THE 2D MUSKAT PROBLEM II:
STABLE REGIME SMALL DATA SINGULARITY ON THE HALF-PLANE

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ABSTRACT. We study the Muskat problem on the half-plane, which models motion of an interface between two fluids of distinct densities (e.g., oil and water) in a porous medium (e.g., an aquifer) that sits atop an impermeable layer (e.g., bedrock). We show that unlike on the whole plane, finite time stable regime singularities do arise in this setting, including from arbitrarily small smooth initial data. To obtain this result, we establish maximum principles for both the potential energy and the slope of solutions in this model, as well as develop a general local well-posedness theory in the companion paper [21].

1. INTRODUCTION AND MAIN RESULTS

The *Muskat problem* models motion of the interface between two incompressible immiscible fluids of different densities $\rho_1 > \rho_0$ (such as water and oil, or salt water and fresh water) inside a porous medium (such as a sand or sandstone aquifer) [1, 16]. It involves the PDE

$$\partial_t \rho + u \cdot \nabla \rho = 0 \tag{1.1}$$

for the fluid density

$$\rho(\mathbf{x}, t) = \rho_1 - (\rho_1 - \rho_0)\chi_{\Omega_t}(\mathbf{x}), \tag{1.2}$$

where u is the fluid velocity and Ω_t is the (time-dependent) region of the lighter fluid. The velocity is determined via *Darcy's law*

$$u := -\nabla p - (0, \rho) \qquad \text{and} \qquad \nabla \cdot u = 0, \tag{1.3}$$

which takes this form after temporal scaling of (1.2) by a factor that is the product of the gravitational constant, permeability of the medium, and the reciprocal of the fluid viscosity. Here p is the fluid pressure, which is uniquely determined by the incompressibility constraint $\nabla \cdot u = 0$, and $-(0, \rho)$ represents downward motion of denser fluid regions due to gravity.

The Muskat problem is related to the Hele-Shaw cell problem [14, 19], and has even been used to model cell velocity in the growth of tumors [11,17]. One can also consider it for a single fluid with a continuously varying density, when the model is known as the *incompressible porous medium (IPM) equation*.

We consider here the Muskat problem in two dimensions, where it has been studied extensively on the whole plane $\mathbb{R}^2$. We mention here only the works most relevant to the present paper and refer the reader to our companion paper [21] for additional recent references. Local regularity on $\mathbb{R}^2$ was proved in [5,7] in the classes of fluid interfaces that are graphs of either $H^3(\mathbb{R})$ or $H^2(\mathbb{R})$ functions of the horizontal variable, provided the lighter fluid lies above the heavier one. This is the *stable regime*, when the Rayleigh-Taylor condition holds [18,19], and the problem is ill-posed in Sobolev spaces when the heavier fluid lies above the lighter one anywhere along the interface [7,20].

This limitation does not apply in the space of analytic interfaces [3], and it was in fact shown in [2, 3] that there are initial $H^4(\mathbb{R})$ interfaces that instantly become analytic, then develop overturning in finite time, and later lose analyticity (and even C^4 regularity) while remaining in the (unstable) overturned regime (cf. [9]). Hence these papers demonstrated existence of overturning in finite time in Sobolev spaces, at which point local regularity breaks down and the interface has a vertical tangent — while remaining a smooth planar *curve* — as well as (post-overturning) finite time loss of interface regularity in the space of analytic curves. Nevertheless, the latter is due to Rayleigh-Taylor instability when the model is in the unstable regime, rather than a result of *super-criticality* of the PDE (which is due to u having the same level of regularity as ρ). One cannot do "better" on $\mathbb{R}^2$, at least within the class of $H^4(\mathbb{R})$ interfaces, because all stable regime solutions remain analytic prior to overturning [3]. In fact, it is known that even corners in the initial interface experience instant smoothing [12, 13].

However, since aquifers typically sit atop impermeable rocky layers, the Muskat problem has a particular physical relevance when considered on the upper half-plane $\mathbb{R} \times \mathbb{R}^+$. An additional appeal of this setting is that it allows one to model dynamically important situations when the heavier fluid *invades* regions initially occupied exclusively by the lighter fluid by flowing underneath the latter along the impermeable bottom. The local regularity proof in $H^3(\mathbb{R})$ from [7] does extend to the half plane if the interface stays uniformly away from the bottom, but constants in the available estimates blow up as the interface approaches $\mathbb{R} \times \{0\}$. Hence this does not allow one to extend the theory to the most interesting setting when the interface touches the bottom, which includes the above invasion scenario.

Nevertheless, it turns out that local regularity for H^3 interfaces does hold on the half-plane, and we provide a (much more involved) proof of this in the companion paper [21]. It is likely that the approach in [3] can then be adjusted to show analyticity of the fluid interface before overturning and in the region where it does not touch the bottom. But this would still leave open possible emergence of finite time interface singularities at $\mathbb{R} \times \{0\}$, and in particular in the stable regime as well as within the well-posedness theory in Sobolev spaces.

The main result of the present paper is Theorem 1.2 below, which shows that *finite time stable regime interface singularities do develop* for the Muskat problem on the half-plane. Moreover, this can happen for *arbitrarily small smooth initial data* (in fact, it happens for all small nonzero data from a fairly general class) and after an arbitrarily long time. The mechanism for this is essentially the invasion scenario described above, specifically when the invasion happens from both directions, and the singularity likely develops at the meeting point of the two invading "fronts".

Our proof has three main ingredients. The first is the local regularity result from [21], which in particular shows that if the interface touches the bottom initially, it cannot "peel off" the bottom unless a singularity occurs. The second is a *quantitative L^2 maximum principle* for the fluid interface in Theorem 1.4, which shows that unless the interface is close to flat (except on a small set), its L^2 norm (when it is appropriately defined and finite) decreases at a uniformly positive rate. Such a maximum principle is known for the Muskat problem on $\mathbb{R}^2$ [6], and since this L^2 norm is also the potential energy of the interface, one can even think of this result as monotonicity of the latter. This was recently derived for the IPM equation in [15], and used there to prove that solutions must be asymptotically stratified.

Since topological constraints force Sobolev norms of some solutions to grow as they stratify (unless they become singular), [15] also demonstrated unbounded growth of certain solutions to IPM on a horizontal strip.

The same conclusion can be derived for the Muskat problem on the half plane, once the local well-posedness theory from [21] and Theorem 1.4 are available, because mass conservation forces interfaces that become more and more flat while still touching the bottom to have near-vertical tangent lines somewhere. But just as in [15], this is not sufficient to prove development of singularity (of the interface as a curve) or even of interface overturning (while it remains a smooth curve) in finite time.

Instead, we conclude the proof of existence of finite time singularity using our third ingredient, a maximum principle for the slope of the interface on the half-plane from Theorem 1.3(ii) below. We show that if the initial interface is not too steep, then it remains such as long as it remains regular and so its tangents cannot become near-vertical. It follows that a finite time interface singularity must occur, with no overturning and so in the stable regime, for any initial interface (from an appropriate class) that is not too steep and touches the bottom. It is possible that some version of our approach could be also applicable to the question of finite time singularity for IPM on the half-plane or on horizontal strips, although this will first require extension of the local well-posedness theory to solutions with more general boundary behaviors than what is currently available [4].

We note that a simple proof of the maximum principle for interface slopes on the whole plane was provided in [8], but the half-plane version is much more involved and its proof in fact forms the bulk of this paper. To illustrate the challenges involved, we note that a version of it on strips $\mathbb{R} \times (0, l)$ was obtained in [10], but this result has fairly restrictive hypotheses and only applies to interfaces that remain *far away* from the strip boundaries. In contrast, Theorem 1.3(ii) applies to all interfaces on the half-plane with slopes bounded by $\frac{3}{10}$.

Since this result is of independent interest, we prove it in more generality than we need here, namely for interfaces that need not vanish at $\pm\infty$ or be periodic, and may even have $O(|x|^{1-})$ growth as $x \to \pm\infty$. This is the optimal power for a general local well-posedness result, and one of the contributions of [21] is extension of existing theory for the Muskat problem to such interfaces (on the plane, half-plane, and horizontal strips); previously, all local regularity results required interfaces that are either from $H^k(\mathbb{R})$ or from $H^k_{\mathrm{loc}}(\mathbb{R})$ and periodic ($k \geq 2$).

Basic setup and local well-posedness. From (1.3) we see that the *vorticity* $\nabla^\perp \cdot u$ equals ρ_{x_1} if we define $\nabla^\perp := (\partial_{x_2}, -\partial_{x_1})$. Since we have the *no-flow boundary condition* $u_2 \equiv 0$ on $\mathbb{R} \times \{0\}$, it follows that

$$u(\mathbf{x}, t) := \nabla^\perp \Delta^{-1} \rho_{x_1}(\mathbf{x}, t) = \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}^+} \left(\frac{(\mathbf{x} - \mathbf{y})^\perp}{|\mathbf{x} - \mathbf{y}|^2} - \frac{(\mathbf{x} - \bar{\mathbf{y}})^\perp}{|\mathbf{x} - \bar{\mathbf{y}}|^2} \right) \rho_{x_1}(\mathbf{y}, t) \, d\mathbf{y}, \qquad (1.4)$$

where Δ is the Dirichlet Laplacian on $\mathbb{R} \times \mathbb{R}^+$, $\bar{\mathbf{y}} := (y_1, -y_2)$, and $\mathbf{y}^\perp := (y_2, -y_1)$. Motion of the fluid interface $\partial\Omega_t$ (see (1.2)) determines the full dynamic for the Muskat problem. One can use this to derive from (1.1)–(1.3) an equivalnet PDE for the fluid interface $x_1 \mapsto f(x_1, t) \geq 0$, with $\Omega_t = \{x_2 > f(x_1, t)\}$ for $t \geq 0$ when the fluids are in the stable regime.

This is done in [21] via the same argument as on the plane, and we get

$$f_t(x,t) = PV \int_{\mathbb{R}} \left[\frac{y \left(f_x(x,t) - f_x(x-y,t) \right)}{y^2 + (f(x,t) - f(x-y,t))^2} + \frac{y \left(f_x(x,t) + f_x(x-y,t) \right)}{y^2 + (f(x,t) + f(x-y,t))^2} \right] dy, \quad (1.5)$$

where we also assumed without loss that $\rho_1 - \rho_0 = 2\pi$. The corresponding whole plane version is instead

$$f_t(x,t) = PV \int_{\mathbb{R}} \frac{y \left(f_x(x,t) - f_x(x-y,t) \right)}{y^2 + (f(x,t) - f(x-y,t))^2} dy. \quad (1.6)$$

Before stating our main results, let us first introduce the general spaces of interface functions that we will consider in Theorem 1.3. For $k = 0, 1, \dots$ define the (local) norms

$$\|g\|_{\tilde{L}^2(\mathbb{R})} := \sup_{x \in \mathbb{R}} \|g\|_{L^2([x-1,x+1])} \qquad \text{and} \qquad \|g\|_{\tilde{H}^k(\mathbb{R})} := \sum_{j=0}^{k} \|g^{(j)}\|_{\tilde{L}^2(\mathbb{R})}$$

on the spaces of those $g \in L^2_{\text{loc}}(\mathbb{R})$ for which these are finite. Then for $k \geq 1$ and any $\gamma \in [0,1]$ define the seminorms

$$\|g\|_{\ddot{C}^\gamma(\mathbb{R})} := \sup_{|x-y| \geq 1} \frac{|g(x) - g(y)|}{|x-y|^\gamma} \qquad \text{and} \qquad \|g\|_{\tilde{H}^k_\gamma(\mathbb{R})} := \|g'\|_{\tilde{H}^{k-1}(\mathbb{R})} + \|g\|_{\ddot{C}^{1-\gamma}(\mathbb{R})},$$

which vanish for constant functions. We note that $\|g\|_{\tilde{H}^k_\gamma(\mathbb{R})} < \infty$ allows g to have $O(|x|^{1-\gamma})$ growth at $\pm\infty$, and refer the reader to [21] for further discussion. We now have the following local regularity result, which holds on both the plane and the half-plane.

