Name: _____

PID: _____

1. Please justify all answers and show your work.
2. You may work with classmates, but all answers must be your own.

1. Make sure that when submitting your homework to Gradescope you indicate which page of your homework each question is on.
2. Compute $\cos^{-1}(\sin(-5\pi/3))$ without a calculator. Show your work.

Solution: We know that $-\frac{5\pi}{3}$ is coterminal to $-\frac{5\pi}{3} + 2\pi = \frac{\pi}{3}$, so

$$\sin\left(-\frac{5\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

Therefore,

$$\cos^{-1}\left(\sin\left(-\frac{5\pi}{3}\right)\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

since $\frac{\pi}{6}$ is an angle in the interval $[0, \pi]$ (which is the range of $\cos^{-1}$ = the domain we restricted cosine to in order to make it one-to-one) which satisfies $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

3. Find a simplified expression (which doesn't involve trig functions) for the function $f(x) = \cos\big(\tan^{-1}(4x+1)\big)$. (This problem originally said "Find a simplified expression (which doesn't involve trig functions) for the function $f(x) = \sin\big(\tan^{-1}(4x+1)\big)$". You may solve either problem.)

Solution: Let $\theta = \tan^{-1}(4x+1)$, so that $\tan(\theta) = 4x+1$ and θ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, which tells us that

$$\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) = 4x+1.$$

and therefore

$$\sin(\theta) = \cos(\theta)(4x+1).$$

We can then compute

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 && \text{by the Pythagorean Identity} \\
(\cos(\theta)(4x+1))^2 + \cos^2(\theta) &= 1 && \text{since } \sin(\theta) = \cos(\theta)(4x+1) \\
\cos^2(\theta)(4x+1)^2 + \cos^2(\theta) &= 1 \\
\cos^2(\theta)((4x+1)^2+1) &= 1 && \text{factor } \cos(\theta) \text{ out of the left side} \\
\cos^2(\theta) &= \frac{1}{(4x+1)^2+1} \\
\cos(\theta) &= \pm\sqrt{\frac{1}{(4x+1)^2+1}}.
\end{aligned}$$

Now, since θ is in either the first or fourth quadrant (by the definition of $\tan^{-1}$ - see the first sentence of the solution), this tells us that $\cos(\theta)$ is positive, so we should pick the plus sign above. Therefore

$$\cos(\theta) = \sqrt{\frac{1}{(4x+1)^2+1}} = \frac{1}{\sqrt{(4x+1)^2+1}}.$$

4. Compute $\tan\big(\cos^{-1}(3/8)\big)$ without a calculator. Show your work.

Solution: We will solve this problem algebraically using the Pythagorean identity; you can also solve this problem by drawing a picture. Let $\theta = \cos^{-1}(3/8)$, so $\cos(\theta) = 3/8$ and θ is in the interval $[0, \pi]$. If we can compute $\sin(\theta)$, then we can compute $\tan(\theta)$ by using the fact that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

We have

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 && \text{by the Pythagorean identity} \\
\sin^2(\theta) + (3/8)^2 &= 1 && \text{since } \cos(\theta) = 3/8 \\
\sin^2(\theta) + 9/64 &= 1 \\
\sin^2(\theta) &= 55/64 \\
\sin(\theta) &= \pm\sqrt{55/64}.
\end{aligned}$$

Do we pick the plus sign or minus sign? Since θ is in the interval $[0, \pi]$ (so it is in either quadrant 1 or quadrant 2), and $\cos(\theta) = 3/8$ is positive (so θ is in either quadrant 1 or quadrant 4), this tells us that θ must be in the first quadrant. Therefore $\sin(\theta)$ is positive as well, so

$$\sin(\theta) = \sqrt{\frac{55}{64}} = \frac{\sqrt{55}}{8}.$$

Therefore,

$$\tan\big(\cos^{-1}(3/8)\big) = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sqrt{55}/8}{3/8} = \frac{\sqrt{55}}{3}.$$

5. Find all solutions to the equation $\sin(\theta) = 2/9$ without a calculator. Leave your answer in terms of inverse trigonometric functions. Show your work.

Solution: One solution is $\sin^{-1}(2/9)$. This solution is in the first quadrant. There will be another solution in the second quadrant, where sine is also positive, which has the same reference angle as $\sin^{-1}(2/9)$ (which is equal to its own reference angle since it is in the first quadrant). The angle $\pi - \sin^{-1}(2/9)$ is in the second quadrant and has reference angle equal to $\sin^{-1}(2/9)$ and is therefore the other solution we are looking for. To get all solutions we add all integer multiples of 2π to these two solutions. Therefore, the full set of solutions to the equation $\sin(\theta) = 2/9$ consists of all angles of the form

$$\theta = \sin^{-1}(2/9) + k(2\pi), \text{ where } k \text{ is an integer}$$
$$\text{or } \theta = (\pi - \sin^{-1}(2/9)) + k(2\pi), \text{ where } k \text{ is an integer.}$$

6. Find all solutions to the equation $3\cos(3x+5) = -7/5$ without a calculator. Leave your answer in terms of inverse trigonometric functions. Show your work.

