LOCAL REGULARITY AND FINITE TIME SINGULARITY FOR THE GENERALIZED SQG EQUATION ON THE HALF-PLANE

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ABSTRACT. We show that the generalized SQG equation with $\alpha \in (0, \frac{1}{4}]$ is locally well-posed on the half-plane in spaces of bounded integrable locally Lipschitz functions that are natural for its dynamic on domains with boundaries, and allow for some power growth of the derivative in the normal direction at the boundary. We also show existence of solutions with smooth initial data that exhibit finite time blow-up in the whole local well-posedness parameter regime $\alpha \in (0, \frac{1}{4}]$, which is the first finite time singularity result for equations (as opposed to patch models) of this type. Moreover, we prove sharpness of both these results by showing ill-posedness of the PDE in the above spaces when $\alpha > \frac{1}{4}$.

1. INTRODUCTION AND MAIN RESULTS

The 2D Euler and (inviscid) SQG equations of fluid dynamics are both active scalar PDE

$$\partial_t \theta + u \cdot \nabla \theta = 0 \tag{1.1}$$

with the Biot-Savart law

$$u := -\nabla^\perp (-\Delta)^{-1+\alpha} \theta, \tag{1.2}$$

where $\alpha = 0$ in the Euler case and $\alpha = \frac{1}{2}$ in the SQG case, $\nabla^\perp := (\partial_{x_2}, -\partial_{x_1})$, and Δ is the Dirichlet Laplacian if the PDE is stated on a domain with a boundary. When $\alpha \in (0, \frac{1}{2})$, (1.1) is called the generalized SQG equation (g-SQG), and this family interpolates between Euler and SQG. The Euler equation was first written down by Euler in the 1750s, while SQG and g-SQG appeared in mathematics and physics literature in the last half-century, in works going back to Blumen [1], Pedlosky [23], and Constantin, Majda, and Tabak [8] for SQG, and Pierrehumbert, Held, and Swanson [24], Smith et al. [26], and Constantin, Iyer, and Wu [7] for g-SQG.

The Euler equation is globally well-posed on both $\mathbb{R}^2$ and domains with (smooth enough) boundaries. For SQG and g-SQG on $\mathbb{R}^2$, local well-posedness results in appropriately regular spaces go back to Wu [28] and Chae, Constantin, and Wu [3], while the question of global well-posedness vs. blow-up remains one of the central open problems in fluid dynamics; see also Resnick [25] for existence of global weak L^2 solutions, Buckmaster, Shkoller, and Vicol [2] for non-uniqueness of weak solutions (via convex integration techniques), and Córdoba and Martínez-Zoroa [11] for ill-posedness results in C^k for $k \geq 2$ (the last three papers all involve SQG). But so far, no well-posedness theory was developed for the SQG and g-SQG equations on domains with boundaries, other than by Constantin and Nguyen for SQG and regular enough solutions that vanish at the boundary [9]. In fact, only existence of weak solutions for general L^2 initial data was obtained by Constantin and Nguyen [10] and by Constantin, Ignatova, and Nguyen [6] for SQG, and by Nguyen [22] for g-SQG, but such solutions might not be unique (these four papers all considered bounded domains with smooth boundaries).

The reason behind this limited progress on domains with boundaries may be underpinned by the fact that while the Biot-Savart law is non-local for all α, the relationship between the solution and its (time-dependent) stream function $\psi := -(-\Delta)^{-1+\alpha}\theta$, satisfying $\psi \equiv 0$ on the boundary, is local in one direction when $\alpha = 0$ (i.e., $\theta = \Delta\psi$). This allows for ψ, and hence also for $u = \nabla^\perp \psi$ to be regular up to the boundary in the Euler case, as long as the boundary itself is regular (cf. [14, 15] and references therein), even if θ does not vanish there.

This property does not carry over to $\alpha > 0$, and thus complicates development of the corresponding theory on domains with boundaries. In fact, it is easy to see that when θ does not vanish on the boundary, its associated velocity u will only be Hölder continuous there, with the normal derivative of the tangential component of u growing as distance to the boundary to the power -2α (see (2.5) and (3.3) below). As a result, any such theory for (1.1) has to involve solutions whose regularity deteriorates at the boundary in some sense.

In contrast, Kiselev, Yao, and the author did establish in 2016 local well-posedness for the corresponding patch problem on the half-plane when $\alpha \in (0, \frac{1}{24})$, where patches with H^3 boundaries are allowed to touch the domain boundary [20] (Gancedo and Patel recently proved this for H^2 patches and $\alpha \in (0, \frac{1}{6})$ [13]). In obtaining this result, they leveraged the fact that angles of the tangent lines of smooth enough patch boundaries must vanish at the domain boundary, so the lower regularity of the tangential velocity at that boundary will not distort the patches critically, at least for small enough $\alpha > 0$. Hence even though the velocity u does not remain smooth up to the boundary, the patches themselves do stay regular, so [20] could avoid having to study solutions with singular boundary behaviors in this setting.

The first main contribution of the present paper is the proof that there is in fact a family of spaces in which local well-posedness does hold for (1.1) with $\alpha \in (0, \frac{1}{4}]$. Moreover, this family is natural for the PDE in the sense that it captures the boundary growth of derivatives of solutions dictated by the dynamic of the equation — and hence also captures the level of boundary regularity exhibited by solutions with *up-to-the-boundary smooth* initial data. We do this on the half-plane $\mathbb{R} \times \mathbb{R}^+$ here, where (1.2) becomes

$$u(t,x) := \int_{\mathbb{R} \times \mathbb{R}^+} \left(\frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-\bar{y})^\perp}{|x-\bar{y}|^{2+2\alpha}} \right) \theta(t,y)dy, \qquad (1.3)$$

with $\bar{y} := (y_1, -y_2)$ and $y^\perp := (y_2, -y_1)$ (the latter will be more convenient for us than the more standard definition $y^\perp := (-y_2, y_1)$, and is equivalent via multiplying θ by -1). We also dropped some factor $c_\alpha > 0$ in (1.3), which can be done via appropriate scaling in time.

Our crucial observation is that in order to obtain local regularity of solutions, one should allow *only* the normal derivative of the solution to deteriorate at the boundary, while requiring its tangential derivative to remain bounded. Our key Lemma 2.2 below shows that this then results in only the normal derivative of the tangential component of the velocity to also deteriorate at the boundary (which in turn only affects the normal derivative of the solution), while the tangential derivative of the velocity as well as the normal derivative of its normal component will remain bounded. In fact, we show that the tangential derivative of the normal component of the velocity then even vanishes at the boundary at an appropriate rate, despite the gradient of the solution (and of the velocity) being unbounded there. This is again dictated by the dynamic of the PDE, and plays a crucial role in our proof of local well-posedness.

We therefore pick some $\beta \in [0, 1)$ and let $W^{1,\infty}_\beta(\mathbb{R} \times \mathbb{R}^+)$ be the space of all $\theta : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\|\theta\|_{L^\infty} < \infty \qquad \text{and} \qquad \|\theta\|_{\dot{W}^{1,\infty}_\beta} := \|\partial_{x_1}\theta\|_{L^\infty} + \|\min\{x_2^\beta, 1\}\partial_{x_2}\theta\|_{L^\infty} < \infty,$$

with the norm $\|\theta\|_{W^{1,\infty}_\beta} := \|\theta\|_{L^\infty} + \|\theta\|_{\dot{W}^{1,\infty}_\beta}$. Note that if we define

$$\tilde{\theta}(x_1, x_2) := \theta\left(x_1, \lambda_\beta(x_2)\right) \tag{1.4}$$

on $\mathbb{R} \times \mathbb{R}^+$, with

$$\lambda_\beta(x_2) := \begin{cases} (1-\beta)^{1/(1-\beta)}x_2^{1/(1-\beta)} & x_2 \in (0, (1-\beta)^{-1}), \\ x_2 - \frac{\beta}{1-\beta} & x_2 \geq (1-\beta)^{-1}, \end{cases} \tag{1.5}$$

then

$$\|\theta\|_{L^\infty} = \|\tilde{\theta}\|_{L^\infty} \qquad \text{and} \qquad \|\theta\|_{\dot{W}^{1,\infty}_\beta} = \|\tilde{\theta}\|_{\dot{W}^{1,\infty}}. \tag{1.6}$$

This means that if we "stretch" the strip $\mathbb{R} \times [0, 1]$ in the vertical direction according to (1.4), then $W^{1,\infty}_\beta$ becomes precisely $W^{1,\infty}$ (also note that clearly $W^{1,\infty}_\beta \subseteq W^{1,\infty}_{\beta'}$ when $\beta \leq \beta'$). In these new coordinates, the PDE is again a transport equation (2.12), but with velocity $\tilde{u}$ from (2.11) below. Its dynamic is not incompressible anymore, but it instead preserves the measure $\lambda'_\beta(x_2)dx$ on $\mathbb{R} \times \mathbb{R}^+$. Moreover, Lemma 2.2 can be used to show that $\tilde{u}$ is in fact Lipschitz when $\tilde{\theta}$ is (see (2.15)), provided $\beta \in [2\alpha, 1 - 2\alpha]$, which then yields local well-posedness via standard techniques. This thus yields our main local regularity result, Theorem 1.1(i) below.