Theorem 1.1 ([21]). *Let $\gamma \in (0,1]$ and $\|\psi\|_{\tilde{H}^3_\gamma(\mathbb{R})} < \infty$.*

(i) There is γ-independent $T_\psi \in (0,\infty]$ and a unique classical solution f to (1.6) on $\mathbb{R} \times [0, T_\psi)$ with $f(\cdot, 0) \equiv \psi$ such that

$$\sup_{t \in [0,T]} \|f(\cdot,t)\|_{\tilde{H}^3_\gamma(\mathbb{R})} < \infty \tag{1.7}$$

for each $T \in (0, T_\psi)$. It in fact satisfies

$$\sup_{t \in [0,T]} \|f(\cdot,t) - \psi\|_{\tilde{H}^3(\mathbb{R})} < \infty \qquad \text{and} \qquad \sup_{t \in [0,T]} \|f_t(\cdot,t)\|_{W^{1,\infty}(\mathbb{R})} < \infty \tag{1.8}$$

for each $T \in (0, T_\psi)$, and if there are $\psi_{\pm\infty} \in \mathbb{R}$ such that $\psi - \psi_{\pm\infty} \in H^3(\mathbb{R}^\pm)$, then even

$$\sup_{t \in [0,T]} \|f(\cdot,t) - \psi\|_{H^3(\mathbb{R})} < \infty. \tag{1.9}$$

Finally, if $T_\psi < \infty$, then for each $\gamma' \in (0,1]$ we have

$$\int_0^{T_\psi} \|f_x(\cdot,t)\|^4_{C^{1,\gamma'}(\mathbb{R})} dt = \infty. \tag{1.10}$$

(ii) If $\psi \geq 0$, then (i) holds with $f \geq 0$ solving (1.5) instead. And if $\inf \psi = 0$, then $\inf f(\cdot,t) = 0$ for each $t \in [0, T_\psi)$.

Main results. We are now ready to state our main result, which shows that for the Muskat problem on the half-plane, stable regime finite time singularity develops from all initial data

that initially touch the bottom and do not have large slopes (plus they are either periodic or $H^3(\mathbb{R})$ perturbations of non-zero constant functions). In it we also use the seminorm

$$\|g\|_{\dot{C}^\gamma(\mathbb{R})} := \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\gamma}.$$

Theorem 1.2. *If* $0 \leq \psi \in \tilde{H}^3(\mathbb{R})$ *is periodic,* $\min \psi = 0 \not\equiv \psi$, *and* $\|\psi'\|_{L^\infty(\mathbb{R})} \leq \frac{3}{10}$, *then* $T_\psi < \infty$ *and* f *from Theorem 1.2(ii) satisfies*

$$\|f_x\|_{L^\infty(\mathbb{R} \times [0, T_\psi))} \leq \frac{3}{10} \qquad and \qquad \int_0^{T_\psi} \|f_{xx}(\cdot, t)\|_{\dot{C}^{\gamma'}(\mathbb{R})}^4 \, dt = \infty \qquad (1.11)$$

for each $\gamma' \in (0, 1]$. *The same result holds when periodicity of* ψ *is replaced by* $\psi - \psi_\infty \in H^3(\mathbb{R})$ *for some constant* $\psi_\infty \in (0, \infty)$.

Remarks. 1. Since $\psi \in \tilde{H}^3(\mathbb{R})$, we have $\|\psi\|_{\tilde{H}^3_\gamma(\mathbb{R})} < \infty$ for all $\gamma \in [0, 1]$. This and the first claim in (1.8) show that we do not need to specify γ in Theorem 1.2.

2. So while remaining a Lipschitz graph, the interface leaves $C^{2,\gamma'}(\mathbb{R})$ for each $\gamma' > 0$ (and hence also $H^3_{\text{loc}}(\mathbb{R})$) at time $T_\psi < \infty$ when, for instance, for some (arbitrarily) small $\varepsilon > 0$ we have $\|\psi - \varepsilon\|_{H^3(\mathbb{R})} \leq \varepsilon$. And this can happen after an arbitrarily long time because in [21] we showed that $T_\psi \geq C_\gamma(1 - \ln \|\psi\|_{\tilde{H}^3_\gamma(\mathbb{R})})$ for some $C_\gamma > 0$.

As we explained above, the main ingredient in the proof of Theorem 1.2 is the following L^∞ maximum principle for f_x on the half-plane. Since we prove it in the general case $\|\psi\|_{\tilde{H}^3_\gamma(\mathbb{R})} < \infty$, the result is new even on the plane and we state it here on both domains.

Theorem 1.3. *(i) Let* γ, ψ, f *be as in Theorem 1.1(i). If* $\|f_x(\cdot, t')\|_{L^\infty(\mathbb{R})} \leq 1$ *for some* $t' \in [0, T_\psi)$, *then* $\|f_x(\cdot, t)\|_{L^\infty(\mathbb{R})}$ *is decreasing on* $[t', T_\psi)$ *(strictly unless it is 0).*
(ii) If γ, ψ, f *are as in Theorem 1.1(ii), then (i) holds with* $\|f_x(\cdot, t')\|_{L^\infty(\mathbb{R})} \leq \frac{3}{10}$ *instead.*

Finally, we state the L^2 maximum principle for f (which is new on the half-plane) that we also use in the proof of Theorem 1.2. In it we let

$$\lambda(a, b, c) := \frac{(a + b)^2 (a - b)^2}{2[c^2 + (a + b)^2][c^2 + (a - b)^2]} \geq 0.$$

Theorem 1.4. *Let* $0 \leq \psi \in \tilde{H}^3(\mathbb{R})$ *and let* f *be from Theorem 1.2(ii).*
(i) If ψ *is periodic with period* ν, *then*

$$\frac{d}{dt} \|f(\cdot, t)\|_{L^2([0,\nu])}^2 = -2\|\rho_{x_1}(\cdot, t)\|_{\dot{H}^{-1}([0,\nu] \times \mathbb{R}^+)}^2 \leq -\int_{[0,\nu] \times \mathbb{R}} \lambda(f(x, t), f(y, t), x - y) \, dx \, dy$$

(1.12)

for each $t \in [0, T_\psi)$, *where* $\rho(\cdot, t) := 2\pi \chi_{\{0 < x_2 < f(x_1, t)\}}$.
(ii) If $\psi - \psi_\infty \in H^3(\mathbb{R})$ *for some* $\psi_\infty \in (0, \infty)$, *then* $\tilde{f} := f - \psi_\infty$ *satisfies*

$$\frac{d}{dt} \|\tilde{f}(\cdot, t)\|_{L^2(\mathbb{R})}^2 = -2\|\rho_{x_1}(\cdot, t)\|_{\dot{H}^{-1}(\mathbb{R} \times \mathbb{R}^+)}^2 \leq -\int_{\mathbb{R}^2} \lambda(f(x, t), f(y, t), x - y) \, dx \, dy \qquad (1.13)$$

for each $t \in [0, T_\psi)$, *where* $\rho(\cdot, t) := 2\pi \chi_{\{0 < x_2 < f(x_1, t)\}}$.

Remark. The proof of Theorem 1.4 shows that for (1.6) we instead have

$$\frac{d}{dt}\|\tilde{f}(\cdot,t)\|^2_{L^2(S)} = -\int_{S\times\mathbb{R}} \ln\left(1 + \frac{(f(x,t)-f(y,t))^2}{(x-y)^2}\right) dxdy,$$

where S is $[0,\nu]$ in (i) (with $\psi_\infty := 0$) and it is $\mathbb{R}$ in (ii). This is a result from [6].

The rest of the paper is composed of four sections. We prove Theorem 1.2 in Section 2, using Theorems 1.3 and 1.4. The latter result is then proved in Section 3, while the much longer proof of Theorem 1.3 is split between Sections 4 and 5.

2. Proof of Theorem 1.2

The proofs of both claims are similar, so we only provide here the first one. We note that the proof of the second claim involves norms $\|\tilde{f}(\cdot,t)\|_{L^1(\mathbb{R})}$ and $\|\tilde{f}(\cdot,t)\|_{L^2(\mathbb{R})}$ with $\tilde{f} := f - \psi_\infty$ instead of $\|f(\cdot,t)\|_{L^1([0,1])}$ and $\|f(\cdot,t)\|_{L^2([0,1])}$ (see (1.9) and the end of the next section).

Assume that the period of ψ is 1 because the general case is identical. By the uniqueness claim in Theorem 1.1(i), $f(\cdot,t)$ also has period 1 for each $t \in [0,T_\psi)$.

Theorem 1.3(ii) proves the first claim in (1.11), while (3.2) below shows that

$$\|f(\cdot,t)\|_{L^1([0,1])} = \|\psi\|_{L^1([0,1])} =: L \in \left(0,\frac{3}{40}\right)$$

for all $t \in [0,T_\psi)$. The last claim in Theorem 1.1(ii) and the first claim in (1.11) now show that for each $t \in [0,T_\psi)$, there is $A_t \subseteq [0,1]$ that is an interval or a union of two intervals of total length $\geq 2L$ such that $f(\cdot,t) \leq \frac{L}{3}$ on A_t. These two claims also show that $\|f\|_{L^\infty(\mathbb{R}\times[0,T_\psi))} \leq \frac{3}{20}$, and so there is $B_t \subseteq [0,1]$ that is an interval or a union of two intervals of total length $\geq 2L$ such that $f(\cdot,t) \geq \frac{2L}{3}$ on B_t. But then the integral in (1.12) is at least

$$4L^2 \frac{\frac{4L^2}{9}\frac{L^2}{9}}{2\left(1+(\frac{L}{3}+\frac{3}{20})^2\right)\left(1+\frac{9}{400}\right)} \geq \frac{L^6}{20}$$

for each $t \in [0,T_\psi)$. This and (1.12) now imply that $T_\psi < \infty$ because $\|f(\cdot,t)\|^2_{L^2([0,1])} \geq 0$. If we then combine the first claim in (1.11) with (1.10), we obtain the second claim in (1.11).

So to complete the proof, it remains to prove Theorems 1.3 and 1.4, as well as (3.2).

