomes

$$r' = ar^3, \quad \theta' = 1.$$

The solution with initial data (r_0, θ_0) equals

$$r(t) = (r_0^{-2} - 2at)^{-\frac{1}{2}}, \quad \theta(t) = \theta_0 + t.$$

Hence clearly, as t increases, $r(t) \to \infty$, and at the same time it rotate round the origin counter-clockwisely with constant speed. □

Also let us mention that stars and degenerate nodes can be altered by small nonlinearities, but unlike centers, their stability doesn't change.

We can classify fixed points according to their stability (and neglect the shapes like stars and degenerate nodes).

Robust cases:

1. Repellers (also called sources): both eigenvalues have positive real part.
2. Attractors (also called sinks): both eigenvalues have negative real part.
3. Saddles: one eigenvalue is positive and one is negative.

(Stars and degenerate nodes are also included in the above depending only on the stability of fixed points.)

Marginal cases:

1. Centers: both eigenvalues are pure imaginary.
2. Higher-order and non-isolated fixed points: at least one eigenvalue is zero.

Thus, from the point of view of stability, the marginal cases are those where at least one eigenvalue satisfies $\text{Re}(\lambda) = 0$.

6.4 Rabbits versus sheep

In this section we look at the well-known Lotka-Volterra model of competition between two species. Suppose that both species (say rabbits versus sheep) are competing for the same food supply (grass) and the amount available is limited. We mainly consider the following two effects:

1. Each species would grow to its carrying capacity in the absence of the other (this is like the logistic growth model in Section 2.3; we assume $(\ln x)' = \frac{x'}{x}$ is proportional to the effect coming from the limitation of the food and competition between the other species). Rabbits has a larger reproduce rate.
2. When rabbits and sheep encounter each other, sheep is more likely to nudge the rabbit aside from the food.

Let us use $x(t)$ to denote the population of rabbits, and $y(t)$ the population of sheep. A specific model that incorporates these assumptions is

$$x' = x(3 - x - 2y), \quad y' = y(2 - x - y).$$

We call $f = x(3 - x - 2y), g = y(2 - x - y)$.

It is not hard to get the following four fixed points: $(0, 0)$, $(0, 2)$, $(3, 0)$, and $(1, 1)$. To classify them, we compute the Jacobian and get

$$A := \mathbf{J}(f, g) = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Now we plug in the coordinates' values of fixed points to get that: $(0,0)$ is a unstable node; $(0,2)$, $(3,0)$ are stable nodes; $(1,1)$ is a saddle. The fixed points are all robust.

Then you can try to find out the eigenvectors which will tell how the phase portrait look like locally near the fixed points. And then connect the vector fields smoothly to obtain the whole phase portrait picture. This is done in the textbook.

The other way is to check nullclines. We see that they are exactly $x = 0, 3 - x - 2y = 0, y = 0, 2 - x - y = 0$. After figuring out the directions of vector fields on the nullclines, we are also able to sketch the phase portrait.

The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold (of the saddle point) lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. We call the region that is below the stable manifold *basin of attraction* of fixed point $(3,0)$. Then for any initial data $\mathbf{x}_0$ in this region, the corresponding solutions eventually converge to $(3,0)$ (rabbits survive and sheep die out).

6.5 Conservative systems

Here by conservative system we consider a particle system where the only force acting on the particle comes from some *potential energy* V. The equation is

$$mx'' + \frac{dV}{dx} = 0.$$

It can be shown that the *total energy*

$$E = \frac{1}{2}m(x')^2 + V(x)$$

is preserved, by which we mean that if $x(t)$ is a solution, then $E(x(t))$ is a constant function of t. This can be easily checked by differentiating $E(x(t))$ with respect to t and obtain that $\frac{d}{dt}E(x(t)) = 0$. Here $\frac{1}{2}m(x')^2$ is called the *kinetic energy*.

The most striking fact about conservative system is the following theorem:

Theorem 6.2. *(Nonlinear centers for conservative systems) Consider the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. Suppose there exists a conserved quantity $E(x)$ and suppose that x_* is an isolated fixed point (i.e., there are no other fixed points in a small neighborhood surrounding x_*). If x_* is a local minimum of E, then all trajectories sufficiently close to x_* are closed.*

I will explain an intuitive idea of the theorem at the end of the following example.

Example 6.5.1. *Consider a particle of mass $m = 1$ moving in a double-well potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. Find and classify all the equilibrium points for the system. Then plot the phase portrait and interpret the results physically.*

Solution. The force is $-dV/dx = x - x^3$, so the equation of motion is $x'' = x - x^3$. Letting $y = x'$ (which represents the velocity), we get the following first-order system

$$x' = y, \quad y' = x - x^3.$$

There are three fixed points (x_*, y_*): $(0,0)$ and $(\pm 1, 0)$. To classify these fixed points we compute the Jacobian

$$A = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}.$$

The linear approximation predicts that $(0,0)$ is a saddle, and $(\pm 1, 0)$ are two centers. We have discussed that centers are not robust under small perturbations i.e. we cannot trust the linearization method if the fixed points are predicted to be a center. However conservative system is an exception, and this is due to Theorem 6.2. Because in the example we can check that $(\pm 1, 0)$ is indeed the minimum of E. The theorem implies that the two fixed points are centers.