Having established local regularity, the natural next question is that of global regularity vs. finite time blow-up. The 2D Euler equation is known to be *critical*: Hölder [16] and Wolibner [27] showed in 1933 that gradients of smooth solutions on smooth domains cannot grow faster than double-exponentially in time (see also Yudovich [29]), and Kiselev and Šverák demonstrated such growth to be possible on domains with a boundary (presence of the boundary was crucial in this work, and while the author showed that gradient of $C^{1,\alpha}$ solutions can grow exponentially on $\mathbb{T}^2$ [30], the question of double-exponential gradient growth on $\mathbb{R}^2$ or $\mathbb{T}^2$, or even of exponential growth of smooth solutions on these domains, remains open). This suggests $\alpha = 0$ to be a borderline case for finite time blow-up, at least for domains with boundaries, and this was confirmed for the corresponding patch problem by Kiselev, Ryzhik, Yao, and the author in [21], where they showed finite time blowup for the H^3 g-SQG patch model on the half-plane when $\alpha \in (0, \frac{1}{24})$, which is the interval of α where [20] established local well-posedness (they also proved global well-posedness for the corresponding Euler problem).

Their argument in fact equally applies to the PDE (1.1), and hence yields finite time blow-up for these α once Theorem 1.1(i) is proved. Moreover, the argument works for all $\alpha > 0$ such that $\frac{20^{-\alpha}}{6} - \frac{1}{1-2\alpha} - 2^{-\alpha} > 0$, which means that it breaks down at $\alpha \approx 0.05$ and is far from covering the full range of α for which we proved local well-posedness in Theorem 1.1(i). Gancedo and Patel used in [13] the same approach with a slightly modified setup that applies up to $\alpha \approx 0.17$, which also suffices to obtain finite time blow-up for all $\alpha \in (0, \frac{1}{6})$ (i.e., the range of their local regularity result for H^2 g-SQG patches on the half-plane), but it does not extend beyond that and so still does not cover all $\alpha \in (0, \frac{1}{4}]$.

We revisit here the argument from [21] but perform a much more precise analysis, with two important improvements. First, we obtain much smaller errors, and in fact evaluate the relevant integrals *exactly* in the most critical case; and second, we sharpen the estimate on the "worst case" scenario for the integrands in those integrals. These allow us to obtain almost optimal bounds in this argument (we only give up relatively little in the identification of the worst case scenario), thanks to which we are in fact able to capture the full interval $\alpha \in (0, \frac{1}{4}]$ on which we obtained local well-posedness. Near-optimality of our estimates turns out to be crucial because even then our final bound makes the blow-up argument break down at $\alpha \approx 0.257$, barely past the minimum needed for a complete answer to the finite time blow-up question! See the discussion after Lemma 4.2 for more details on this.

The above two results form the two parts of the following main result of this paper.

Theorem 1.1. *Let* $\alpha \in (0, \frac{1}{4}]$, $\beta \in [2\alpha, 1 - 2\alpha]$, *and* $X_\beta := W_\beta^{1,\infty}(\mathbb{R} \times \mathbb{R}^+) \cap L^1(\mathbb{R} \times \mathbb{R}^+)$.

(i) For any $\theta_0 \in X_\beta$, *there is* $T_{\theta_0} \in (0, \infty]$ *and a unique classical solution* $\theta \in L_{\text{loc}}^\infty([0, T_{\theta_0}); X_\beta)$ *to* (1.1) *with* $\theta(0, \cdot) = \theta_0$, *where* T_{θ_0} *is bounded below by some positive function of* $(\alpha, \beta, \|\theta_0\|_{X_\beta})$ *that is decreasing in the last argument, and* $\lim_{t \to T_{\theta_0}} \|\theta(t, \cdot)\|_{\dot{W}_\beta^{1,\infty}} = \infty$ *whenever* $T_{\theta_0} < \infty$.

(ii) There is $\theta_0 \in X_\beta \cap C^\infty(\mathbb{R} \times \mathbb{R}^+)$ *such that* $T_{\theta_0} < \infty$, *so the unique solution* θ *blows up in finite time.*

Remarks. 1. Since the dynamic of (1.1) is measure-preserving, the last claim is equivalent to $\lim_{t \to T_\infty} \|\theta(t, \cdot)\|_{X_\beta} = \infty$. By (1.6), it is also equivalent to $\lim_{t \to T_\infty} \|\tilde{\theta}(t, \cdot)\|_{\dot{W}^{1,\infty}} = \infty$.

2. Here we refer to locally Lipschitz solutions for which (1.1) holds almost everywhere as *classical* solutions. However, it is not difficult to see that the result extends to spaces and initial data with higher degree of (local) regularity.

3. One has a choice of β when $\alpha \in (0, \frac{1}{4})$, and arguably $\beta = 2\alpha$ could be considered most natural as this reflects precisely the level of boundary regularity exhibited by general solutions with smooth initial data.

Theorem 1.1(ii) seems to be the first result for this general family of fluid PDEs that demonstrates finite time singularity from smooth initial data. We compare it to Elgindi's recent surprising blow-up result for the axisymmetric Euler equation without swirl on $\mathbb{R}^3$ [12], which requires initial data to have Hölder continuous vorticity (it then becomes unbounded in finite time), something that smooth axisymmetric swirl-free solutions to Euler on $\mathbb{R}^3$ cannot achieve because the PDE is globally regular in that setting (this result may be similar in spirit to the exponential growth examples for Euler on $\mathbb{T}^2$ in [30]). We also mention here preprints [4, 5] by Chen and Hou, the second of which was completed shortly after the present work. These together provide finite time smooth initial data blow-up results for both 3D axisymmetric Euler and 2D Boussinesq equations on bounded domains, and their proofs heavily employ computer-assisted methods.

We note that the proof of Theorem 1.1(ii) suggests that the blow-up there is happening via $\partial_{x_1}\theta(t, \cdot)$ becoming unbounded near the origin. Since functions in X_β have $\partial_{x_1}\theta$ bounded up to the boundary, this further illustrates that the blow-up in Theorem 1.1(ii) is truly a non-linear phenomenon, rather than somehow caused by the space X_β.

Our second main result provides a complete answer to the natural question of sharpness of (both parts of) Theorem 1.1.

Theorem 1.2. *PDE* (1.1) *is ill-posed in* X_β *for any* $\alpha \in (0, \frac{1}{2})$ *and* $\beta \in [0, 1) \setminus [2\alpha, 1 - 2\alpha]$ *(with* $[2\alpha, 1 - 2\alpha] := \emptyset$ *when* $\alpha > \frac{1}{4}$*).*

That is, for all the couples (α, β) not covered by Theorem 1.1, the PDE (1.1) is ill-posed in X_β (see the start of Section 3 for details). The reason behind this lies in Lemma 2.2, which through the relationship between u and $\tilde{u}$ in (2.13) suggests potential problems with local regularity due to $\partial_{x_2} \tilde{u}_1$ becoming unbounded near the boundary when $\beta < 2\alpha$ and $\partial_{x_1} \tilde{u}_2$ becoming unbounded there when $\beta > 1 - 2\alpha$. We show that this indeed results in breakdown of local well-posedness in X_β for such (α, β).

The proof in fact shows that when $\beta \in [0, 2\alpha)$, then the vertical domain stretching (1.4) is too weak for obtaining well-posedness, while it is also too strong when $\beta \in (1 - 2\alpha, 1)$. This means that not only X_β with $\beta \in [2\alpha, 1 - 2\alpha]$ are the right spaces for (1.1) on the half-plane with respect to the $W^{1,\infty}$ norm, both also that other rates of stretching will not be helpful when it comes to local well-posedness in this norm.

Moreover, we note that since Lemma 2.2 shows that $\partial_{x_1} u_1$ is also bounded up to the boundary when $\theta \in X_\beta$, it follows that the lemma (and hence also Theorem 1.1(i)) cannot possibly extend to $\alpha \geq \frac{1}{2}$, because u_1 is then unbounded unless θ vanishes on the boundary.

Finally, our blow-up proof applies identically to the g-SQG patch model from [20,21], even though local regularity for that model is presently only known for $\alpha < \frac{1}{24}$ (and $\alpha < \frac{1}{6}$ in [13]).

Theorem 1.3. *For each* $\alpha \in (0, \frac{1}{4}]$*, there are solutions to the* H^3 *gSQG patch model on* $\mathbb{R} \times \mathbb{R}^+$ *from [20,21] (as well as the* H^2 *version from [13]) that cease to exist in finite time. If the local well-posedness result for this model from [20, 21] (resp. [13]) holds for some* $\alpha \in (0, \frac{1}{4}]$*, then there exist such solutions that exhibit finite time blow-up in the sense of their* H^3 *(resp.* H^2*) norms diverging to* ∞ *at some finite time.*

Note that the second claim follows from the first and from the proof of Theorem 1.4 in a recent work of Jeon and the author [17]. Finally, we note that shortly after the present paper was finished, Jeong, Kim, and Yao obtained independently Theorem 1.1(i) as well as Theorem 1.2 for $\beta < \min\{2\alpha, 1 - 2\alpha\}$ [18].