3. Proof of Theorem 1.4

Proof of (i). Assume that the period $\nu = 1$ because the general case is identical. By the uniqueness claim in Theorem 1.1(i), $f(\cdot,t)$ also has period 1 for each $t \in [0,T_\psi)$. Since

$$\frac{y\left(f'(x)\pm f'(x-y)\right)}{y^2+(f(x)\pm f(x-y))^2} = \frac{d}{dx}\arctan\frac{f(x)\pm f(x-y)}{y} \tag{3.1}$$

for any $y \in \mathbb{R}$, we see that

$$\frac{d}{dt}\|f(\cdot,t)\|_{L^1([0,1])} = 0 \tag{3.2}$$

for $t \in [0,T_\psi)$. From (3.1) and integration by parts in x we obtain (dropping t in the notation)

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2([0,1])} = -\sum_{\pm} PV\int_{[0,1]\times\mathbb{R}} f'(x)\arctan\frac{f(x)\pm f(x-y)}{y} dxdy$$

$$= -\sum_{\pm} \int_{[0,1]^2} PV \sum_{n \in \mathbb{Z}} f'(x) \arctan \frac{f(x) \pm f(x-y)}{y+n} \, dx dy$$

$$= -\sum_{\pm} \int_{[0,1]^2} PV \sum_{n \in \mathbb{Z}} f'(x) \arctan \frac{f(x) \pm f(y)}{x-y-n} \, dx dy$$

$$= -\sum_{\pm} PV \int_{[0,1] \times \mathbb{R}} f'(x) \arctan \frac{f(x) \pm f(y)}{x-y} \, dx dy.$$

Let us write the integral in x as

$$\int_0^1 \left[(x-y) \frac{d}{dx} G \left(\frac{f(x) \pm f(y)}{x-y} \right) + \frac{f(x) \pm f(y)}{x-y} \arctan \frac{f(x) \pm f(y)}{x-y} \right] dx, \qquad (3.3)$$

where

$$G(s) := s \arctan s - \ln \sqrt{1+s^2} = \int_0^s \arctan r \, dr.$$

We integrate the first term in (3.3) by parts in x again and find that (3.3) equals

$$\int_0^1 \frac{1}{2} \ln \left(1 + \frac{(f(x) \pm f(y))^2}{(x-y)^2} \right) dx + (1-y) G \left(\frac{f(1) \pm f(y)}{1-y} \right) - (-y) G \left(\frac{f(0) \pm f(y)}{-y} \right)$$

plus the additional "boundary" term $-2\pi f(y)$ when $y \in (0,1)$ and $\pm$ is $+$ (from the singularity at $x = y$). The last two terms above cancel with the same terms when y is increased and decreased by 1 because f is 1-periodic, so we obtain

$$\frac{d}{dt} \|f\|_{L^2}^2 = 4\pi \int_0^1 f(y) \, dy - \sum_{\pm} \int_{[0,1] \times \mathbb{R}} \ln \left(1 + \frac{(f(x) \pm f(y))^2}{(x-y)^2} \right) dx dy. \qquad (3.4)$$

We now use the formulas

$$\ln(1+p) + \ln(1+q) - \ln(1+p+q) \geq \int_1^{1+q} \left(\frac{1}{z} - \frac{1}{p+z} \right) dz \geq \frac{pq}{(1+q)(1+p+q)}$$

with $p := \frac{(f(x)+f(y))^2}{(x-y)^2}$ and $q := \frac{(f(x)-f(y))^2}{(x-y)^2}$, and then

$$2\ln(1+p+q) - \ln(1+p+q+r) - \ln(1+p+q-r) \geq 0$$

with $r := \frac{2f(x)^2 - 2f(y)^2}{(x-y)^2}$ to get

$$2\sum_{\pm} \ln \left(1 + \frac{(f(x) \pm f(y))^2}{(x-y)^2} \right) \geq \ln \left(1 + \frac{4f(x)^2}{(x-y)^2} \right) + \ln \left(1 + \frac{4f(y)^2}{(x-y)^2} \right) + 2\tilde{\lambda}(f(x), f(y), x-y),$$

where

$$\tilde{\lambda}(a,b,c) := \frac{(a+b)^2(a-b)^2}{[c^2 + 2a^2 + 2b^2][c^2 + (a-b)^2]} \geq 0. \qquad (3.5)$$

Since also

$$\int_{\mathbb{R}} \ln \left(1 + \frac{C^2}{y^2} \right) dy = \int_{\mathbb{R}} \frac{d}{dy} \left[y \ln \left(1 + \frac{C^2}{y^2} \right) - 2C \arctan \frac{C}{y} \right] dy = 2\pi C$$

and

$$\int_{[0,1]^2} \sum_{n \in \mathbb{Z}} \ln\left(1 + \frac{4f(y+n)^2}{(x-(y+n))^2}\right) dxdy = \int_{[0,1]^2} \sum_{n \in \mathbb{Z}} \ln\left(1 + \frac{4f(y)^2}{((x-n)-y)^2}\right) dxdy$$

for $y \in (0,1)$ (by 1-periodicity of f), this and (3.4) yield

$$\frac{d}{dt}\|f\|^2_{L^2([0,1])} \leq -\int_{[0,1]\times\mathbb{R}} \tilde{\lambda}(f(x), f(y), x-y)\, dxdy. \tag{3.6}$$

Now (1.12) follows from $\tilde{\lambda} \geq \lambda$.

Let $\Gamma_t := \Gamma_f(t)$ and

$$\Omega_t' := \left\{\mathbf{x} \in (0,1) \times (0,\infty) \,\middle|\, x_2 < f(x_1, t)\right\}.$$

We have

$$\int_0^1 f(x)\partial_t f(x)dx = \int_{\Gamma_t} \frac{x_2\, \partial_t f(x_1)}{\sqrt{1+f'(x_1)^2}}\, d\sigma(\mathbf{x}) = \int_{\Gamma_t} x_2\,(0, \partial_t f(x_1)) \cdot \frac{(-f'(x_1), 1)}{\sqrt{1+f'(x_1)^2}}\, d\sigma(\mathbf{x}).$$

We now use the fact that $\partial_t f(x_1)$ is precisely the number such that at time t we have

$$(0, \partial_t f(x_1)) \cdot \frac{(-f'(x_1), 1)}{\sqrt{1+f'(x_1)^2}} = u(\mathbf{x}) \cdot \frac{(-f'(x_1), 1)}{\sqrt{1+f'(x_1)^2}},$$

where the second vector on both sides is the unit outer normal $n(\mathbf{x})$ to Ω_t' at $\mathbf{x} = (x_1, x_2)$. So 1-periodicity of f, the divergence theorem, and $\nabla \cdot u \equiv 0$ yield

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2([0,1])} = \int_{\partial\Omega_t'} x_2 u(\mathbf{x}) \cdot n(\mathbf{x})\, d\mathbf{x} = \int_{\Omega_t'} u_2(\mathbf{x})\, d\mathbf{x} = \int_{[0,1]\times[0,\infty)} \rho(\mathbf{x})u_2(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{[0,1]\times[0,\infty)} \rho(\mathbf{x})\, \partial_{x_1}(-\Delta)^{-1}\rho_{x_1}(\mathbf{x}, t) = -\|\rho_{x_1}\|^2_{\dot{H}^{-1}([0,1]\times[0,\infty))}.$$

The integration by parts at the end can be justified by approximating ρ by smooth characteristic functions ρ_n of Ω_t' with $u^n := \partial_{x_1}(-\Delta)^{-1}(\rho_n)_{x_1}$, and then $\rho_n \to \rho$ in $L^2([0,1] \times [0,\infty))$ implies $u^n \to u$ in $L^2([0,1] \times [0,\infty))$ and $(\rho_n)_{x_1} \to \rho_{x_1}$ in $\dot{H}^{-1}([0,1] \times [0,\infty))$.

Proof of (ii). This proof is almost identical to the one above, with all integrals over $[0,1]$ being now over $\mathbb{R}$ (which simplifies some steps) and with the whole argument estimating instead

$$\frac{d}{dt}\|\tilde{f}(\cdot, t)\|_{L^1(\mathbb{R})} \qquad \text{and} \qquad \frac{d}{dt}\|\tilde{f}(\cdot, t)\|^2_{L^2(\mathbb{R})}.$$

Since $\tilde{f}_x = f_x$ and $\tilde{f}_t = f_t$, all steps in the proof easily extend to this setting. We note that in the last part of the proof we use

$$\Omega_t' := \left\{\mathbf{x} \in \mathbb{R} \times (0,\infty) \,\middle|\, x_2 < f(x_1, t)\right\},$$
$$\Omega_t^{\pm} := \left\{\mathbf{x} \in \mathbb{R} \times (0,\infty) \,\middle|\, \pm\psi_\infty < \pm x_2 < \pm f(x_1, t)\right\},$$

and also $x_2 - \psi_\infty$ in place of x_2. This then yields

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(\mathbb{R})} = \int_{\partial\Omega_t^+} (x_2 - \psi_\infty)u(\mathbf{x}) \cdot n(\mathbf{x})\, d\mathbf{x} - \int_{\partial\Omega_t^-} (x_2 - \psi_\infty)u(\mathbf{x}) \cdot n(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{\Omega_t^+} u_2(\mathbf{x}) \, d\mathbf{x} - \int_{\Omega_t^-} u_2(\mathbf{x}) \, d\mathbf{x} = PV \int_{\Omega_t'} u_2(\mathbf{x}) \, d\mathbf{x}$$

because $PV \int_{\mathbb{R} \times [0,\psi_\infty]} u_2(\mathbf{x}) \, d\mathbf{x} = 0$.

4. Proof of Theorem 1.3

For $k = 0, 1, \dots$ and $\gamma \in [0, 1]$ let

$$\|g\|_{C^{k,\gamma}(\mathbb{R})} := \sum_{j=0}^{k} \|g^{(j)}\|_{L^\infty(\mathbb{R})} + \|g^{(k)}\|_{\dot{C}^\gamma(\mathbb{R})}$$

when $g \in L^2_{\mathrm{loc}}(\mathbb{R})$ (note that then $C^{k,0}(\mathbb{R}) = W^{k,\infty}(\mathbb{R})$), and for $k \geq 1$ define the seminorms (vanish on constant functions)

$$\|g\|_{C^{k,\gamma}_\gamma(\mathbb{R})} := \|g'\|_{C^{k-1,\gamma}(\mathbb{R})} + \|g\|_{\ddot{C}^{1-\gamma}(\mathbb{R})} \qquad \text{and} \qquad \|g\|_{C^k_\gamma(\mathbb{R})} := \|g'\|_{C^{k-1,0}(\mathbb{R})} + \|g\|_{\ddot{C}^{1-\gamma}(\mathbb{R})}.$$

Note that when $\gamma \in (0, \frac{1}{2}]$ and $k \geq 1$, then

$$\|g\|_{C^{k-1,\gamma}_\gamma(\mathbb{R})} \leq C_{k,\gamma} \|g\|_{\tilde{H}^k_\gamma(\mathbb{R})} \tag{4.1}$$

holds for all $g \in L^2_{\mathrm{loc}}(\mathbb{R})$, with some constant $C_{k,\gamma} < \infty$. We will assume without loss that $\gamma \in (0, \frac{1}{2}]$ because $\|g\|_{\tilde{H}^3_\gamma(\mathbb{R})} < \infty$ implies $\|g\|_{\tilde{H}^3_{\gamma'}(\mathbb{R})} < \infty$ for all $\gamma' \in (0, \gamma]$.

Since

$$\frac{f(x) \pm f(x-y) \pm y \, f'(x-y)}{y^2 + (f(x) \pm f(x-y))^2} = -\frac{d}{dy} \arctan \frac{f(x) \pm f(x-y)}{y},$$

we see that

$$\int_{\mathbb{R}} \frac{f(x) \pm f(x-y) \pm y \, f'(x-y)}{y^2 + (f(x) \pm f(x-y))^2} \, dy = \begin{cases} \pi & \pm \text{ is } + \text{ and } f(x) > 0, \\ 0 & \pm \text{ is } - \text{ or } f(x) = 0 \end{cases}$$

when $\|f\|_{C^2_\gamma(\mathbb{R})} < \infty$ (and $f \geq 0$ when $\pm$ is $+$). Hence (1.6) can be equivalently written as

$$f_t(x,t) = PV \int_{\mathbb{R}} \frac{(x-y)f_x(x,t) - (f(x,t) - f(y,t))}{(x-y)^2 + (f(x,t) - f(y,t))^2} \, dy, \tag{4.2}$$

where we also changed variables $y \leftrightarrow x - y$, while (1.5) is equivalent to

$$f_t(x,t) = \pi \chi_{(0,\infty)}(f(x,t)) + \sum_\pm PV \int_{\mathbb{R}} \frac{(x-y)f_x(x,t) - (f(x,t) \pm f(y,t))}{(x-y)^2 + (f(x,t) \pm f(y,t))^2} \, dy. \tag{4.3}$$

Note that the last integral with $\pm$ being $+$ is discontinuous at all points in $\partial\{x \mid f(x,t) = 0\}$, but this will not cause a problem because our analysis of (4.3) will be performed at points where $f_x(x,t) \neq 0$ and therefore $f(x,t) > 0$. Note also that "PV" is only needed in (4.2) and (4.3) as $|y| \to \infty$ but not as $|y| \to 0$ when $\|f(\cdot,t)\|_{C^2_\gamma(\mathbb{R})} < \infty$.