We can actually connect the phase portrait to the motion of an undamped particle in a double-well potential. The fixed point $(1,0)$ corresponds to the scenario that the particle stays at one of the two bottom areas of the double-well potentials. While if $y \neq 0$ and $|y|$ is small, then it corresponds to that the particle has a small speed. Then the particle will not stay at the bottom anymore, but instead it will oscillate in the bottom area. The oscillation is represented by a closed orbits in the phase portrait, and we have a center there. The large orbits represent more energetic oscillations that repeatedly take the particle back and forth over the hump. If the velocity of the particle is just right, it can climb up the hill and stay at the local maximum of the double-well potential. And this corresponds to the stable manifold in the phase portrait that is connecting to the saddle point. □

Example 6.5.2. *Suppose that E is not a constant on any open set. Show that conservative system cannot have any attracting fixed points.*

The proof is given in the textbook in Example 6.5.1.

6.6 Reversible systems

By reversible system we mean that the dynamics look the same when we run the system backward in time. People also refer to it as the *time-reversal symmetry*.

Firstly for a first-order 2-dimensional system, we say that it is reversible, if it is invariant under $t \to -t$ and $(x, y) \to (x, -y)$. This definition seems a little bit artificial at first glance. Let us use the following general example to explain.

Example 6.6.1. *Any mechanical system of the form $mx'' = F(x)$ is symmetric under time reversal.*

Solution. By introducing $y = x'$, the system can be rewritten into

$$x' = y, \quad y' = \frac{1}{m} F(x).$$

Then if we change the variables: $t \to -t$, $(x, y) \to (x, -y)$, both equations stay the same. Hence if $(x(t), y(t))$ is a solution, then so is $(x(-t), -y(-t))$. Equivalently if denoting $\pi(x, y) = (x, -y)$, we have

$$\frac{d}{dt}(\pi \mathbf{x}(-t)) = -(x', -y')(-t) = -\pi \mathbf{f}(\mathbf{x}(-t)) = \left(-y(-t), \frac{F(x(-t))}{m}\right)^T = \mathbf{f}(\pi(\mathbf{x}(-t))).$$

□

Reversible systems are different from conservative systems, but they share some similar properties. For instance, centers are robust in reversible systems as well.

Theorem 6.3. *Suppose the origin $x_* = 0$ is a linear center for the continuously differentiable system*

$$x' = f(x, y), \quad y' = g(x, y), \tag{2}$$

and suppose that the system is reversible. Then all trajectories that are sufficiently close to the origin, are closed curves.

Here we remark that in order to have the equation (2) being reversible, we need f to be odd in y, and g to be even in y (i.e. $f(x, y) = -f(x, -y)$, $g(x, y) = g(x, -y)$). Can you prove it?

The key idea of the proof of the theorem is this: Consider a trajectory that starts on the positive x-axis near the origin. Due to the dominant influence of the linear center, if sufficiently near the origin, the trajectory must intersect the negative x-axis. By reflecting the trajectory across the x-axis, and changing the sign of t, we obtain a twin trajectory with the same endpoints but with its arrow reversed (Figure 6.6.3). This makes the origin a center.

Example 6.6.2. *Show that the system*

$$x' = y - y^3, \quad y' = -x - y^2$$

has a nonlinear center at the origin, and plot the phase portrait.

Solution. Firstly it is easy to check that the system is reversible, since the equations are invariant under the transformation $t \to -t$, $(x, y) \to (x, -y)$. Next we compute the Jacobian at the origin which is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and so the origin is a center for the linearized system. By Theorem 6.3, the origin is a nonlinear center.

The other fixed points of the system are $(-1, 1)$ and $(-1, -1)$ and they are saddle points, as is easily checked by computing the linearization. The phase portrait is given in Figure 6.6.4. Notice that in the picture the trajectories above the x-axis have twins below the x-axis, with arrows reversed. □

In general there is a definition of reversibility which extends nicely to higher-order systems. Suppose there is a map π in the phase space, and it satisfies $\pi \circ \pi = id$. Then a dynamic system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is time reversible, if it is invariant under the change of variables $(t, \mathbf{x}) \to (-t, \pi(\mathbf{x}))$. For example in our previous definition, the map can be taken to be

$$\pi(x, y) = (x, -y).$$

We have the following example

Example 6.6.3. *Show that the system*

$$x' = -2\cos x - \cos y, \quad y' = -2\cos y - \cos x$$

is reversible, but not conservative. Then plot the phase portrait.

Solution. The system is invariant under the change of variables $t \to -t$, $(x, y) \to (-x, -y)$. Hence the system is reversible, with $\pi(x, y) = (-x, -y)$. Due to this symmetry, we know that every trajectory has its twin trajectory that is no longer symmetric about the x-axis, but is symmetric about $x = y$.