We prove Theorem 1.1(i), Theorem 1.2, and Theorem 1.1(ii) in Sections 2, 3, and 4, respectively. Sections 3 and 4 are independent of each other, so either can be read first.

2. Proof of Theorem 1.1(i)

Below, dependence of all constants $C. < \infty$ will always be specified in their subscripts (so C will be a universal constant), and the constants are always some finite numbers but may change from line to line. The following two lemmas are stated at any fixed time t, so we will suppress t in the notation.

Lemma 2.1. *If* $\theta \in L^1(\mathbb{R} \times \mathbb{R}^+) \cap L^\infty(\mathbb{R} \times \mathbb{R}^+)$ *and* $\alpha \in (0, \frac{1}{2})$*, then* u *from* (1.2) *satisfies*

$$\|u\|_{C^{1-2\alpha}} \leq C_\alpha \|\theta\|_{L^1 \cap L^\infty} \tag{2.1}$$

and $\lim_{x \to (s,0)} u_2(x) = 0$ *for all* $s \in \mathbb{R}$.

Proof. Estimate (2.1) is just Lemma 3.1 in [21]. It means that u extends continuously to $\mathbb{R} \times \{0\}$, and there we have $u_2 \equiv 0$ because the second coordinate of the parenthesis in (1.3) vanishes when $x_2 = 0$. $\square$

Lemma 2.2. *If* $\theta \in X_\beta$ *with* $\alpha \in [0, \frac{1}{2})$ *and* $\beta \in [0, 1)$, *then* u *from* (1.2) *satisfies*

$$\|\partial_{x_1} u_1\|_{L^\infty} \leq C_\alpha (\|\partial_{x_1} \theta\|_{L^\infty} + \|\theta\|_{L^1 \cap L^\infty}), \tag{2.2}$$

$$\| \max\{x_2^{2\alpha-1}, 1\} \partial_{x_1} u_2 \|_{L^\infty} \leq C_\alpha (\|\partial_{x_1} \theta\|_{L^\infty} + \|\theta\|_{L^1 \cap L^\infty}), \tag{2.3}$$

$$\|\partial_{x_2} u_2\|_{L^\infty} \leq C_\alpha (\|\partial_{x_1} \theta\|_{L^\infty} + \|\theta\|_{L^1 \cap L^\infty}), \tag{2.4}$$

$$\| \min\{x_2^{2\alpha}, 1\} \partial_{x_2} u_1 \|_{L^\infty} \leq C_{\alpha,\beta} \big(\| \min\{x_2^{\beta}, 1\} \partial_{x_2} \theta \|_{L^\infty} + \|\theta\|_{L^1 \cap L^\infty} \big). \tag{2.5}$$

Proof. From (1.3) we have

$$u(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \theta(y) dy = \int_{\mathbb{R}^2} \theta(x-y) \frac{y^\perp}{|y|^{2+2\alpha}} dy, \tag{2.6}$$

provided we extend θ oddly onto $\mathbb{R} \times \mathbb{R}^-$. Thus for any $x \in \mathbb{R} \times \mathbb{R}^+$ and $h \in [0, 1]$ we have

$$|u(x + he_1) - u(x)| \leq \int_{B_2(0)} \frac{\|\partial_{x_1} \theta\|_{L^\infty} h}{|y|^{1+2\alpha}} dy + \int_{\mathbb{R}^2 \setminus B_2(x)} Ch|\theta(y)| dy + \int_{B_{2+h}(x) \setminus B_{2-h}(x)} |\theta(y)| dy$$

$$\leq C_\alpha (\|\partial_{x_1} \theta\|_{L^\infty} + \|\theta\|_{L^1} + \|\theta\|_{L^\infty}) h,$$

where the first integral is from the second integral in (2.6), the second integral is from the first integral in (2.6) (in it we used that $|\nabla_y \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}}| \leq C$ when $y \notin B_1(x)$), and the third integral bounds the error in the evaluation of $u(x + he_1)$ that was introduced by adding them. This now yields (2.2) as well as

$$\|\partial_{x_1} u_2\|_{L^\infty} \leq C_\alpha (\|\partial_{x_1} \theta\|_{L^\infty} + \|\theta\|_{L^1 \cap L^\infty}). \tag{2.7}$$

Next assume that $x_2 \in (0, 1)$ and note that oddness of θ in x_2 yields

$$u_2(x + he_1) - u_2(x) = \int_{\mathbb{R}^2} (\theta(x-y) - \theta(x + he_1 - y)) \frac{y_1}{|y|^{2+2\alpha}} dy$$

$$= \int_{\mathbb{R} \times (-\infty, x_2]} (\theta(x-y) - \theta(x + he_1 - y)) \left(\frac{y_1}{|y|^{2+2\alpha}} - \frac{y_1}{|y - 2x_2 e_2|^{2+2\alpha}} \right) dy.$$

Hence

$$|u_2(x + he_1) - u_2(x)| \leq \int_{[-x_2, x_2]^2} \|\partial_{x_1} \theta\|_{L^\infty} h \frac{|y_1|}{|y|^{2+2\alpha}} dy$$

$$+ \int_{\mathbb{R} \times (-\infty, x_2] \setminus [-x_2, x_2]^2} \|\partial_{x_1} \theta\|_{L^\infty} h|y| C_\alpha x_2 |y|^{-3-2\alpha} dy \leq C_\alpha \|\partial_{x_1} \theta\|_{L^\infty} h x_2^{1-2\alpha}.$$

This and (2.7) imply (2.3).

We used neither information about $\partial_{x_2} \theta$ nor oddness of θ in x_2 in the proof of (2.2), which will also be the case in the proof of (2.4). For any $x \in \mathbb{R} \times \mathbb{R}^+$ (without loss assume $x_1 = 0$) and $h \in [0, \min\{1, \|\theta\|_{L^\infty} \|\partial_{x_1} \theta\|_{L^\infty}^{-1}\}]$ we have

$$u_2(x + he_2) - u_2(x) = \int_{x + [-2,2]^2} y_1 \left(\frac{1}{|x + he_2 - y|^{2+2\alpha}} - \frac{1}{|x-y|^{2+2\alpha}} \right) \theta(y) dy + A_1$$

for some $|A_1| \leq Ch\|\theta\|_{L^1}$. The integral above equals

$$\int_{x + [0,2] \times [-2,2]} y_1 \left(\frac{1}{|x + he_2 - y|^{2+2\alpha}} - \frac{1}{|x-y|^{2+2\alpha}} \right) (\theta(y) - \theta(-y_1, y_2)) dy.$$

The parenthesis has the same sign as $y_2 - x_2 - \frac{h}{2}$ so absolute value of this integral is at most

$$\int_{x+[0,2]\times[-2,2]} y_1 \left(\frac{1}{|x+he_2-y|^{2+2\alpha}} - \frac{1}{|x-y|^{2+2\alpha}} \right) \mathrm{sgn}\left(y_2 - x_2 - \frac{h}{2} \right) 2M(y_1) dy$$

$$= \int_{x+[-2,2]^2} y_1 \left(\frac{1}{|x+he_2-y|^{2+2\alpha}} - \frac{1}{|x-y|^{2+2\alpha}} \right) \mathrm{sgn}\left(y_1 \left(y_2 - x_2 - \frac{h}{2} \right) \right) M(y_1) dy,$$

where $M(s) := \min\{|s| \|\partial_{x_1}\theta\|_{L^\infty}, \|\theta\|_{L^\infty}\}$. But this is precisely $u_2(x+he_2) - u_2(x)$ when θ is replaced by

$$\theta^*(y) := \chi_{x+[-2,2]^2}(y) \, \mathrm{sgn}\left(y_1 \left(y_2 - x_2 - \frac{h}{2} \right) \right) M(y_1)$$