Finally, we observe that the second claim in (1.8) implies

$$\sup_{x \in \mathbb{R}} \|f_x(x,\cdot)\|_{W^{1,\infty}([0,T])} \leq \|f_x\|_{L^\infty(\mathbb{R} \times [0,T])} + \sup_{t \in [0,T]} \|f_t(\cdot,t)\|_{W^{1,\infty}(\mathbb{R})} < \infty$$

for each $T \in [0, T_\psi)$. This and (1.7) yield $f_x \in W^{1,\infty}(\mathbb{R} \times [0,T])$, so $M_t := \|f_x(\cdot,t)\|_{L^\infty}$ is locally Lipschitz on $[0, T_\psi)$.

Proof of (i). Let M_t be as above and assume that $M_t \in (0,1]$ for some t (if $M_t = 0$, then $f(\cdot, t) \equiv C$ for some $C \in \mathbb{R}$, and then this holds for all $t' \geq t$). For the sake of simplicity, in the following argument we drop t from the notation (other than in M_t) and denote $g' := g_x$.

For any $(x,t) \in \mathbb{R} \times [0, T_\psi)$ with $f'(x) \neq 0$ we have

$$
\partial_t f'(x) = \frac{d}{dx} PV \int_{\mathbb{R}} \frac{(x-y)f'(x) - (f(x) - f(y))}{(x-y)^2 + (f(x) - f(y))^2} \, dy
$$
$$
= PV \int_{\mathbb{R}} \frac{(x-y)f''(x)}{(x-y)^2 + (f(x) - f(y))^2} \, dy - 2f'(x) PV \int_{\mathbb{R}} h_{f'(x)}(y, f(x) - f(x-y)) \, dy,
$$

where

$$
h_a(y, r) := \frac{y^2 - r^2 - (a^{-1} - a)yr}{(y^2 + r^2)^2}
$$

and so the last integral is just

$$
\frac{1}{f'(x)} PV \int_{\mathbb{R}} \frac{[(x-y)f'(x) - (f(x) - f(y))] \, [x - y + f'(x)(f(x) - f(y))]}{[(x-y)^2 + (f(x) - f(y))^2]^2} \, dy
$$

after changing variables $y \leftrightarrow x - y$. We note that (4.6) and (4.8) below show that both integrals converge in the principal value sense.

When $|f'(x)| = M_t \in (0,1]$, then $f''(x) = 0$ and $h_{f'(x)}(y, f(x) - f(x-y)) \geq 0$ for all $y \in \mathbb{R}$ (see (4.4) below), with the inequality being sharp when $|y|$ is large (since equality only holds when $f(x) - f(x-y) = f'(x)(x-y)$). Hence $\partial_t f'(x) < 0$, which yields the result when such x exists for each $t \in [0, T_\psi)$. But this need not be the case when only $\|f\|_{\tilde{H}^3_\gamma(\mathbb{R})} < \infty$. One could circumvent this issue by looking at points where $|f'(x)|$ is maximal on $[x - L, x + L]$ for large L, but this will not be the case for the much more involved argument in (ii) because an estimate like (4.4) is not available for the analogous term coming from the second integral in (4.3). We will therefore instead estimate $\partial_t f'(x)$ at all points where $|f'(x)|$ is close to M_t, which will also work in (ii). Our argument is based on replacing $h_{f'(x)}(y, f(x) - f(x-y))$ by $h_{\mathrm{sgn}(f'(x))M_t}(y, f(x) - f(x-y))$ and estimating the difference, and then using that $h_{\mathrm{sgn}(f'(x))M_t}(y, f(x) - f(x-y)) \geq 0$ for all $y \in \mathbb{R}$ if also $M_t \leq 1$.

Let us consider any $\delta \in (0, \frac{1}{2}M_t]$ and any $x \in \mathbb{R}$ such that $|f'(x)| \geq M_t - \delta$. We have

$$
h_{\mathrm{sgn}(f'(x))M_t}(y, f(x) - f(x-y)) \geq \frac{(1 - M_t^2 - |M_t^{-1} - M_t| \, M_t) \, y^2}{y^4} \geq 0 \qquad (4.4)
$$

for all $y \in \mathbb{R}$ because $|f(x) - f(x-y)| \leq M_t |y|$ and $M_t \leq 1$. We also have

$$
\left| h_{f'(x)}(y, f(x) - f(x-y)) - h_{\mathrm{sgn}(f'(x))M_t}(y, f(x) - f(x-y)) \right| \leq \frac{M_t^2 + 4}{M_t y^2} \delta \qquad (4.5)
$$

for $y \in \mathbb{R}$ because

$$
|(b^{-1} - b) - (a^{-1} - a)| \leq |b - a| \, (1 + \min\{a, b\}^{-2})
$$

for $a, b > 0$. Writing

$$
f(x) - f(x-y) = f'(x)y - \frac{f''(z)}{2} y^2
$$

for some z between x and $x - y$ shows that the numerator of $h_{f'(x)}(y, f(x) - f(x - y))$ is

$$\frac{(1 + f'(x)^2)f''(x)}{2f'(x)} y^3 + \frac{(1 + f'(x)^2)(f''(z) - f''(x))}{2f'(x)} y^3 - \frac{f''(z)^2}{4} y^4.$$

Estimating the last two terms by

$$\frac{2(1 + \|f'\|_{C^{1,\gamma}}^3)}{|f'(x)|} |y|^{3+\gamma}$$

for $|y| \leq 1$, and using

$$\left| \frac{y^3}{(y^2 + a_-^2)^2} - \frac{y^3}{(y^2 + a_+^2)^2} \right| \leq \frac{2(a_- + a_+)(a_- - a_+)}{|y|^3}$$

with $a_\pm := f(x) - f(x \pm y)$, we see that there is a constant C_γ (depending only on γ) such that for any $\varepsilon \in [0, 1]$ we have

$$\left| PV \int_{-\varepsilon}^{\varepsilon} h_{f'(x)}(y, f(x) - f(x - y)) \, dy \right| \leq \frac{C_\gamma(1 + \|f'\|_{C^1}^5)}{|f'(x)|} \varepsilon + \frac{C_\gamma(1 + \|f'\|_{C^{1,\gamma}}^3)}{|f'(x)|} \varepsilon^\gamma. \qquad (4.6)$$

Finally, we have

$$\left| PV \int_{\mathbb{R}} \frac{x - y}{(x - y)^2 + (f(x) \pm f(y))^2} \, dy \right| \leq C_\gamma \|f\|_{C_\gamma^2}$$

for some new C_γ (see (2.2) in [21]), as well as

$$|f''(x)| \leq 2\|f''\|_{\dot{C}^\gamma}^{1/(1+\gamma)} (M_t - |f'(x)|)^{\gamma/(1+\gamma)}. \qquad (4.7)$$

The latter holds because $|f''(y) - f''(x)| \leq \frac{1}{2}|f''(x)|$ whenever $|y - x| \leq |f''(x)|^{1/\gamma}(2\|f''\|_{\dot{C}^\gamma})^{-1/\gamma}$, and so

$$M_t - |f'(x)| \geq \text{sgn}(f'(x)) \left[f'\left(x + \text{sgn}(f'(x))\frac{f''(x)^{1/\gamma}}{(2\|f''\|_{\dot{C}^\gamma})^{1/\gamma}} \right) - f'(x) \right] \geq \frac{f''(x)^{1/\gamma}}{(2\|f''\|_{\dot{C}^\gamma})^{1/\gamma}} \frac{f''(x)}{2}.$$

It follows that (recall also (4.1))

$$\left| PV \int_{\mathbb{R}} \frac{(x - y)f''(x)}{(x - y)^2 + (f(x) \pm f(y))^2} \, dy \right| \leq 2C_\gamma \|f\|_{C_\gamma^{2,\gamma}}^{(2+\gamma)/(1+\gamma)} \delta^{\gamma/(1+\gamma)}. \qquad (4.8)$$

Estimates (4.4), (4.5), (4.6), and (4.8) show that with some new C_γ we have

$$\text{sgn}(f_x(x, t)) \, \partial_t f_x(x, t) \leq C_\gamma \left(1 + \|f(\cdot, t)\|_{C_\gamma^{2,\gamma}}^5 \right) \delta^{\gamma/2} \qquad (4.9)$$

when $M_t > 0$, $\delta \in (0, \frac{1}{2}M_t]$, and $|f'(x)| \geq M_t - \delta$ (use $\varepsilon := \sqrt{\delta}$ in (4.6), and then $|y| \geq \varepsilon$ in (4.4) and (4.5)). Taking $\delta \to 0$ in (4.9) and using (1.7) (together with $\gamma \in (0, \frac{1}{2}]$) now yields $\frac{d}{dt}M_t \leq 0$. Moreover, we have

$$h_{\text{sgn}(f'(x))M_t}(y, f(x) - f(x - y)) \geq \frac{\left[1 - \left(\frac{1}{2}M_t\right)^2 - |M_t^{-1} - M_t| \frac{1}{2}M_t \right] y^2}{y^4} \geq \frac{1}{2y^2} \qquad (4.10)$$

for $|y| \geq (2\|f\|_{\dot{C}^{1-\gamma}})^{1/\gamma} M_t^{-1/\gamma}$, since $|f(x) - f(x - y)| \leq \frac{M_t}{2}|y|$ for these y. Joining this with (4.4) adds a negative constant to the right-hand side of (4.9) that is uniform in δ and x,

which makes the right-hand side of (4.9) negative when δ is small enough. Thus $\frac{d}{dt}M_t < 0$, which finishes the proof.

Proof of (ii). We proceed as in (i), again assuming that $M_t \in (0, 1]$, but this time we have

$$\partial_t f'(x) = \sum_{\pm} PV \int_{\mathbb{R}} \frac{(x-y)f''(x)}{(x-y)^2 + (f(x) \pm f(y))^2} \, dy$$

$$- 2f'(x) \sum_{\pm} PV \int_{\mathbb{R}} h_{f'(x)}(y, f(x) \pm f(x-y)) \, dy.$$

Estimates from the proof of (i) apply to three of the integrals above. As for the fourth, similarly to (4.5) we obtain

$$\left| h_{f'(x)}(y, f(x) + f(x-y)) - h_{\mathrm{sgn}(f'(x))M_t}(y, f(x) + f(x-y)) \right| \leq \frac{M_t^2 + 4}{M_t^2(y^2 + f(x)^2)} \delta \quad (4.11)$$

for $y \in \mathbb{R}$, where we used $|yr| \leq y^2 + r^2$ instead of $|f(x) - f(x-y)| \leq M_t|y|$, as well as that $f(x) + f(x-y) \geq f(x)$ because now $f \geq 0$. We also have

$$|f'(x)| \leq 2\|f''\|_{L^\infty}^{1/2} \sqrt{f(x)},$$

which is proved similarly to (4.7) (using $f \geq 0$), so

$$f(x) \geq \frac{M_t^2}{16\|f''\|_{L^\infty}} \tag{4.12}$$

when $\delta \in (0, \frac{1}{2}M_t]$, and $|f'(x)| \geq M_t - \delta$.