Solution: We first divide both sides of the equation by 3, to get $\cos(3x+5) = -\frac{7}{15}$. We then make the substitution $u = 3x+5$ and solve the equation $\cos(u) = -\frac{7}{15}$. One solution to this equation is given by $u = \cos^{-1}\left(-\frac{7}{15}\right)$. This solution is in the second quadrant, and there will be another solution in the third quadrant (where cosine is also negative) which has the same reference angle as $\cos^{-1}\left(-\frac{7}{15}\right)$. This second solution is $2\pi - \cos^{-1}\left(-\frac{7}{15}\right)$. We get all solutions to the equation $\cos(u) = -\frac{7}{15}$ by adding integer multiples of 2π (which is the period of cosine). Therefore, the solutions to $\cos(u) = -\frac{7}{15}$ are all angles of the form

$$u = \cos^{-1}\left(-\frac{7}{15}\right) + k(2\pi), \text{ where } k \text{ is an integer}$$
$$\text{or } u = \left((2\pi - \cos^{-1}\left(-\frac{7}{15}\right)\right) + k(2\pi), \text{ where } k \text{ is an integer.}$$

To get solutions in terms of x, we need to undo the substitution we made earlier:

$$3x+5 = u = \cos^{-1}\left(-\frac{7}{15}\right) + k(2\pi), \text{ where } k \text{ is an integer}$$
$$\text{or } 3x+5 = u = \left(2\pi - \cos^{-1}\left(-\frac{7}{15}\right)\right) + k(2\pi), \text{ where } k \text{ is an integer.}$$

Solving the above equations for x, we see that the solutions to $3\cos(3x+5) = -\frac{7}{5}$ are all angles of the form

$$x = \frac{\cos^{-1}\left(-\frac{7}{15}\right) + k(2\pi) - 5}{3}, \text{ where } k \text{ is an integer}$$
$$\text{or } x = \frac{\left(2\pi - \cos^{-1}\left(-\frac{7}{15}\right)\right) + k(2\pi) - 5}{3}, \text{ where } k \text{ is an integer.}$$

7. Find all solutions to the equation $5\sin\left(\frac{\pi}{3}t\right) - 1 = -4$ that are in the interval $[0, 12]$ without a calculator. Leave your answer in terms of inverse trigonometric functions. Show your work.

Solution: We first isolate the sine term to get the equation $\sin\left(\frac{\pi}{3}t\right) = -\frac{3}{5}$. Then, we make the substitution $u = \frac{\pi}{3}t$ and solve the equation $\sin(u) = -\frac{3}{5}$. One solution is given by $u = \sin^{-1}\left(-\frac{3}{5}\right)$, which is in the fourth quadrant. This angle is negative; to get an angle in the interval $[0, 2\pi)$, we will use the coterminal angle $\sin^{-1}\left(-\frac{3}{5}\right) + 2\pi$ instead. There will be another solution in the third quadrant which has the same reference angle as $\sin^{-1}\left(-\frac{3}{5}\right) + 2\pi$. The reference angle of $\sin^{-1}\left(-\frac{3}{5}\right) + 2\pi$ is $-\sin^{-1}\left(-\frac{3}{5}\right)$. An angle in the third quadrant (in the interval $[0, 2\pi)$) with this reference angle is $\pi - \sin^{-1}\left(-\frac{3}{5}\right)$. We get all other solutions to $\sin(u) = -\frac{3}{5}$ by adding all integer multiples of 2π (the period of $\sin(u)$) to these two solutions. Therefore, our solutions to $\sin(u) = -\frac{3}{5}$ are

$$u = \left(\sin^{-1}\left(-\frac{3}{5}\right) + 2\pi\right) + k(2\pi), \text{ where } k \text{ is an integer}$$
$$\text{or } u = \left(\pi - \sin^{-1}\left(-\frac{3}{5}\right)\right) + k(2\pi), \text{ where } k \text{ is an integer.}$$

When we undo the substitution $\frac{\pi}{3}t = u$, we get that the solutions to $\sin\left(\frac{\pi}{3}t\right) = -\frac{3}{5}$ are

$$t = \left(\left(\sin^{-1}\left(-\frac{3}{5}\right) + 2\pi\right) + k(2\pi)\right)\left(\frac{3}{\pi}\right), \text{ where } k \text{ is an integer}$$
$$\text{or } t = \left(\left(\pi - \sin^{-1}\left(-\frac{3}{5}\right)\right) + k(2\pi)\right)\left(\frac{3}{\pi}\right), \text{ where } k \text{ is an integer.}$$

However, we only want the solutions in $[0, 12]$. Note that the period of $\sin\left(\frac{\pi}{3}t\right)$ is $\frac{2\pi}{\pi/3} = 6$, so we are looking for the solutions in the "first" two cycles of $\left(\frac{\pi}{3}t\right)$. We made sure that the first two solutions we found were those in the "first" cycle of sine, so we can get the solutions in the "second" cycle as well by setting $k = 1$ in the above equations. Therefore the solutions in $[0, 12]$ are

$$t = \left(\left(\sin^{-1}\left(-\frac{3}{5}\right) + 2\pi\right) + 0 \cdot (2\pi)\right)\left(\frac{3}{\pi}\right)$$
$$t = \left(\left(\pi - \sin^{-1}\left(-\frac{3}{5}\right)\right) + 0 \cdot (2\pi)\right)\left(\frac{3}{\pi}\right)$$
$$t = \left(\left(\sin^{-1}\left(-\frac{3}{5}\right) + 2\pi\right) + 1 \cdot (2\pi)\right)\left(\frac{3}{\pi}\right)$$
$$t = \left(\left(\pi - \sin^{-1}\left(-\frac{3}{5}\right)\right) + 1 \cdot (2\pi)\right)\left(\frac{3}{\pi}\right)$$