(we will denote u corresponding to θ^* by u^*; also note that θ^* is not odd in y_2). Since this is constant in y_2 between the jumps at $y_2 = x_2 - 2, x_2 + \frac{h}{2}, x_2 + 2$, (2.6) now yields

$$u_2^*(x+he_2) - u_2^*(x) = \int_{\mathbb{R}^2} (\theta^*(x-y) - \theta^*(x+he_2-y)) \frac{y_1}{|y|^{2+2\alpha}} dy$$

$$= 2 \int_{[-2,2]\times[-h/2,h/2]} \theta^*(x-y) \frac{y_1}{|y|^{2+2\alpha}} dy + A_2,$$

where $|A_2| \le 8h\|\theta^*\|_{L^\infty} \le 8h\|\theta\|_{L^\infty}$. Let $m := \min\{\|\theta\|_{L^\infty}\|\partial_{x_1}\theta\|_{L^\infty}^{-1}, 2\} \ge h$. Then the last integral equals

$$\int_{[-2,2]\times[-h/2,h/2]} M(y_1) \frac{|y_1|}{|y|^{2+2\alpha}} dy$$

$$\le \int_{B_h(0)} \frac{\|\partial_{x_1}\theta\|_{L^\infty}}{|y|^{2\alpha}} dy + 2h \int_h^m \|\partial_{x_1}\theta\|_{L^\infty} \frac{1}{y_1^{2\alpha}} dy_1 + 2h \int_m^2 \|\theta\|_{L^\infty} \frac{1}{y_1^{1+2\alpha}} dy_1$$

$$\le C_\alpha \|\partial_{x_1}\theta\|_{L^\infty} h^{2-2\alpha} + C_\alpha \|\partial_{x_1}\theta\|_{L^\infty} m^{1-2\alpha} h + C_\alpha \|\theta\|_{L^\infty}^{1-2\alpha} \|\partial_{x_1}\theta\|_{L^\infty}^{2\alpha} h$$

$$\le C_\alpha (\|\partial_{x_1}\theta\|_{L^\infty} + \|\theta\|_{L^\infty}) h,$$

where we used the fact that the last integral is zero if $m = 2$. We thus obtain

$$|u_2(x+he_2) - u_2(x)| \le C_\alpha(\|\partial_{x_1}\theta\|_{L^\infty} + \|\theta\|_{L^\infty})h + |A_1| + |A_2| \le C_\alpha(\|\partial_{x_1}\theta\|_{L^\infty} + \|\theta\|_{L^1} + \|\theta\|_{L^\infty})h,$$

and (2.4) follows.

It remains to prove (2.5). For any $x \in \mathbb{R} \times \mathbb{R}^+$ (without loss assume again $x_1 = 0$) and $h \in [0, \min\{1, \frac{x_2}{3}\}]$ we have similarly to the proof of (2.2),

$$u_1(x+he_2) - u_1(x) = \int_{[-2,2]^2} (\theta(x+he_2-y) - \theta(x-y)) \frac{y_2}{|y|^{2+2\alpha}} dy$$

$$+ \int_{\mathbb{R}^2\setminus(x+[-2,2]^2)} \left(\frac{y_2}{|x+he_2-y|^{2+2\alpha}} - \frac{y_2}{|x-y|^{2+2\alpha}} \right) \theta(y) dy + A_3$$

for some $|A_3| \le 8h\|\theta\|_{L^\infty}$. Letting A_4 be the second integral above, we have $|A_4| \le Ch\|\theta\|_{L^1}$, so let I be the first integral above and let

$$M := \| \min\{x_2^\beta, 1\}\partial_{x_2}\theta\|_{L^\infty}.$$

If $x_2 \geq 3$, then clearly $|I| \leq C_\alpha h M$. Otherwise we can use $2|x_2 + h - y_2| \geq |x_2 - y_2|$ for $y_2 \notin [x_2, x_2 + 2h]$ to estimate

$$|I| \leq \int_{[-2,2]^2} 2hM|x_2 - y_2|^{-\beta} \frac{|y_2|}{|y|^{2+2\alpha}} dy + \int_{[-2,2] \times [x_2, x_2+2h]} 2\|\theta\|_{L^\infty} \frac{|y_2|}{|y|^{2+2\alpha}} dy. \tag{2.8}$$

The first integral is no more than

$$C_\alpha h M \int_{[-2,2]} |x_2 - y_2|^{-\beta} |y_2|^{-2\alpha} dy_2 \leq C_{\alpha,\beta} h M \max\{x_2^{1-2\alpha-\beta}, 1\}.$$

The second integral in (2.8) can be estimated by

$$2\|\theta\|_{L^\infty} \left(4h \int_{x_2}^2 \frac{2x_2}{y_1^{2+2\alpha}} dy_1 + 4hx_2 \left(\frac{x_2}{3}\right)^{-1-2\alpha} \right) \leq C_\alpha h \|\theta\|_{L^\infty} x_2^{-2\alpha}.$$

So we obtain

$$|u_1(x + he_2) - u_1(x)| \leq C_{\alpha,\beta} M(1 + \min\{x_2, 1\}^{-2\alpha+1-\beta})h + C_\alpha \left(\|\theta\|_{L^\infty} \max\{x_2^{-2\alpha}, 1\} + \|\theta\|_{L^1} \right) h$$

and (2.5) follows. $\square$

Lemmas 2.1 and 2.2 mean that for $\alpha \in (0, \frac{1}{4}]$ and $\beta \in [2\alpha, 1 - 2\alpha]$, solutions to (1.1) satisfy the a priori estimate

$$\frac{d}{dt} \|\theta(t, \cdot)\|_{X_\beta} \leq C_{\alpha,\beta} \|\theta(t, \cdot)\|_{X_\beta}^2. \tag{2.9}$$

To see this, let $\kappa_\beta(x_2) := \min\{x_2^\beta, 1\}$, so that λ_β from (1.5) satisfies

$$\lambda'_\beta(x_2) = \kappa_\beta(\lambda_\beta(x_2)) \tag{2.10}$$

on $(0, \infty)$. If θ solves (1.1)–(1.2) and we let

$$\tilde{u}(t, x_1, x_2) := \left(u_1(t, x_1, \lambda_\beta(x_2)), \kappa_\beta(\lambda_\beta(x_2))^{-1} u_2(t, x_1, \lambda_\beta(x_2)) \right), \tag{2.11}$$

then (2.10) shows that we have

$$\partial_t \tilde{\theta} + \tilde{u} \cdot \nabla \tilde{\theta} = 0 \tag{2.12}$$

on $\mathbb{R} \times \mathbb{R}^+$. Note that the dynamic of this PDE preserves the measure $\lambda'_\beta(x_2) dx_1 dx_2$. We now obtain

$$\partial_{x_1} \tilde{u}_1(t, x_1, x_2) = \partial_{x_1} u_1(t, x_1, \lambda_\beta(x_2)),$$

$$\partial_{x_1} \tilde{u}_2(t, x_1, x_2) = \kappa_\beta(\lambda_\beta(x_2))^{-1} \partial_{x_1} u_2(t, x_1, \lambda_\beta(x_2)),$$

$$\partial_{x_2} \tilde{u}_1(t, x_1, x_2) = \kappa_\beta(\lambda_\beta(x_2)) \partial_{x_2} u_1(t, x_1, \lambda_\beta(x_2)), \tag{2.13}$$

$$\partial_{x_2} \tilde{u}_2(t, x_1, x_2) = \partial_{x_2} u_2(t, x_1, \lambda_\beta(x_2)) - \frac{\kappa'_\beta(\lambda_\beta(x_2))}{\kappa_\beta(\lambda_\beta(x_2))} u_2(t, x_1, \lambda_\beta(x_2)),$$

where we also have

$$\frac{\kappa'_\beta(\lambda_\beta(x_2))}{\kappa_\beta(\lambda_\beta(x_2))} = \begin{cases} \beta \lambda_\beta(x_2)^{-1} & x_2 \in (0, (1-\beta)^{-1}), \\ 0 & x_2 \geq (1-\beta)^{-1}. \end{cases} \tag{2.14}$$

Hence Lemma 2.2, the definitions of $\kappa_\beta, \lambda_\beta$, and $\|\kappa_\beta(x_2) \partial_{x_2} \theta(t, \cdot)\|_{L^\infty} = \|\partial_{x_2} \tilde{\theta}(t, \cdot)\|_{L^\infty}$ yield

$$\|\nabla \tilde{u}(t, \cdot)\|_{L^\infty} \leq C_{\alpha,\beta}(\|\nabla \tilde{\theta}(t, \cdot)\|_{L^\infty} + \|\theta(t, \cdot)\|_{L^1 \cap L^\infty}). \tag{2.15}$$

Indeed, this follows using $\beta \in [2\alpha, 1 - 2\alpha]$ and

$$\kappa_\beta(\lambda_\beta(x_2)) = (1 - \beta)^{\beta/(1-\beta)} x_2^{\beta/(1-\beta)} \tag{2.16}$$

for $x_2 \in [0, (1 - \beta)^{-1}]$, as well as

$$|u_2(t, x)| \leq C_\alpha (\|\partial_{x_1} \theta(t, \cdot)\|_{L^\infty} + \|\theta(t, \cdot)\|_{L^1 \cap L^\infty}) \, x_2, \tag{2.17}$$

which holds by (2.4) and the last claim in Lemma 2.1.

Since $\|\theta(t, \cdot)\|_{L^1}$ and $\|\theta(t, \cdot)\|_{L^\infty}$ stay constant in time because (1.1) is a transport equation with divergence-free velocity, (2.15) shows that

$$\frac{d}{dt} \|\nabla \tilde{\theta}(t, \cdot)\|_{L^\infty} \leq C_{\alpha,\beta} \|\nabla \tilde{\theta}(t, \cdot)\|_{L^\infty} \left(\|\nabla \tilde{\theta}(t, \cdot)\|_{L^\infty} + \|\theta_0\|_{L^1 \cap L^\infty} \right).$$

This now yields (2.9).