Hence (4.5), (4.6), (4.8), (4.11), and (4.12) (but not (4.4)!) show that

$$\mathrm{sgn}(f_x(x,t)) \, \partial_t f_x(x,t) \leq C_\gamma \left(1 + \|f(\cdot, t)\|_{C_\gamma^{2,\gamma}}^5 \right) \delta^{\gamma/2} + \frac{C_\gamma}{M_t} \sqrt{\delta} + \frac{C_\gamma \|f''(\cdot, t)\|_{L^\infty}^2}{M_t^4} \sqrt{\delta}$$

$$- 2f'(x) \sum_{\pm} \int_{|y| \geq \sqrt{\delta}} h_{\mathrm{sgn}(f'(x))M_t}(y, f(x) \pm f(x-y)) \, dy \tag{4.13}$$

when $\delta \in (0, \frac{1}{2}M_t]$ and $|f'(x)| \geq M_t - \delta$, where $C_\gamma M_t^{-1} \sqrt{\delta}$ comes from (4.11) for $|y| \geq \sqrt{\delta}$, and we also used that (4.12) and $f \geq 0$ imply

$$\int_{-\sqrt{\delta}}^{\sqrt{\delta}} h_{f'(x)}(y, f(x) + f(x-y)) \, dy \geq -2\sqrt{\delta} \frac{1 + M_t^{-1}}{(M_t^2/16\|f''\|_{L^\infty})^2}.$$

Taking $\delta \to 0$ in (4.13) will conclude $\frac{d}{dt}M_t < 0$ when $M_t \in (0, \frac{3}{10}]$ (and hence finish the proof) if we can prove

$$\liminf_{\delta \to 0} \inf_{f \in \mathcal{F}_{a,c,\delta}} \sum_{\pm} \int_{|y| \geq \sqrt{\delta}} h_a(y, f(0) \pm f(y)) \, dy > 0 \tag{4.14}$$

for any $a \in (0, \frac{3}{10}]$ and $c \in [1, \infty)$, where

$$\mathcal{F}_{a,c,\delta} := \left\{ f \geq 0 \,\middle|\, \|f'\|_{L^\infty} = a \, \& \, f'(0) \leq -a + \delta \, \& \, \|f\|_{C_\gamma^2} \leq c \right\}.$$

Here we assume without loss that $x = 0$ and $f'(0) < 0$, which can be achieved by translating and reflecting f, and we also changed variables $y \leftrightarrow -y$, which replaces h_{-a} by h_a.

Fix a, c as above, and for each $\delta > 0$ and each $f \in \mathcal{F}_{a,c,\delta}$ let

$$
f_\delta(y) = \begin{cases} f(0) - ay & y \in \left(\frac{f(0)-f(-\sqrt{\delta})}{a}, \frac{f(0)-f(\sqrt{\delta})}{a} \right), \\ f(\sqrt{\delta}) & y \in \left(\frac{f(0)-f(\sqrt{\delta})}{a}, \sqrt{\delta} \right), \\ f(-\sqrt{\delta}) & y \in \left(-\sqrt{\delta}, \frac{f(0)-f(-\sqrt{\delta})}{a} \right), \\ f(y) & |y| \geq \sqrt{\delta} \end{cases}
$$

(generally $f_\delta \notin \mathcal{F}_{a,c,\delta}$). Then $h_a(y, f_\delta(0) - f_\delta(y)) = 0$ for $y \in (\frac{f(0)-f(-\sqrt{\delta})}{a}, \frac{f(0)-f(\sqrt{\delta})}{a})$, and

$$
\sqrt{\delta} - \frac{c+1}{a}\delta \leq \pm \frac{f(0) - f(\pm\sqrt{\delta})}{a} \leq \sqrt{\delta} \tag{4.15}
$$

because

$$
a \geq f'(y) \geq a - \delta - c\sqrt{\delta} \geq a - (c+1)\sqrt{\delta}
$$

for $|y| \leq \sqrt{\delta}$. Hence

$$
\sup_{f \in \mathcal{F}_{a,c,\delta}} \left| \int_{|y| \leq \sqrt{\delta}} h_a(y, f_\delta(0) - f_\delta(y)) \, dy \right| \leq \frac{2(c+1)}{a}\delta \sup_{az,w \in [0,(c+1)\delta]} \left| h_a \left(\sqrt{\delta} - z, a\sqrt{\delta} - w \right) \right| \to 0
$$

as $\delta \to 0$ because the sup is $O(\delta^{-1/2})$ (the $O(\delta)$ terms in the numerator of $h_a(\sqrt{\delta} - z, a\sqrt{\delta} - w)$ cancel). Since (4.12) (with $M_t = a$) shows that for all $f \in \mathcal{F}_{a,c,\delta}$ and all $|y| \leq \frac{a^3}{16c}$ ($\leq a f(0)$) we have

$$
|h_a(y, f_\delta(0) + f_\delta(y))| \leq \frac{2(f(0) + f_\delta(y))^2}{(f(0) + f_\delta(y))^4} \leq \frac{2}{f(0)^2} \leq \frac{512c^2}{a^4},
$$

for all small δ we obtain

$$
\sup_{f \in \mathcal{F}_{a,c,\delta}} \left| \int_{|y| \leq \sqrt{\delta}} h_a(y, f_\delta(0) + f_\delta(y)) \, dy \right| \leq \frac{1024c^2}{a^4}\sqrt{\delta} \to 0.
$$

Hence (4.14) will follow from

$$
\inf_{f \in \mathcal{F}_{a,c}} \sum_\pm \int_\mathbb{R} h_a(y, f(0) \pm f(y)) \, dy > 0 \tag{4.16}
$$

where

$$
\mathcal{F}_{a,c} := \left\{ f \geq 0 \,\middle|\, \|f'\|_{L^\infty} = a = -f'(0) \ \& \ \|f\|_{\dot{C}^{1-\gamma}} \leq c \ \& \ f(0) \geq a^2(16c)^{-1} \right\} \supseteq \{ f_\delta \,|\, f \in \mathcal{F}_{a,c,\delta} \}
$$

It turns out that the constraints involving c are not needed if we only want to prove (4.16) with the $\geq$ inequality, which we do next. We then derive (4.16) as stated at the end of this section. Let us from now on fix any $a \in (0, \frac{3}{10}]$ and denote

$$
h(y, r) := h_a(y, r) = \frac{y^2 - r^2 - (a^{-1} - a)yr}{(y^2 + r^2)^2}.
$$

Lemma 4.1. *For $a \in (0, \frac{3}{10}]$, let $\mathcal{F}$ be the set of all $f \geq 0$ such that $f'(0) = -a = \|f'\|_{L^\infty}$. Then*

$$
H(f) := \int_\mathbb{R} \left[h(y, f(0) + f(y)) + h(y, f(0) - f(y)) \right] dy \geq 0 \tag{4.17}
$$

for all $f \in \mathcal{F}$, and equality in (4.17) holds only when $f(y) \equiv a|y - f(0)a^{-1}|$.

Remark. Note that there is no PV needed in (4.17) for $f \in \mathcal{F}$. Indeed, $h(y, f(0) + f(y))$ is integrable on $\mathbb{R}$ because $f(0) > 0$, while with

$$\alpha(y) := a - \frac{f(0) - f(y)}{y} \in [0, 2a]$$

we have

$$h(y, f(0) - f(y)) \geq \frac{a^{-1} - a}{[1 + (a - \alpha(y))^2]^2} \frac{\alpha(y)}{y^2} \geq 0 \tag{4.18}$$

because $a \in (0, 1]$, so $\int_{\mathbb{R}} h(y, f(0) - f(y)) \, dy \in [0, \infty]$ also exists.

Proof. The remark above shows that the integral in (4.17) exists and belongs to $\mathbb{R} \cup \{\infty\}$. Next, if $f \in \mathcal{F}$ and $\tilde{f}(y) := \frac{f(f(0)y)}{f(0)}$, then $\tilde{f} \in \mathcal{F}$ and $H(\tilde{f}) = f(0)H(f)$. Therefore it suffices to consider the case $f(0) = 1$, which we do from now on.

Then for any $S \subseteq \mathbb{R}$ we have

$$\int_S h(y, 1 + f(y)) \, dy \geq -2 \int_0^\infty \frac{y^2 + (2 + ay)^2 + a^{-1}y(2 + ay)}{\max\{y, 2 - ay\}^4} \, dy =: I \in (-\infty, 0), \tag{4.19}$$

which together with (4.18) imples $H(f) \geq I$. Since we also have

$$H(a|y - a^{-1}|) = \int_{\mathbb{R}} \left[h(y, 2 - ay) + h(y, ay) \right] dy = 0 \tag{4.20}$$

because $h(y, ay) = 0$ for all $y \in \mathbb{R}$ and

$$a(1 + a^2) \, h(y, 2 - ay) = \frac{d}{dy} \frac{1 - 3a^2 - 2a(1 - a^2)y}{y^2 + (2 - ay)^2}, \tag{4.21}$$

it suffices to consider only f with $H(f) \leq 1$, which then implies

$$1 - I \geq \int_{\mathbb{R}} h(y, 1 - f(y)) \, dy = \int_{\mathbb{R}} \frac{a^{-1} - a}{[1 + (a - \alpha(y))^2]^2} \frac{\alpha(y)}{y^2} \, dy.$$

If for some $y_0 \in (0, 1]$ we have $y_0 \alpha(y_0) \geq C y_0^{3/2}$ with $C := 1 - I$, then from $(y\alpha(y))' = a + f'(y) \in [0, 2a]$ we obtain

$$y\alpha(y) \geq (C - 2a)y_0^{3/2} \geq (C - 2a)y^{3/2}$$

for $y \in [y_0 - y_0^{3/2}, y_0]$, and hence (4.18) shows that

$$\int_{\mathbb{R}} h(y, 1 - f(y)) \, dy \geq \int_{y_0 - y_0^{3/2}}^{y_0} \frac{a^{-1} - a}{[1 + a^2]^2} \frac{C - 2a}{y^{3/2}} \, dy \geq \frac{(a^{-1} - a)(1 - 2a)}{[1 + a^2]^2} (1 - I) > 1 - I$$

because $a \in (0, \frac{3}{10}]$ and $I \leq 0$. The same argument applies to $y_0 \in [-1, 0)$, so we see that $H(f) \leq 1$ forces $|y|\alpha(y) \leq C|y|^{3/2}$ and so

$$0 \leq \operatorname{sgn}(y) \left(f(y) - (1 - ay) \right) \leq C|y|^{3/2} \tag{4.22}$$

for all $y \in [-1, 1]$. We will make use of this estimate in the next section.

Next, (4.21) shows that it suffices to show

$$H(f) > H(a|y - a^{-1}|) \tag{4.23}$$

when f is not $a|y - a^{-1}|$ (recall that we assume $f \in \mathcal{F}$ and $f(0) = 1$). Letting $A := a^{-1} - a$, we have

$$\partial_r h(y, r) = \frac{2r^3 + 3Ayr^2 - 6y^2 r - Ay^3}{(y^2 + r^2)^3} = \frac{1}{y^3} g\left(\frac{r}{y}\right),$$

where

$$g(s) := \frac{2s^3 + 3As^2 - 6s - A}{(1 + s^2)^3}. \tag{4.24}$$

We will separately study the integral in (4.17) over $y < 0$ and $y > 0$. We start with the former, which will be much simpler to minimize. This is because at the end of this proof we will show that

$$\int_{s_0 - a}^{s_0 - b} g(s) ds > \int_b^a g(s) ds \tag{4.25}$$

for any $b \in [-a, a)$ and $s_0 < 0$. This now implies that if $y < 0$ and we let $b := \frac{1 - f(y)}{y} \in [-a, a]$ and $s_0 := \frac{2}{y} < 0$, then

$$y^2 \left[h(y, 1 + f(y)) - h(y, 2 - ay) + h(y, 1 - f(y)) - h(y, ay) \right]$$

$$= \frac{1}{y} \int_{2 - ay}^{1 + f(y)} g\left(\frac{r}{y}\right) dr + \frac{1}{y} \int_{ay}^{1 - f(y)} g\left(\frac{r}{y}\right) dr$$

$$= \int_{s_0 - a}^{s_0 - b} g(s) ds + \int_a^b g(s) ds \geq 0,$$

with equality at the end only when $f(y) = 1 - ay$ (i.e., $b = a$). This shows that $1 - ay$ is the unique minimizer of

$$H^-(f) := \int_{-\infty}^0 \left[h(y, 1 + f(y)) + h(y, 1 - f(y)) \right] dy$$

among all Lipschitz $f : (-\infty, 0] \to [0, \infty)$ such that $f'(0) = -a = \|f'\|_{L^\infty}$ and $f(0) = 1$. Recalling (4.23), it now suffices to show that $a|y - a^{-1}|$ is the unique minimizer of

$$H^+(f) := \int_0^\infty \left[h(y, 1 + f(y)) + h(y, 1 - f(y)) \right] dy \tag{4.26}$$

on the set of all Lipschitz $f : [0, \infty) \to [0, \infty)$ such that $f'(0) = -a = \|f'\|_{L^\infty}$ and $f(0) = 1$. This is the claim of the next lemma, which we prove in the next section, so the present proof will be finished once we establish (4.25).