The idea of showing that the system is not conservative, is to show that it has an attracting fixed point. We will do it below. It is also worthy thinking about whether or not we can find a unstable node to prove this since it is like the "reverse" of attracting fixed point.

Now we find the fixed point which is amount to solve $2\cos x = -\cos y, 2\cos y = -\cos x$. We get $\cos x = \cos y = 0$. Then by periodicity we get the following four groups of fixed points:

$$(\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi), \quad (-\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi), \quad (\frac{\pi}{2}+2k\pi, -\frac{\pi}{2}+2k\pi), \quad (-\frac{\pi}{2}+2k\pi, -\frac{\pi}{2}+2k\pi)$$

with $k \in \mathbb{N}$ (all integers). Since the Jacobian matrix is $\begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin y \end{pmatrix}$, the four groups of fixed points are unstable nodes, saddles, saddles, and stable nodes respectively.

□

Finally let me mention that symmetry is always very important in both mathematics and physics. For example in physics, Noether's theorem states that every differentiable symmetry of the action of a physical system has a corresponding conservation law.

6.7 Pendulum

(Skipped)

6.8 Index theory

There are a lot of index theory in mathematics and physics. Here we introduce one of them.

Suppose that $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is a smooth vector field on the phase plane. Consider a closed curve C which is not necessarily a trajectory. We might only take simple closed curve (i.e. C doesn't intersect itself) and that it doesn't pass through any fixed points of the system. Then at each point x on C, the vector field $\mathbf{x}' = (x', y')$ makes a well-defined angle $\phi = \tan^{-1}(y'/x')$ with the positive x-axis. As x moves counterclockwise around C, the angle ϕ changes continuously since the vector field is smooth. Notice that when x returns to its starting place, ϕ returns to its original direction. Hence, after going over C once, ϕ changed by $2k\pi$ (denoted as $[\phi]_C$) with $k \in \mathbb{Z}$. Finally we **define** this k (which can be written as $\frac{1}{2\pi}[\phi]_C$) to be the index of the closed curve C with respect to the vector field $\mathbf{f}$.

Example 6.8.1. *Suppose the vector field is $\mathbf{f}(x, y) = (x, y)$. Show that for any simple closed curve C that contains the origin, $I_C = 1$. Show that if C does not contain the origin, then $I_C = 0$.*

Example 6.8.2. *Given the vector field in Figure 6.8.4a, compute I_C.*

Computing the index is useful. It encodes some important properties of the dynamic system. For instance if for a system, there is a closed curve C such that $I_C = 1$. Then there must be a fixed point inside the curve. This statement will be clear after we introduce the properties of index. But before that we look at the following example.

Example 6.8.3. *Show that the system $x' = xe^{-x}, y' = 1 + x + y^2$ has no closed orbits.*

Solution. Clearly the system has no fixed point. Now if there is a closed orbit, then taking the orbit as C, we see that the vector field is always the tangent vector on C. As a point going over the closed curve, the tangent rotate by 2π. Therefore the definition of the index yields that $I_C = 1$. This implies that there is at least one fixed point inside C which is a contradiction. □

Now we introduce some properties of index:

1. Suppose that C can be continuously deformed into C' without passing through a fixed point. Then $I_C = I_{C'}$. (Continuous deformation is one the most important and basic idea in topology. When continuously deforming the curve, I_C also varies continuously. However since I_C is always an integer, it has to be stay the same.)

2. If C doesn't enclose any fixed points, then $I_C = 0$. (You can use (1) and deform C to be a tiny circle along which the vector field is almost the same as a constant vector field. For the later $I_C \approx 0$ and so since I_C is an integer, $I_C = 0$.)

3. If we reverse all the arrows in the vector field by changing $t \to -t$, the index is unchanged. (All angles just change from ϕ to $\phi + \pi$.)

4. Suppose that the closed curve C is actually a trajectory for the system, i.e., C is a closed orbit. Then $I_C = 1$.

Suppose x_* is an isolated fixed point. Then the index I of x_* is defined as I_C, where C is a small circle which encloses x_*. By property (1) above, I_C is independent of the C we choose as long as it does not intersect or enclose any other fixed point, and so it is a property of x_* alone. We will call it the index $I = I(x_*)$ of the fixed point x_*.

Example 6.8.4. *Find the index of a stable node, an unstable node, and a saddle.*

Solution. They are $1, 1, -1$. □

Theorem 6.4. *If a closed curve C surrounds n isolated fixed points $x_1, x_2, ..., x_n$ and there is no fixed point on C, then*

$$I_C = I_1 + I_2 + ... + I_n$$

where I_k is the index of x_k.

As a corollary of the theorem, we have any closed orbit in the phase plane must enclose fixed points whose indices sum to $+1$.

Example 6.8.5. *Show that closed orbits are impossible for the rabbit vs. sheep system*

$$x' = x(3 - x - 2y), \quad y' = y(2 - x - y)$$

with $x \geq 0, y \geq 0$.

The proof is given in the textbook.