Having the a priori estimate (2.9), it is easy to construct local solutions in X_β and show their uniqueness. One can for instance approximate the Biot-Savart kernel $\frac{y^\perp}{|y|^{2+2\alpha}}$ by a sequence of regularized kernels, (e.g., by multiplying it by a smooth approximation of $\chi_{\mathbb{R}^2 \setminus B_{1/n}(0)}$), construct solutions to these regularized equations, and then recover the solution to (1.1) in the limit (because (2.9) equally holds for solutions to all the regularizations). Uniqueness then follows from (2.12) and (2.15). This proves Theorem 1.1(i) because (2.9) and constancy of $\|\theta(t, \cdot)\|_{L^1}$ and $\|\theta(t, \cdot)\|_{L^\infty}$ show that the unique solution can only cease to exist at some time T when $\lim_{t \to T_{\theta_0}} \|\theta(t, \cdot)\|_{\dot{W}^{1,\infty}_\beta} = \infty$, as well as that such a time is bounded below by some function of α, β, and $\|\theta_0\|_{X_\beta}$ (which can clearly be chosen to be decreasing in the last argument).

3. Proof of Theorem 1.2

Since the proof of Theorem 1.1(ii) is completely independent of this section, a reader interested in the singularity result can first read Section 4.

All constants in this section may depend on α. We split the analysis into two cases: $\beta \in [0, 2\alpha)$ and $\beta \in (1 - 2\alpha, 1)$. In the first we provide initial data in X_β for which no solutions exist in $L^\infty([0, T]; X_\beta)$ for any $T > 0$. In the second case we construct a sequence of initial data that converges in X_β but no sequence of solutions for these data can converge to a solution for the limiting datum in $L^\infty([0, T]; X_\beta)$ for any $T > 0$.

Case $\beta \in [0, 2\alpha)$. For $x \in \mathbb{R} \times \mathbb{R}^+$, let

$$\phi_0(x) := \min \left\{ 1, d\big(x, \mathbb{R}^2 \setminus ([-3, 3] \times [-1, 2 + (1 - \beta)^{-1}])\big) \right\},$$

with d the distance on $\mathbb{R}^2$, and let $\phi_1(x) := \max\{0, 1 - |x|\}$. With some $a_n \in (0, 2^{-4n-1}]$ $(n = 1, 2, \dots)$, and with λ_β from (1.5) and $\Lambda_\beta(x) := (x_1, \lambda_\beta(x_2))$, define

$$\tilde{\theta}_0 := \phi_0 + \phi_\infty \qquad \text{and} \qquad \theta_0 := \tilde{\theta}_0 \circ \Lambda_\beta^{-1} \in X_\beta, \tag{3.1}$$

where

$$\phi_\infty(x) := \sum_{n=1}^{\infty} a_n \phi_1 \left(\frac{x - (0, 2^{-4n})}{a_n} \right). \tag{3.2}$$

It is easy to see that $\tilde{\theta}_0$ is Lipschitz with constant 1 because supports of the summands in the series above are disjoint, and are all contained in $[-1, 1] \times (0, 1]$. We will now show
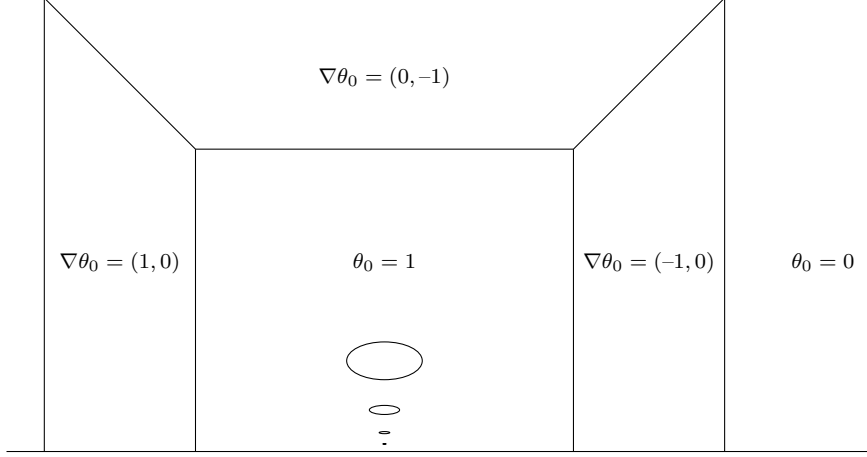
FIGURE 1. Function θ_0 for $\beta < 2\alpha$, with $\phi_\infty \circ \Lambda_\beta^{-1}$ supported on the "ellipses".

that when the a_n are small enough, a classical solution θ to (1.1) with initial datum θ_0 would have to instantly leave X_β due to $\tilde\theta(t,x) := \theta(t, \Lambda_\beta(x))$ not being Lipschitz on $[0,T] \times \mathbb{R} \times \mathbb{R}^+$ for any $T > 0$. Specifically, we will essentially obtain $\partial_{x_2} u_1(t,x) \sim x_2^{-2\alpha}$ for small $t, x_2 > 0$ and $x_1 \sim 0$ (with the other three partial derivatives of u sufficiently controlled), which will result in fast horizontal shearing of the "cap" functions from the series in (3.2) (and hence instantaneous loss of Lipschitz continuity of $\tilde\theta$).

As before, extend $\theta_0, \phi_0, \phi_\infty$ oddly onto $\mathbb{R}^2$. Let us first look at the velocity

$$v^0(x) := \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \phi_0(\Lambda_\beta^{-1}(y))dy,$$

generated by only $\phi_0 \circ \Lambda_\beta^{-1}$. Lemma 2.2 shows that $\partial_{x_1} v_1^0$ and $\partial_{x_2} v_2^0$ are bounded on $\mathbb{R} \times \mathbb{R}^+$. Moreover, from

$$\partial_{x_1} v_2^0(x) = -\int_{\mathbb{R}^2} \frac{d}{dx_1} \phi_0(\Lambda_\beta^{-1}(x-y)) \frac{y_1}{|y|^{2+2\alpha}} dy$$

$$= \int_{\mathbb{R} \times (-\infty, x_2]} \frac{d}{dx_1} \phi_0(\Lambda_\beta^{-1}(x-y)) \left(\frac{y_1}{|y - 2x_2 e_2|^{2+2\alpha}} - \frac{y_1}{|y|^{2+2\alpha}} \right) dy$$

and constancy of $\phi_0 \circ \Lambda_\beta^{-1}$ on $[-2,2] \times (0,2]$ and on $[-2,2] \times [-2,0)$ (so the integrand vanishes when $x \in [-1,1] \times (0,1]$ and $|y| \le 1$) we see that $x_2^{-1} \partial_{x_1} v_2^0$ is bounded on $[-1,1] \times (0,1]$.

For the last partial derivative, we can use the same constancy of ϕ_0 and

$$v_1^0(x + he_2) - v_1^0(x) = \int_{\mathbb{R}^2} \left[\phi_0(\Lambda_\beta^{-1}(x + he_2 - y)) - \phi_0(\Lambda_\beta^{-1}(x-y)) \right] \frac{y_2}{|y|^{2+2\alpha}} dy$$

to see that now instead

$$\partial_{x_2} v_1^0(x) - 2 \int_{-1}^1 \frac{x_2}{|(x_2, y_1)|^{2+2\alpha}} dy_1$$

is bounded on $[-1,1] \times (0,1]$ (because of the jump ϕ_0 has at the x_1 axis). This means that on this rectangle we have

$$\partial_{x_2} v_1^0(x) \ge C^{-1} x_2^{-2\alpha} - C \tag{3.3}$$

for some constant $C \geq 1$, which we pick so we also have

$$\max \left\{ \partial_{x_1} v_1^0(x), \partial_{x_2} v_2^0(x), x_2^{-1} \partial_{x_1} v_2^0(x) \right\} \leq C \qquad (3.4)$$

there. We will now show that when a_n above are small enough, then these bounds will hold for a short time for the actual velocity u, yielding the desired shearing dynamic.