To this end we note that from

$$g'(s) = -6 \frac{s^4 + 2As^3 - 6s^2 - 2As + 1}{(1 + s^2)^4}$$

it follows that g has at most 4 local extrema and $\text{sgn}(s) g(s) > 0$ when $|s|$ is large enough. Since $a \in (0, \frac{3}{10}]$ also implies $a^{-1} > 10a$ and so

$$g(-a^{-1}) = (a^{-3} + 2a^{-1} + a)(1 + a^{-2})^{-3} \ > 0,$$
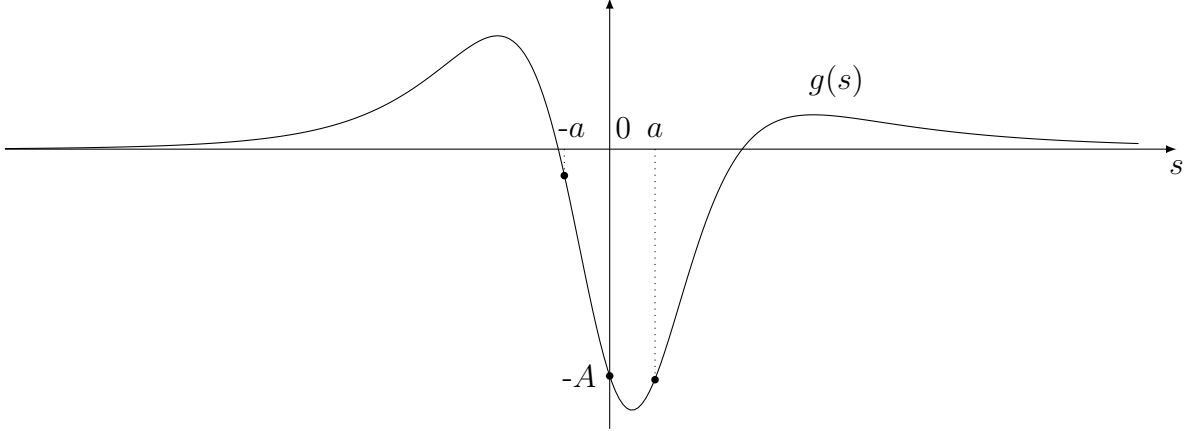$$g(-a) = -(a^{-1} - 10a + 5a^3)(1 + a^2)^{-3} \ < 0,$$
$$g(0) = -A \ < g(-a),$$

FIGURE 1. The function g for $a = \frac{3}{10}$ (and so $A = 3$).

$$g(a) = -(a^{-1} + 2a + a^3)(1 + a^2)^{-3} \; < g(0),$$
$$g(a^{-1}) = (5a^{-3} - 10a^{-1} + a)(1 + a^{-2})^{-3} \; > 0,$$
$$g'(a) = 6\frac{1 - 2a + 4a^2 + a^4}{(1 + a^2)^2} \; > 0,$$

we see that g does have 4 local extrema at

$$s_1 \in (-\infty, -a^{-1}), \qquad s_2 \in (-a^{-1}, -a), \qquad s_3 \in (0, a), \qquad s_4 \in (a, \infty), \tag{4.27}$$

and these are negative local minima at s_1, s_3 and positive local maxima at s_2, s_4. Figure 1 shows g when $a = \frac{3}{10}$, and hence $A = 3$.

We next note that

$$g(s_1) \geq \frac{2s_1^3 - A}{(1 + s_1^2)^3} \geq 3\left(\frac{s_1}{1 + s_1^2}\right)^3 \geq 3\left(\frac{-a^{-1}}{1 + a^{-2}}\right)^3 = -\frac{3a^3}{(1 + a^2)^3} > g(-a),$$

so $g(s) > g(-a)$ for all $s < -a$. This and $g(s) < g(-a)$ for $s \in (-a, a]$ show that

$$\int_{s_0 - a}^{s_0 - b} g(s)ds > (a - b)g(-a) > \int_{-a}^{-b} g(s)ds$$

when $b \in [-a, a)$ and $s_0 - b \leq -a$. We also have

$$\frac{d}{ds_0} \int_{s_0 - a}^{s_0 - b} g(s)ds = g(s_0 - b) - g(s_0 - a) < 0$$

when $s_0 - b \in [-a, a - (b)_+]$ (i.e., $s_0 \in [b - a, a - (b)_-]$) because $g(s) > g(-a)$ for all $s < -a$, g is decreasing on $[-a, 0]$, and $g(s) < g(0)$ for $s \in (0, a]$. Hence if $b \leq 0$, then for all $s_0 < a + b$ $(= a - (b)_-)$ we have

$$\int_{s_0 - a}^{s_0 - b} g(s)ds > \int_{b}^{a} g(s)ds,$$

while if $b \geq 0$, then for all $s_0 < b \ (\leq a - (b)_-)$ we have

$$\int_{s_0-a}^{s_0-b} g(s)ds > \int_{b-a}^0 g(s)ds > (a-b)g(0) > \int_b^a g(s)ds.$$

In both cases, the range of s_0 for which the inequality holds includes all $s_0 < 0$, therefore we proved (4.25). $\qquad\square$

Let us next show (4.16). Assume that $f_n \in \mathcal{F}_{a,c}$ is a sequence such that $\lim_{n\to\infty} H(f_n) = 0$ (and $H(f_n) \geq 0$ for all n by Lemma 4.1). If $\limsup_{n\to\infty} f_n(0) = \infty$, then clearly

$$\limsup_{n\to\infty} \int_0^\infty h(y, f_n(0) + f_n(y))\, dy \geq 0,$$

while (4.4) and (4.10) show that

$$\inf_n \int_0^\infty h(y, f_n(0) - f_n(y))\, dy > 0.$$

Hence $\limsup_{n\to\infty} H(f_n) > 0$, a contradiction, so we have $m := \sup_n f_n(0) < \infty$. Since also $f_n(0) \geq \frac{a^2}{16c}$, we see that

$$\tilde{f}_n(y) := \frac{f_n(f_n(0)y)}{f_n(0)} \in \mathcal{F}_{a,c'}$$

with $c' := c(\frac{a^2}{16c})^{-\gamma}$. Now $H(\tilde{f}_n) = f_n(0)H(f_n) \leq m\, H(f_n) \to 0$ and $\tilde{f}_n(0) = 1$. So the argument from the start of the proof of the next lemma, applied to H and on $\mathbb{R}$ instead to H^+ on $\mathbb{R}^+$, shows that some subsequence $\{f_{n_k}\}$ converges to some $f \in \mathcal{F}_{a,c'}$ with $f(0) = 1$ such that $H(f) = 0$. But then Lemma 4.1 shows that $f(y) \equiv a|y - a^{-1}| \notin \mathcal{F}_{a,c'}$, a contradiction.

It now remains to prove the following result, which was used in the proof of Lemma 4.1. We do this in the next section.

Lemma 4.2. *For $a \in (0, \frac{3}{10}]$, let $\mathcal{F}^+$ be the set of all Lipschitz $f : [0, \infty) \to [0, \infty)$ such that $f'(0) = -a = \|f'\|_{L^\infty}$ and $f(0) = 1$. Then $a|y - a^{-1}|$ is the unique minimizer of H^+ on $\mathcal{F}^+$.*

5. Proof of Lemma 4.2

From (4.19) and (4.18) we again see that

$$J := \inf_{f \in \mathcal{F}^+} H^+(f) \geq I$$

and (4.22) holds for all $y \in [0, 1]$ whenever $H^+(f) \leq J + 1$. Let $f_n \in \mathcal{F}^+$ be such that $H^+(f_n) \leq J + \frac{1}{n}$. The Arzelà-Ascoli Theorem and (4.22) show that there is a subsequence $\{f_{n_k}\}$ that converges to some $f \in \mathcal{F}^+$ in $L^\infty_{\text{loc}}([0, \infty))$. Since for all $\tilde{f} \in \mathcal{F}^+$ we have

$$\int_M^\infty \left| h(y, 1 + \tilde{f}(y)) + h(y, 1 - \tilde{f}(y)) \right| dy \leq \int_M^\infty \frac{y^2 + (2 + ay)^2 + a^{-1}y(2 + ay)}{y^4}\, dy \to 0$$

as $M \to \infty$, it follows that for each $\sigma \in (0, a)$ we have

$$I_{n_k,\sigma} := \int_\sigma^\infty \left[h(y, 1 + f_{n_k}(y)) + h(y, 1 - f_{n_k}(y)) \right] dy \to \int_\sigma^\infty \left[h(y, 1 + f(y)) + h(y, 1 - f(y)) \right] dy$$

as $k \to \infty$. Since $I_{n_k,\sigma} \leq H^+(f_{n_k}) + 7\sigma \leq J + \frac{1}{n_k} + 7\sigma$ for all k, which holds by (4.18) and $h(y, 1 + f_n(y)) \geq -7$ for all $(y,n) \in [0,a] \times \mathbb{N}$, we see that

$$\int_\sigma^\infty \left[h(y, 1 + f(y)) + h(y, 1 - f(y)) \right] dy \leq J + 7\sigma$$

for each $\sigma \in (0,a)$. Hence $H^+(f) \leq J$, which means that H^+ has a minimizer in $\mathcal{F}^+$.

Consider now any such minimizer f, and for $0 < p < q$ and $\delta > 0$ let

$$f_{p,q,\delta}^+(s) := \begin{cases} f(s) & s \notin (p,q), \\ \min\{f(s) + \delta, f(p) + a(s-p), f(q) + a(q-s)\} & s \in (p,q) \end{cases}$$

and

$$f_{p,q,\delta}^-(s) := \begin{cases} f(s) & s \notin (p,q), \\ \max\{f(s) - \delta, f(p) - a(s-p), f(q) - a(q-s)\} & s \in (p,q). \end{cases}$$

Then clearly $f_{p,q,\delta}^+ \in \mathcal{F}^+$, while $f_{p,q,\delta}^- \in \mathcal{F}^+$ as long as $f \geq \delta$ on $[p,q]$, and we let (with every statement below that includes $\pm$ being two separate statements)

$$\begin{aligned} p_{0+}^\pm &:= \inf \left\{ s \in (p,q) \,\middle|\, f' \not\equiv \pm a \text{ on } (p,s) \right\}, \\ p_{\delta+}^\pm &:= \inf \left\{ s \in (p,q) \,\middle|\, \int_p^s (a \mp f'(s)) ds = \delta \right\} \quad \searrow p_{0+}^\pm \text{ as } \delta \to 0^+, \\ q_{0-}^\pm &:= \sup \left\{ s \in (p,q) \,\middle|\, f' \not\equiv \mp a \text{ on } (s,q) \right\}, \\ q_{\delta-}^\pm &:= \sup \left\{ s \in (p,q) \,\middle|\, \int_s^q (a \pm f'(s)) ds = \delta \right\} \quad \nearrow q_{0-}^\pm \text{ as } \delta \to 0^+ \end{aligned}$$