Lemma 2.1 applies uniformly to all solutions θ as above (let us call the constant in it again C) because $\|\theta_0\|_{L^1 \cap L^\infty} \leq \frac{15}{1-\beta}$ regardless of our choice of a_n. So consider any divergence-free u on $[0, T] \times \mathbb{R}^2$ for some T, satisfying $u(t, x_1, -x_2) = (u_1(t, x), -u_2(t, x))$,

$$\sup_{t \in [0,T]} \|u(t, \cdot)\|_{C^{1-2\alpha}(\mathbb{R} \times \mathbb{R}^+)} \leq C, \qquad (3.5)$$

$\lim_{x \to (s,0)} u_2(t, x) = 0$ for each $(t, s) \in [0, T] \times \mathbb{R}$, and such that the left-hand sides of (2.2)–(2.5) are all finite (with the L^∞ norms taken over $[0, T] \times \mathbb{R} \times \mathbb{R}^+$). Note that then $\|x_2^{-1} u_2\|_{L^\infty}$ is also finite, similarly to (2.17), and all these estimates show that $\frac{d}{dt} z^{t,x} = u(t, z^{t,x})$ and $z^{0,x} := x$ has a unique solution on $[0, T]$ for each $x \in \mathbb{R}^2$ and $z^{t,\cdot}$ is a measure preserving bijection on $\mathbb{R} \times \mathbb{R}^+$ (and so also on $\mathbb{R} \times \mathbb{R}^-$) for each $t \in [0, T]$. If we let $z^{-t,\cdot}$ be its inverse, then the gSQG velocity

$$v(t, x) := \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \phi_0(\Lambda_\beta^{-1}(z^{-t,y})) dy \qquad (3.6)$$

corresponding to the u-transported function $\phi_0 \circ \Lambda_\beta^{-1}$ satisfies

$$\partial_{x_j} v(t, x) = \text{p.v.} \int_{\mathbb{R}^2} \partial_{x_j} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \phi_0(\Lambda_\beta^{-1}(z^{-t,y})) dy.$$

Since $\|u(t, \cdot)\|_{L^\infty} \leq C$, we see that $\phi_0(\Lambda_\beta^{-1}(z^{-t,y})) = \text{sgn}(y_2)$ for $(t, y) \in [0, \frac{1}{2C}] \times [-\frac{3}{2}, \frac{3}{2}]^2$. We can then even remove "p.v." in

$$\partial_{x_j} v(t, x) - \partial_{x_j} v^0(x) = \int_{\mathbb{R}^2} \partial_{x_j} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \left[\phi_0(\Lambda_\beta^{-1}(z^{-t,y})) - \phi_0(\Lambda_\beta^{-1}(y)) \right] dy$$

$$= \int_{\mathbb{R}^2 \setminus B_{1/2}(x)} \partial_{x_j} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \left[\phi_0(\Lambda_\beta^{-1}(z^{-t,y})) - \phi_0(\Lambda_\beta^{-1}(y)) \right] dy$$

when $(t, x) \in [0, \frac{1}{2C}] \times [-1, 1] \times (0, 1]$. From this, (3.3), and (3.4) we see that after increasing C (it still only depends on α), we obtain

$$\partial_{x_2} v_1(t, x) \geq C^{-1} x_2^{-2\alpha} - C \qquad (3.7)$$

and

$$\max \left\{ \partial_{x_1} v_1(t, x), \partial_{x_2} v_2(t, x), x_2^{-1} \partial_{x_1} v_2(t, x) \right\} \leq C \qquad (3.8)$$

for $(t, x) \in [0, \min\{\frac{1}{2C}, T\}] \times [-1, 1] \times (0, 1]$, where the bound on $x_2^{-1} \partial_{x_1} v_2(t, x)$ also uses

$$\partial_{x_1} v_2(t, x) - \partial_{x_1} v_2^0(x)$$

$$= \int_{\mathbb{R} \times (-\infty, x_2]} \left(\partial_{x_1} \frac{x_1 - y_1}{|x-y|^{2+2\alpha}} - \partial_{x_1} \frac{x_1 - y_1}{|\bar{x} - y|^{2+2\alpha}} \right) \left[\phi_0(\Lambda_\beta^{-1}(y)) - \phi_0(\Lambda_\beta^{-1}(z^{-t,y})) \right] dy.$$

Let us now assume that $\theta \in L^\infty([0, T]; X_\beta)$ is a classical solution to (1.1) with initial datum given by (3.1) and (3.2), with some $T \in (0, 1]$ and some a_n to be determined later. That

is, $\tilde\theta(t,x) := \theta(t,\Lambda_\beta(x))$ is Lipschitz in x with some constant L. We extend θ oddly onto $[0,T] \times \mathbb{R}^2$, let

$$u(t,x) := \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \theta(t,y) dy \tag{3.9}$$

be the velocity corresponding to θ, and define $z^{t,x}$ as above. Now $\theta(t,x) = \theta_0(z^{-t,x})$ shows that $u = v + w$, where v is from (3.6) and

$$w(t,x) := \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \phi_\infty(\Lambda_\beta^{-1}(z^{-t,y})) dy.$$

Since (3.7) and (3.8) hold by Lemma 2.1 and will yield the desired shearing dynamic, we only need to show that w will not distort it too much at small times when the a_n are small.

From (2.17) we see that there is $T_\theta \in (0, \min\{\frac{1}{2C}, T\}]$ (depending on α, T, L) such that

$$z_2^{t,x} x_2^{-1} \in [2^{-1}, 2] \tag{3.10}$$

for all $(t,x) \in [0, T_\theta] \times \mathbb{R} \times \mathbb{R}^+$. This, $(\lambda_\beta^{-1})' \geq 1$, and $z^{-t,\cdot}$ being measure-preserving show that there is $C' < \infty$ such that if $x, x' \in \mathbb{R} \times (0, \lambda_\beta(2^{1-4N})]$ for some $N \geq 1$, then

$$|w(t,x) - w(t,x')| \leq C' \sum_{n=1}^{N-1} a_n^3 \lambda_\beta(2^{-4n-3})^{-2-2\alpha} |x - x'| + C' \sum_{n=N}^{\infty} a_n^{2-2\alpha} \tag{3.11}$$

for all $t \in [0, T_\theta]$. This is because $y_2 \geq \lambda_\beta(2^{-4n-2})$ when $\Lambda_\beta^{-1}(z^{-t,y}) \in B_{2^{-4n-1}}((0, 2^{-4n}))$, due to (3.10) and convexity of λ_β, which means that

$$\min\{|x-y|, |x'-y|\} \geq \lambda_\beta(2^{-4n-2}) - \lambda_\beta(2^{-4n-3}) \geq \lambda_\beta(2^{-4n-3})$$

for such y when $n \leq N - 1$, while for $n \geq N$ we simply used the bound

$$\left| \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} a_n \phi_1 \left(\frac{\Lambda_\beta^{-1}(z^{-t,y}) - (0, 2^{-4n})}{a_n} \right) dy \right| \leq a_n \int_{B_{a_n}(0)} |y|^{-1-2\alpha} dy \leq C'' a_n^{2-2\alpha}.$$

If we pick $a_n := a 2^{-8n/(1-2\alpha)^2}$ with small enough $a \in (0, (1-\beta)^{1/(1-\beta)}]$ (note that then also $a_n \leq \lambda_\beta(2^{-4n-1})$ because $0 \leq \beta < 2\alpha < 1$), then for any $N \geq 1$ we obtain

$$|w(t,x) - w(t,x')| \leq |x - x'| + a_N^{(3-2\alpha)/2} \tag{3.12}$$

whenever $t \in [0, T_\theta]$ and $x, x' \in \mathbb{R} \times (0, \lambda_\beta(2^{1-4N})]$ (we decreased the power of a_N here just to remove C'). In fact, when $N = 1$ we can clearly even take any $x, x' \in \mathbb{R} \times \mathbb{R}^+$.

We will also need a better estimate on $w_2(t,x) - w_2(t,x')$ when $x_2 = x_2'$. Oddness of $\phi_\infty(\Lambda_\beta^{-1}(z^{-t,y}))$ in y_2 shows that the contribution of the nth summand in (3.2) and of its reflection across the x_1 axis to $\partial_{x_1} w_2(t,x)$ is

$$\left| \int_{\mathbb{R}^2} \left(\partial_{x_1} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} - \partial_{x_1} \frac{y_1 - x_1}{|\bar{x}-y|^{2+2\alpha}} \right) a_n \phi_1 \left(\frac{\Lambda_\beta^{-1}(z^{-t,y}) - (0, 2^{-4n})}{a_n} \right) dy \right|$$
$$\leq C'' a_n^3 \lambda_\beta(2^{-4n-3})^{-3-2\alpha} x_2$$

when $n \leq N - 1$ and $x \in \mathbb{R} \times (0, \lambda_\beta(2^{1-4N})]$. Then, similarly to (3.12), we obtain

$$|w_2(t,x) - w_2(t,x')| \leq x_2 |x - x'| + a_N^{(3-2\alpha)/2}$$

when also $x_2 = x_2'$ (and $a > 0$ is small enough), which combines with (3.12) to show that

$$|w_2(t, x) - w_2(t, x')| \leq |x_2 - x_2'| + \max\{x_2, x_2'\}|x_1 - x_1'| + 2a_N^{(3-2\alpha)/2} \qquad (3.13)$$

whenever $t \in [0, T_\theta]$ and $x, x' \in \mathbb{R} \times (0, \lambda_\beta(2^{1-4N})]$.