(with $\inf \emptyset := \infty$ and $\sup \emptyset := -\infty$). Then for each $\delta > 0$ we have

$$\begin{aligned} f_{p,q,\delta}^\pm &\equiv f & \text{on } (-\infty, p_{0+}^\pm] \cup [q_{0-}^\pm, \infty), \\ f_{p,q,\delta}^\pm &\equiv f \pm \delta & \text{on } [p_{\delta+}^\pm, q_{\delta-}^\pm] \text{ if } p_{\delta+}^\pm \leq q_{\delta-}^\pm. \end{aligned}$$

Since f minimizes H^+, it follows that whenever $p_{0+}^\pm < q_{0-}^\pm$ (as well as $\min_{s \in [p,q]} f(s) > 0$ in the "$-$" case), we have

$$\begin{aligned} 0 \leq \lim_{\delta \to 0^+} \frac{H^+(f_{p,q,\delta}^\pm) - H^+(f)}{\delta} &= \pm \int_{p_{0+}^\pm}^{q_{0-}^\pm} \left[h_r(y, 1 + f(y)) - h_r(y, 1 - f(y)) \right] dy \\ &= \pm \int_{p_{0+}^\pm}^{q_{0-}^\pm} \frac{1}{y^3} \left[g\left(\frac{1 + f(y)}{y} \right) - g\left(\frac{1 - f(y)}{y} \right) \right] dy. \end{aligned}$$

This shows that if $f' \not\equiv \pm a$ on $(p, p + \varepsilon)$ and $f' \not\equiv \mp a$ on $(q - \varepsilon, q)$ for all $\varepsilon > 0$, and also $\min_{s \in [p,q]} f(s) > 0$ in the "lower sign" case, then

$$\pm \int_p^q \frac{1}{y^3} \left[g\left(\frac{1 + f(y)}{y} \right) - g\left(\frac{1 - f(y)}{y} \right) \right] dy \geq 0. \tag{5.1}$$

Assume that $f' \not\equiv a$ and $f' \not\equiv -a$ on $(p, p + \varepsilon)$ for all $\varepsilon > 0$. Then there are $q_n \searrow p$ such that $f' \not\equiv -a$ on $(q_n - \varepsilon, q_n)$ for all $\varepsilon > 0$, so (5.1) yields

$$0 \leq \frac{p^3}{q_n - p} \int_p^{q_n} \frac{1}{y^3} \left[g\left(\frac{1 + f(y)}{y}\right) - g\left(\frac{1 - f(y)}{y}\right) \right] dy \to g\left(\frac{1 + f(p)}{p}\right) - g\left(\frac{1 - f(p)}{p}\right).$$

There are also $q'_n \searrow p$ such that $f' \not\equiv a$ on $(q'_n - \varepsilon, q'_n)$ for all $\varepsilon > 0$, so (5.1) again yields

$$0 \geq \frac{p^3}{q'_n - p} \int_p^{q'_n} \frac{1}{y^3} \left[g\left(\frac{1 + f(y)}{y}\right) - g\left(\frac{1 - f(y)}{y}\right) \right] dy \to g\left(\frac{1 + f(p)}{p}\right) - g\left(\frac{1 - f(p)}{p}\right),$$

provided $f(p) > 0$. This will yield part (i) of the following result.

Lemma 5.1. *Let f be a minimizer of H^+ on $\mathcal{F}^+$ and $p \in (0, \infty)$.*
  (i) If $f' \not\equiv a$ and $f' \not\equiv -a$ on $(p, p + \varepsilon)$ for all $\varepsilon > 0$, then

$$g\left(\frac{1 + f(p)}{p}\right) = g\left(\frac{1 - f(p)}{p}\right). \tag{5.2}$$

  (ii) If $f' \not\equiv a$ and $f' \not\equiv -a$ on $(p, p - \varepsilon)$ for all $\varepsilon > 0$, then (5.2) holds.
  (iii) If $f' \equiv -a$ on $(p - \varepsilon, p)$ and $f' \equiv a$ on $(p, p + \varepsilon)$ for some $\varepsilon > 0$, then

$$g\left(\frac{1 + f(p)}{p}\right) \geq g\left(\frac{1 - f(p)}{p}\right). \tag{5.3}$$

  (iv) If $f' \equiv a$ on $(p - \varepsilon, p)$ and $f' \equiv -a$ on $(p, p + \varepsilon)$ for some $\varepsilon > 0$, then

$$g\left(\frac{1 + f(p)}{p}\right) \leq g\left(\frac{1 - f(p)}{p}\right).$$

Proof. We proved (i) when $f(p) > 0$, but it obviously also holds when $f(p) = 0$. We can obtain (ii) via a similar argument (with $q_n \nearrow p$). In (iii) we see that (5.1) holds with $\pm$ being $+$ and with $(p - \varepsilon, p + \varepsilon)$ in place of (p, q) whenever $\varepsilon > 0$ is small enough, so multiplying it by $\frac{p^3}{2\varepsilon}$ and taking $\varepsilon \to 0$ yields (5.3). Finally, (iv) is obtained similarly when $f(p) > 0$, via (5.1) with $\pm$ being $-$, while it holds trivially when $f(p) = 0$. $\qquad \square$

Consider the open sets

$$P_\pm := \left\{ y > 0 \,\middle|\, \pm g\left(\frac{1 + f(y)}{y}\right) > \pm g\left(\frac{1 - f(y)}{y}\right) \right\}$$

and the set

$$P_0 := \left\{ y > 0 \,\middle|\, g\left(\frac{1 + f(y)}{y}\right) = g\left(\frac{1 - f(y)}{y}\right) \right\}.$$

Lemma 5.1 is now clearly equivalent to (i) and (ii) in the following result. In it we also let $s_y^\pm := \frac{1 \pm f(y)}{y}$, which satisfy

$$-a \leq s_y^- \leq \min\left\{ a, \frac{1}{y} \right\} \leq \frac{1}{y} \leq s_y^+. \tag{5.4}$$

Lemma 5.2. *(i) For each $y \in P_+$ there is $\varepsilon > 0$ such that either $f' \equiv -a$ on $(y - \varepsilon, y + \varepsilon)$, or $f' \equiv a$ on $(y - \varepsilon, y + \varepsilon)$, or $f' \equiv -a$ on $(y - \varepsilon, y)$ and $f' \equiv a$ on $(y, y + \varepsilon)$.*

(ii) For each $y \in P_-$ there is $\varepsilon > 0$ such that either $f' \equiv a$ on $(y - \varepsilon, y + \varepsilon)$, or $f' \equiv -a$ on $(y - \varepsilon, y + \varepsilon)$, or $f' \equiv a$ on $(y - \varepsilon, y)$ and $f' \equiv -a$ on $(y, y + \varepsilon)$.

(iii) If $\frac{1 + f(y_0)}{y_0} \leq a$ for some $y_0 \in P_0$, then with $y_1 := \inf\{y \geq y_0 \,|\, f(y) = 0\}$ we have $(y_0, y_1) \in P_-$.

Proof. It remains to prove (iii), so assume that y_0 is as above, and also that $f(y_0) > 0$ (otherwise the result is trivial). First note that $s^+_{y_0} \leq a$ implies $s^+_y \leq a$ for all $y \geq y_0$. Then $y_0 \in P_0$ and $f(y_0) > 0$ show that if $s' \in (0, s_3)$ is the unique number with $g(s') = g(a)$ (recall that $g(0) > g(a)$), then

$$0 < s' \leq s^-_{y_0} < s_3 < s^+_{y_0} \leq a$$

(and in particular, $g'(s^-_{y_0}) < 0 < g'(s^+_{y_0})$). We have

$$y \frac{d}{dy} \left(g(s^+_y) - g(s^-_y) \right) = -\left(s^+_y g'(s^+_y) + s^-_y |g'(s^-_y)| \right) + f'(y) \left(g'(s^+_y) + g'(s^-_y) \right) \tag{5.5}$$

for all $y > 0$, and we also claim that

$$r g'(r) + q |g'(q)| > a \,|g'(r) + g'(q)| \tag{5.6}$$

whenever $s' \leq q < r \leq a$ and $g(r) = g(q)$. From this and (5.5), and $g(s^+_{y_0}) = g(s^-_{y_0})$ we see that $\frac{d}{dy} \left(g(s^+_y) - g(s^-_y) \right) < 0$ holds for $y = y_0$ as well as for any other $y \in [y_0, y_1) \cap P_0$. Therefore we must have $(y_0, y_1) \in P_-$, and it remains to prove (5.6) assuming that $s' \leq q < r \leq a$ and $g(r) = g(q)$.

If g'' were constant on $[0, a]$, we would have $g'(r) = -g'(q) \neq 0$ and (5.6) would follow trivially. We therefore want to show that g'' does not vary too much on $[0, a]$. We note that this can be done easily for all small enough $a > 0$, but we provide here a quantitative argument that applies to all $a \in (0, \frac{3}{10}]$. Since

$$\frac{g''(s)}{12} = \frac{A + 10s - 10As^2 - 20s^3 + 5As^4 + 2s^5}{(1 + s^2)^5},$$

$$\frac{g'''(s)}{120} = \frac{1 - 3As - 15s^2 + 10As^3 + 15s^4 - 3As^5 - s^6}{(1 + s^2)^6},$$

and $a \leq \frac{3}{10}$, we see that g'' increases on $[0, s'']$ and decreases on $[s''', a]$ for some $s'' < s'''$ and both close to $\frac{a}{3}$ (because the second numerator above is approximately $1 - 3a^{-1}s$). In fact, a simple analysis of the three parentheses in

$$3as - 15s^2 + 10As^3 + 15s^4 - 3As^5 - s^6$$
$$= (3as - 15s^2 + 10a^{-1}s^3) - (10as - 15s^2 + 3a^{-1}s^3)s^2 + (3as - s^2)s^4$$

shows that when $s \in [0, a]$, they take values in $[-2a^2, \frac{a^2}{5}]$, $[-2a^2, 2a^2]$, and $[0, 2a^2]$, respectively. Thus the absolute value of the whole expression is bounded by $2a^2(1 + a^2 + a^4) \leq \frac{11a^2}{5}$, which means that for $s \in [0, a]$ we have $\mathrm{sgn}(s - \frac{a}{3})g'''(s) \geq 0$ when $3a^{-1}|s - \frac{a}{3}| \geq \frac{11a^2}{5}$. This holds when $|s - \frac{a}{3}| \geq \frac{a}{15}$ because $a^2 \leq \frac{9}{100}$, so we can pick $s'' := \frac{a}{3} - \frac{a}{15}$ and $s''' := \frac{a}{3} + \frac{a}{15}$ above.

Then $|g'''(s)| \leq 120(3a^{-1}\frac{a}{15} + \frac{11a^2}{5}) \leq 48$ on $[s'', s''']$, so $|g''(s) - g''(\frac{a}{3})| \leq \frac{16a}{5} \leq 1$ there. Hence $g''(s)$ takes values in $[g''(0), g''(\frac{a}{3}) + 1]$ on $[0, s_3]$ and in $[g''(a), g''(\frac{a}{3}) + 1]$ on $[s_3, a]$.

A simple estimate thus yields

$$7.5A - 1 \leq 12\left(A - \frac{3}{20}\right)(1 + a^2)^{-5} \leq g''(s) \leq 12(A + 0.7) + 1 \leq 12A + 10$$

for $s \in [0, a]$. This shows that if $g(r) = g(q)$ for some $s' \leq q < r \leq a$, then

$$\max\left\{\frac{|q - s_3|}{|r - s_3|}, \frac{|g'(q)|}{|g'(r)|}\right\} \leq \sqrt{\frac{12A + 10}{7.5A - 1}} \leq \frac{3}{2}$$

because $A \geq 3$. Then $s_3 \geq s' + \frac{2}{5}(a - s') \geq \frac{2a}{5}$ and

$$rg'(r) + q|g'(q)| \geq \frac{2ag'(r)}{5} > \frac{ag'(r)}{3} \geq a\,|g'(r) + g'(q)|\,.$$

Hence we proved (5.6) and the proof of (iii) is finished. $\qquad\square$

From (5.4) we see that $g(s_y^-) \leq g(-a) < 0 < g(s_y^+)$ holds for all small enough $y > 0$ (recall that $g(a) < g(-a)$) and so $(0, \sigma) \subseteq P_+$ for some $\sigma > 0$. Let us pick the largest such σ. Since also $f'(0) = -a$, it follows from Lemma 5.2(i) that $f(y) = 1 - ay$ on $[0, p]$ for some $p \in (0, a^{-1}]$ (again pick the largest such p). We first claim that $p = a^{-1}$.