From (3.7), (3.8), (3.12), and (3.13) we see that for any $n \geq 1$, and all $t \in [0, T_\theta]$ and $x, x' \in [-1, 1] \times (0, \lambda_\beta(2^{1-4n})]$ with $x_2 \geq x_2'$, we have the bounds

$$u_1(t, x) - u_1(t, x') \geq C^{-1}x_2^{-2\alpha}(x_2 - x_2') - 3C|x - x'| - a_n^{(3-2\alpha)/2} \qquad (3.14)$$

and

$$|u_2(t, x) - u_2(t, x')| \leq 3C|x_2 - x_2'| + 3Cx_2|x_1 - x_1'| + 2a_n^{(3-2\alpha)/2}. \qquad (3.15)$$

We now take $y_n := (0, \lambda_\beta(2^{-4n}))$ and $y_n' := (a_n, \lambda_\beta(2^{-4n}) - 2b_n)$, with $b_n := 2^{-2(2\alpha+\beta)n/(1-\beta)}a_n$. From (3.10) we see that for $t \in [0, T_\theta]$ we have

$$\max\{z_2^{t,y_n}, z_2^{t,y_n'}\} \leq 2\lambda_\beta(2^{-4n}) \leq \lambda_\beta(2^{1-4n}),$$

so (3.14) and (3.15) hold with $(x, x') = (z^{t,y_n}, z^{t,y_n'})$ when $t \in [0, T_\theta]$. Let

$$T_n := \min\{T_\theta, T_n', T_n''\},$$

with $T_n' \geq 0$ the first time when $z_1^{T_n',y_n} = z_1^{T_n',y_n'}$ and $T_n'' \geq 0$ the first time when $z_2^{T_n'',y_n} - z_2^{T_n'',y_n'} \notin (b_n, 3b_n)$ (we let $T_n' := \infty$ resp. $T_n'' := \infty$ if there is no such time). From (3.14) we see that

$$u_1(t, z^{t,y_n}) - u_1(t, z^{t,y_n'}) \geq (2C)^{-1}\lambda_\beta(2^{1-4n})^{-2\alpha}b_n \qquad \text{and} \qquad z_1^{t,y_n'} - z_1^{t,y_n} \in [0, a_n] \qquad (3.16)$$

when $t \in [0, T_n]$ and n is large enough because $b_n \geq 2C\lambda_\beta(2^{1-4n})^{2\alpha}(3C(a_n + 3b_n) + a_n)$ for large n (note that $b_n \leq a_n$). Then (3.16) yields

$$T_n \leq a_n 2C\lambda_\beta(2^{1-4n})^{2\alpha}b_n^{-1} \leq 2^{1+1/(1-\beta)}C2^{-2(2\alpha-\beta)n/(1-\beta)} \to 0, \qquad (3.17)$$

so $T_n < T_\theta$ for large n. Moreover, if $T_n = T_n''$ for large enough n, then (3.15) and (3.16) yield

$$T_n \geq b_n \left(9Cb_n + 3C\lambda_\beta(2^{1-4n})a_n + 2a_n^{(3-2\alpha)/2}\right)^{-1} \to (9C)^{-1},$$

which cannot hold for any large enough n by (3.17). Hence $T_n = T_n'$ for all large n, which means that $|z^{T_n,y_n} - z^{T_n,y_n'}| \leq 3b_n$. Since

$$\tilde{\theta}(T_n, \Lambda_\beta^{-1}(z^{T_n,y_n})) = \theta_0(y_n) = 1 + a_n \qquad \text{and} \qquad \tilde{\theta}(T_n, \Lambda_\beta^{-1}(z^{T_n,y_n'})) = \theta_0(y_n') = 1,$$

we see from this and (3.10) that a Lipschitz constant for $\tilde{\theta}(T_n, \cdot)$ on $\mathbb{R} \times \mathbb{R}^+$ is no less than

$$a_n \left(3b_n\lambda_\beta'(2^{-4n-1})^{-1}\right)^{-1} = 3^{-1}(2 - 2\beta)^{\beta/(1-\beta)}2^{2(2\alpha-\beta)n/(1-\beta)} \to \infty$$

for all large n. But this contradicts our hypothesis that $\tilde{\theta}$ is Lipschitz in x on $[0, T] \times \mathbb{R} \times \mathbb{R}^+$. The proof for $\beta \in [0, 2\alpha)$ is thus finished.

Case $\beta \in (1 - 2\alpha, 1)$. We use here a similar approach as for $\beta < 2\alpha$, but it will be more complicated due the fact that this time the supports of ϕ_∞ and $\nabla\phi_0$ will have to touch. Since now $\beta \leq 1 - 2\alpha$ does not hold, the bound (2.3) will be too weak to control $\nabla\tilde{u}$, and so the fast shearing dynamic will be happening in the vertical direction. However, (2.3) in fact shows that this dynamic will only be fast in the scaling of $\tilde{\theta}$, and it will therefore be sufficient (as
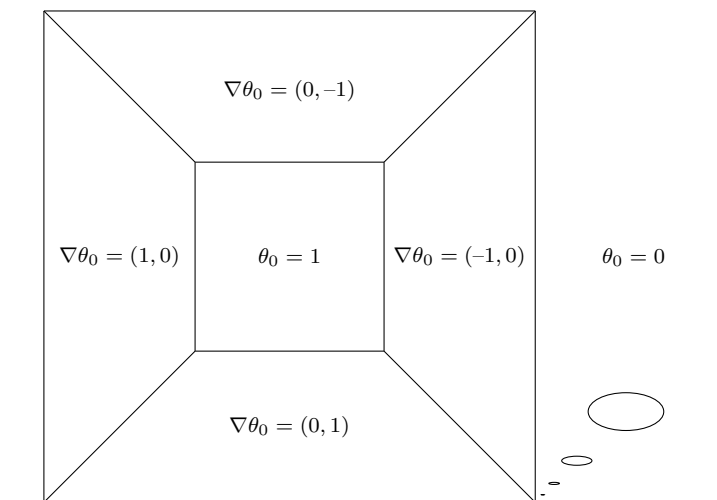
FIGURE 2. Function θ_0 for $\beta > 1 - 2\alpha$, with $\phi_\infty \circ \Lambda_\beta^{-1}$ supported on the "ellipses".

well as convenient) to let the background ϕ_0 be Lipschitz on all of $\mathbb{R}^2$ in the scaling of θ. So now let

$$\phi_0(x) := \min\left\{1, d\big(x, \mathbb{R}^2 \setminus ([-3,0] \times [0,3])\big)\right\}$$

for $x \in \mathbb{R} \times \mathbb{R}^+$, let again $\phi_1(x) := \max\{0, 1 - |x|\}$, and with some $a_n \in (0, 2^{-4n-1}]$ let

$$\tilde{\theta}_0 := \phi_0 \circ \Lambda_\beta + \phi_\infty \qquad \text{and} \qquad \theta_0 := \tilde{\theta}_0 \circ \Lambda_\beta^{-1} = \phi_0 + \phi_\infty \circ \Lambda_\beta^{-1} \in X_\beta, \qquad (3.18)$$

where now

$$\phi_\infty(x) := \sum_{n=1}^{\infty} \frac{a_n}{n} \phi_1\left(\frac{x - (\lambda_\beta(2^{-4n}), 2^{-4n})}{a_n}\right). \qquad (3.19)$$

Then $\tilde{\theta}_0$ is again Lipschitz with constant 1 because $\lambda'_\beta \leq 1$. We will show that when the a_n are small enough, then a solution $\theta \in L^\infty_{\text{loc}}([0, T_{\theta_0}); X_\beta)$ to (1.1) with initial datum θ_0 cannot be a limit of solutions θ_{n_0} obtained by truncating the sum in (3.19) at $n = n_0$ (whose initial data converge to θ_0 in X_β) because the Lipschitz constants of the functions $\tilde{\theta}_{n_0}(t, x) := \theta_{n_0}(t, \Lambda_\beta(x))$ on $[0, T] \times \mathbb{R} \times \mathbb{R}^+$ diverge to ∞ for any $T > 0$. The reason for this will now be that $\partial_{x_1} u_2(t, x) \sim x_2^{1-2\alpha}$ for small $t \geq 0$ and small $x_1 \sim x_2 > 0$ (with the other three partial derivatives of u sufficiently controlled), which together with (2.13) will result in $\partial_{x_1} \tilde{u}_2(t, x)$ becoming arbitrarily large near the origin at small times and hence in fast vertical shearing of the "cap" functions from the series in (3.19).

Again extend $\theta_0, \phi_0, \phi_\infty$ oddly onto $\mathbb{R}^2$. Then ϕ_0 and ϕ_∞ are Lipschitz, and so the velocity

$$v^0(x) := \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^{2+2\alpha}} \phi_0(y) dy,$$

generated by only ϕ_0 is also Lipschitz with some constant C by the argument in the proof of (2.2). Moreover, the simple form of $\partial_{x_1}\phi_0$ and easy cancellations due to oddness in x_2 yield

$$\partial_{x_1} v_2^0(x) = -\int_{\mathbb{R}^2} \partial_{x_1}\phi_0(x - y) \frac{y_1}{|y|^{2+2\alpha}} dy = \int_{x_1}^{x_1+1} \int_{x_1-y_1-x_2}^{x_1-y_1+x_2} \frac{y_1}{|y|^{2+2\alpha}} dy_2 dy_1 + A_1$$

when $x_1, x_2 \geq 0$, where A_1 is the integral of $\frac{y_1}{|y|^{2+2\alpha}}$ over the region

$$\{y_1 \in [x_1 + 2, x_1 + 3] \ \& \ y_2 \in [x_1 - y_1 - x_2, x_1 - y_1 + x_2]\}$$

minus the integral of $\frac{y_1}{|y|^{2+2\alpha}}$ over

$$\{y_1 \in [x_1, x_1 + 1] \cup [x_1 + 2, x_1 + 3] \ \& \ y_2 \in [y_1 - x_1 - 3 - x_2, y_1 - x_1 - 3 + x_2]\}.$$