Assume therefore that $p < a^{-1}$. Then $s_y^- = a < s_y^+$ for all $y \leq p$, so $\sigma > p$. Lemma 5.2(i) then shows that $f(y) = 1 + a(y - 2p)$ on $[p, \sigma]$. Let $q \geq \sigma$ be the largest number such that $f(y) = 1 + a(y - 2p)$ on $[p, q]$ (or let $q := \infty$ if $f(y) = 1 + a(y - 2p)$ on $[p, \infty)$). For all $\delta \in (0, a^{-1} - p]$ let

$$\mathcal{F}^+ \ni f_\delta(y) := \begin{cases} 1 - ay & y \in [0, p + \delta], \\ \min\{1 + a(y - 2(p + \delta)), f(y)\} & y > p + \delta. \end{cases}$$

Then $f_\delta = f - 2a\delta$ on $[p + \delta, q)$ and $f_\delta = f$ on $[q_\delta, \infty)$ for some $q_\delta \searrow q$ as $\delta \to 0$ (the latter if $q < \infty$). It follows that

$$0 \leq \lim_{\delta \to 0^+} \frac{H^+(f_\delta) - H^+(f)}{2a\delta} = \int_p^q \left[-h_r(y, 1 + f(y)) + h_r(y, 1 - f(y))\right] dy$$

$$= \int_p^q \frac{-g((2 - 2ap)y^{-1} + a) + g(2apy^{-1} - a)}{y^3}\, dy. \tag{5.7}$$

Arguments of both functions in the last integral are strictly decreasing in y, and for $y \in [p, \infty)$ we have

$$-a < 2apy^{-1} - a \leq a < (2 - 2ap)y^{-1} + a.$$

The properties of g then imply that there is $q_0 \in [p, \infty)$ such that the integrand is negative on $[p, q_0)$ and positive on (q_0, ∞). If we therefore show that

$$\int_p^\infty \frac{-g((2 - 2ap)y^{-1} + a) + g(2apy^{-1} - a)}{y^3}\, dy < 0, \tag{5.8}$$

it will follow that this integral with any $q \in (p, \infty]$ in place of ∞ is also negative, which will contradict (5.7).

One can directly integrate in (5.8), but it will be simpler to instead keep the limit in (5.7) outside of the integral. We see that the left-hand side of (5.8) is

$$\lim_{\delta \to 0^+} \frac{K_\delta - K_0}{2a\delta},$$

where

$$K_\delta := \int_p^\infty \left[h(y, 2 + a(y - 2p - 2\delta)) + h(y, a(2p + 2\delta - y)) \right] dy.$$

We have

$$h(y, B \pm ay) = \frac{y^2 - (B \pm ay)^2 - Ay(B \pm ay)}{[y^2 + (B \pm ay)^2]^2} = \partial_y G^\pm(B, y), \tag{5.9}$$

where (recall that $A = \frac{1-a^2}{a}$)

$$G^+(B, y) := \frac{1}{2a} \frac{B}{y^2 + (B + ay)^2},$$

$$G^-(B, y) := \frac{1 - 3a^2}{2a(1 + a^2)} \frac{B}{y^2 + (B - ay)^2} - \frac{2(1 - a^2)}{1 + a^2} \frac{y}{y^2 + (B - ay)^2},$$

and then

$$\partial_B G^+(B, y) = \frac{1}{2a} \frac{(1 + a^2)y^2 - B^2}{[y^2 + (B + ay)^2]^2},$$

$$\partial_B G^-(B, y) = \frac{1}{2a(1 + a^2)} \frac{(3a^2 - 1)B^2 + 8a(1 - a^2)yB + (1 - 10a^2 + 5a^4)y^2}{[y^2 + (B - ay)^2]^2}.$$

This and $p \in (0, a^{-1})$ means that if we first integrate in y and then take $\delta \to 0^+$, we obtain

$$\lim_{\delta \to 0^+} \frac{K_\delta - K_0}{\delta} = 2a \, \partial_B G^+(2 - 2ap, p) - 2a \, \partial_B G^-(2ap, p)$$

$$= \frac{(1 + a^2)p^2 - 4(1 - ap)^2}{[p^2 + (2 - ap)^2]^2} - \frac{(1 + a^2)p^2}{[p^2 + (ap)^2]^2} < -\frac{4(1 - ap)^2}{[p^2 + (2 - ap)^2]^2} < 0,$$

proving (5.8). This yields a contradiction, so indeed $p = a^{-1}$ and $f(y) = 1 - ay$ on $[0, a^{-1}]$.

Next, let us consider f immediately to the right of $y = a^{-1}$. One option is $f \equiv 0$ on $[a^{-1}, a^{-1} + \varepsilon]$ for some $\varepsilon > 0$, and we analyze it later. Otherwise we have $a^{-1} \in \partial P$, where $P := \{y > a^{-1} \,|\, f(y) > 0\}$. Since $(1 \pm f(a^{-1}))/a^{-1} = a$ and $g'(a) > 0$, it follows that $P \cap (a^{-1}, a^{-1} + \varepsilon) \subseteq P_+$ for some $\varepsilon > 0$. So Lemma 5.2(i) applies on this set and we see that $f(y) = a(y - a^{-1}) = ay - 1$ on $(a^{-1}, a^{-1} + \varepsilon)$.

If $f(y) = ay - 1$ does not hold on all of $[a^{-1}, \infty)$, let $q \in [a^{-1} + \varepsilon, \infty)$ be the largest number such that this equality holds on $[a^{-1}, q]$. We can now proceed similarly to the case $p < a^{-1}$ above to obtain contradiction with f being a minimizer of H^+ on $\mathcal{F}^+$. If

$$f_\delta(y) := \begin{cases} \max\{1 - ay, 0\} & y \in [0, a^{-1} + \delta], \\ \min\{ay - 1 - a\delta, f(y)\} & y > a^{-1} + \delta, \end{cases}$$

then

$$0 \leq \lim_{\delta \to 0^+} \frac{H^+(f_\delta) - H^+(f)}{a\delta} = \int_{a^{-1}}^q \frac{-g(a) + g(2y^{-1} - a)}{y^3} \, dy. \tag{5.10}$$

The integrand is again negative on (a^{-1}, q_0) and positive on (q_0, ∞) for some $q_0 \in (a^{-1}, \infty)$, so it suffices to show that

$$\int_{a^{-1}}^{\infty} \frac{-g(a) + g(2y^{-1} - a)}{y^3} \, dy \leq 0 \tag{5.11}$$

in order to get a contradiction with (5.10) (since also $q < \infty$). Letting

$$K_\delta := \int_{a^{-1}}^{\infty} \left[h(y, ay - a\delta) + h(y, 2 + a\delta - ay)) \right] dy,$$

we again see that the left-hand side of (5.11) equals

$$\lim_{\delta \to 0^+} \frac{K_\delta - K_0}{a\delta} = \partial_B G^+(0, a^{-1}) - \partial_B G^-(2, a^{-1}) = \frac{a}{2(1 + a^2)} - \frac{a}{2(1 + a^2)} = 0.$$

So we proved (5.11), obtaining a contradiction, which then forces $f(y) = a|y - a^{-1}|$ on $[0, \infty)$.

It remains to show that if $f \equiv 0$ on $[a^{-1}, a^{-1} + \varepsilon]$ for some $\varepsilon > 0$, then f cannot be a minimizer of H^+ on $\mathcal{F}^+$. Assume first that $f \not\equiv 0$ on $[a^{-1}, \infty)$, and let $p > a^{-1}$ be the largest number such that $f \equiv 0$ on $[a^{-1}, p]$. Let also $q := \min\{y > p \,|\, f(y) = 0\} \in (p, \infty]$. Then Lemma 5.2(iii) shows that there is at most one number in $(p, q) \cap P_0$, and if such number y_0 does exist, then

$$(p, q) \cap P_+ = (p, y_0) \qquad \text{and} \qquad (p, q) \cap P_- = (y_0, q). \tag{5.12}$$

If there is no such y_0, then (p, q) is one of $P_\pm$, so we can pick $y_0 \in \{p, q\}$ so that (5.12) holds.

Then Lemma 5.2(i,ii) show that either $f(y) = a(y - p)$ for all $y \in [p, \infty)$ (when $q = \infty$) or

$$f(y) = \begin{cases} a(y - p) & y \in [p, \frac{q+p}{2}], \\ a(q - y) & y \in [\frac{q+p}{2}, q] \end{cases}$$

(when $q < \infty$, and then we must have $\frac{q+p}{2} \geq y_0$). In the latter case we let

$$f_\delta(y) := \begin{cases} \max\{f(y), a(y - p)\} & y \in [0, \frac{q+p}{2} + \delta], \\ \max\{f(y), a(q + 2\delta - y)\} & y > \frac{q+p}{2} + \delta, \end{cases}$$

and then

$$0 \leq \lim_{\delta \to 0^+} \frac{H^+(f_\delta) - H^+(f)}{2a\delta} = \int_{\frac{q+p}{2}}^{q} \frac{1}{y^3} \left(g\left(\frac{1 + f(y)}{y} \right) - g\left(\frac{1 - f(y)}{y} \right) \right) dy < 0$$

(the last inequality follows from $(\frac{q+p}{2}, q) \subseteq P_-$), a contradiction. In the former case we let

$$f_\delta(y) := \max\{f(y), a(y - p + \delta)\}$$

and then

$$0 \leq \lim_{\delta \to 0^+} \frac{H^+(f_\delta) - H^+(f)}{a\delta} = -\partial_B G^+(1 - ap, p) + \partial_B G^-(1 + ap, p)$$

$$= -\frac{1}{2a} \frac{p^2 + 2ap - 1}{(p^2 + 1)^2} + \frac{1}{2a(1 + a^2)} \frac{p^2(1 - 3a^2) + p(6a - 2a^3) + 3a^2 - 1}{(p^2 + 1)^2}$$

$$= -\frac{2a}{1 + a^2} \frac{p^2 - Ap - 1}{(p^2 + 1)^2} < 0,$$

where at the end we used $p > a^{-1}$, together with $a^{-2} - Aa^{-1} - 1 = 0$ and $(y^2 - Ay - 1)' > 0$ on (a^{-1}, ∞). This is again a contradiction, so it remains to consider the case $f \equiv 0$ on $[a^{-1}, \infty)$.

In that case we pick any $y_0 > s_3^{-1}$, so that there is $\varepsilon > 0$ such that

$$\partial_{rr} h(y, r) = \frac{1}{y^4} \, g'\left(\frac{r}{y}\right) < 0 \tag{5.13}$$

for all $(y, r) \in B_{2\varepsilon}((y_0, 1))$. Then we pick any $f_0 \in \mathcal{F}^+$ that is equal to f outside $(y_0 - \varepsilon, y_0 + \varepsilon)$ and it is positive on this interval (from $f \in \mathcal{F}^+$ we have $f_0 \leq a\varepsilon$ there). Then

$$H^+(f_0) - H^+(f) = \int_{y_0 - \varepsilon}^{y_0 + \varepsilon} \left[h(y, 1 + f_0(y)) + h(y, 1 - f_0(y)) - 2h(y, 1) \right] dy < 0$$

by (5.13). This again yields a contradiction with the assumption that f is a minimizer of H^+ on $\mathcal{F}^+$, so the proof of Lemma 4.2 is finished.

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