Since these regions are distance more than 1 from the origin and have total area $6x_2$, we get

$$\partial_{x_1} v_2^0(x) \geq x_2 (C^{-1}|x|^{-2\alpha} - C) \tag{3.20}$$

for $x \in [0,1]^2$ and some $C \geq 1$. As we noted above, we also have

$$\|\nabla v^0\|_{L^\infty} \leq C. \tag{3.21}$$

Lemma 2.1 again applies uniformly to all solutions θ as above (let us call the constant in it again C). So consider any divergence-free u on $[0,T] \times \mathbb{R}^2$ for some $T \in (0, \frac{1}{10C}]$ satisfying $u(t, x_1, -x_2) = (u_1(t,x), -u_2(t,x))$,

$$\sup_{t \in [0,T]} \|u(t, \cdot)\|_{C^{1-2\alpha}(\mathbb{R}^2)} \leq C, \tag{3.22}$$

and such that the left-hand sides of (2.2)–(2.5) are all finite (with the L^∞ norms taken over $[0,T] \times \mathbb{R} \times \mathbb{R}^+$). So $\|x_2^{-1} u_2\|_{L^\infty}$ is again finite and we can define $z^{t,x}$ as before. Additionally, we let

$$D_t := \{x \in \mathbb{R}^2 \, | \, |x_2| \geq 3(x_1 - z_1^{t,0})\}$$

and assume

$$\|\nabla u\|_{L^\infty\left(\bigcup_{t \in [0,T]}(\{t\} \times D_t)\right)} \leq 3C \tag{3.23}$$

as well as

$$\left| u(t,x) - \frac{d}{dt} z^{t,0} \right| \leq 3C|x - z^{t,0}| \tag{3.24}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^2$. Note that this implies

$$(1 - 3Ct)|x| \leq |z^{t,x} - z^{t,0}| \leq (1 + 6Ct)|x| \tag{3.25}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^2$, as well as

$$z^{t,x} \in \{x \in \mathbb{R}^2 \, | \, |x_2| \tan(3Ct) \geq x_1 - z_1^{t,0}\} \subseteq D_t \tag{3.26}$$

for any $(t,x) \in [0,T] \times \mathbb{R}_0^- \times \mathbb{R}$ because $T \leq \frac{1}{10C}$ and the set in (3.26) is the complement of a cone in $\mathbb{R}^2$ with vertex at $z^{t,0}$, bisector $[z_1^{t,0}, \infty) \times \{0\}$, and interior half-angle shrinking with speed $3C$.

Since $\operatorname{supp} \phi_0 \subseteq \mathbb{R}_0^- \times \mathbb{R}$, it follows that $\phi_0(z^{-t,\cdot})$ (i.e., ϕ_0 transported by u for time t) is Lipschitz with constant $\frac{1}{1-3Ct} \leq \frac{3}{2}$ for all $t \in [0,T]$. Therefore the gSQG velocity

$$v(t,x) := \int_{\mathbb{R}^2} \phi_0(z^{-t,x-y}) \frac{y^\perp}{|y|^{2+2\alpha}} dy \tag{3.27}$$

corresponding to it satisfies

$$\|\nabla v\|_{L^\infty} \leq 2C \tag{3.28}$$

(analogously to (3.21)). We can also extend (3.20) to v via

$$\partial_{x_1} v_2(t,x) - \partial_{x_1} v_2^0(x - z^{t,0}) = \int_{\mathbb{R}^2} \left[\partial_{x_1} \phi_0(x - y - z^{t,0}) - \frac{d}{dx_1} \phi_0(z^{-t,x-y}) \right] \frac{y_1}{|y|^{2+2\alpha}} dy, \quad (3.29)$$

at least for some (t,x). Note that $\operatorname{supp} \phi_0$ is composed of 10 trapezoids (2 of them are squares) such that $\nabla \phi_0$ is constant in the interior of each of them, with $|\nabla \phi_0| \in \{0,1\}$. This and (3.23) show that if $z^{-t,x-y}$ with $t \in [0,T]$ lies inside one of these trapezoids and $\partial_{x_1} \phi_0 = a$ on that trapezoid, then $|\frac{d}{dx_1} \phi_0(z^{-t,x-y}) - a| \le 6Ct$. It follows that the absolute value of the parenthesis in (3.29) is at most $6Ct$ whenever $x - y - z^{t,0}$ belongs to the same trapezoid as $z^{-t,x-y}$. Also, if $z^{-t,x-y} \notin \operatorname{supp} \phi_0$, then clearly $\frac{d}{dx_1} \phi_0(z^{-t,x-y}) = 0$, so the parenthesis is 0 whenever $x - y - z^{t,0} \notin \operatorname{supp} \phi_0$. This means that if we let $g(x-y)$ be that parenthesis when $z^{-t,x-y}$ and $x - y - z^{t,0}$ belong to the same of the above 11 regions $R_1, \ldots, R_{11}$ (the 10 trapezoids and $\mathbb{R}^2 \setminus \operatorname{supp} \phi_0$) and 0 otherwise, then $\|g\|_{L^\infty} \le 6Ct$ and $\|g\|_{L^1} \le 108Ct$ (since $z^{-t,\cdot}$ is measure-preserving) and so oddness of g in x_2 shows

$$\left| \int_{\mathbb{R}^2} g(x-y) \frac{y_1}{|y|^{2+2\alpha}} dy \right| = \left| \int_{\mathbb{R} \times (-\infty, x_2]} g(x-y) \left(\frac{y_1}{|y|^{2+2\alpha}} - \frac{y_1}{|y - 2x_2 e_2|^{2+2\alpha}} \right) dy \right| \le C'' t x_2^{1-2\alpha}$$

$$(3.30)$$

for some $C'' > 0$, as in the proof of (2.3).

Let us now assume that $z^{-t,x-y}$ and $x - y - z^{t,0}$ do not lie in the same region R_j, in which case the parenthesis in (3.29) is only bounded by 3. Then (3.24) and (3.25) yield

$$|z^{-t,x-y} - (x - y - z^{t,0})| \le 6Ct|x - y - z^{t,0}|, \quad (3.31)$$

which means that $x - y - z^{t,0}$ must be within distance $6Ct|x-y-z^{t,0}|$ of one of the boundaries of the 11 regions (which are 23 straight segments, the rightmost two forming the segment $\{0\} \times [-3,3]$). If (t,x) is such that $|x_2| \le 2(x_1 - z_1^{t,0})$, then the points y such that $x - y - z^{t,0}$ is within distance $6Ct|x-y-z^{t,0}|$ of one of the boundaries of the 11 regions form no more than $C_1 \sqrt{t}$ proportion of $\partial B_r(0)$ for any $r > 0$ (with some $C_1 > 0$). Using this and the computation that yields (3.30) shows that the contribution of the points y considered here to the integral in (3.29) is no more than $C''' \sqrt{t} x_2^{1-2\alpha}$ for some $C''' > 0$.

The above arguments and (3.20) show that there is $C' > 0$ such that

$$\partial_{x_1} v_2(t,x) \ge x_2 \left[\left((2C)^{-1} - C' \sqrt{t} \right) x_2^{-2\alpha} - C \right] \quad (3.32)$$

holds when $(t,x) \in [0,T] \times [0,1]^2$ and $|x_2| \in [\frac{1}{2}(x_1 - z_1^{t,0}), 2(x_1 - z_1^{t,0})]$. This bound and (3.21) will play the same role here as (3.7) and (3.8) did before.

We now again assume that $\theta \in L^\infty(0,T; X_\beta)$ is a solution to (1.1) with initial datum given by (3.18) and (3.19), with some $T \in (0, \frac{1}{10C}]$ and some a_n as above. That is, $\tilde{\theta}(t,x) := \theta(t, \Lambda_\beta(x))$ is Lipschitz in x with some constant L. Let us also assume that θ is the limit of classical solutions θ_{n_0} with initial data θ_{0,n_0} obtained by truncating the sum in (3.19) at $n = n_0$ (we call that sum ϕ_{n_0}). Note that $\theta_{0,n_0} \to \theta_0$ in X_β.

Fix now any n_0, extend θ_{n_0} oddly onto $[0,T] \times \mathbb{R}^2$, and let

$$u(t,x) := \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \theta_{n_0}(t,y) dy$$

be the velocity corresponding to θ_{n_0}, and define $z^{t,x}$ as above. (Then θ_{n_0} is also a unique classical solution to (1.1) on $\mathbb{R}^2$ for at least a short time because θ_{0,n_0} is Lipschitz. But this might cease to be the case at some point, so we just assume that $\theta_{n_0} \in L^\infty([0,T];X_\beta)$ is some solution.) Again $\theta_{n_0}(t,x) = \theta_{0,n_0}(z^{-t,x})$ shows that $u = v + w$, where v is from (3.27) and

$$w(t,x) := \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \phi_{n_0}(\Lambda_\beta^{-1}(z^{-t,y})) dy.$$

One can then show as before that if $a_n := 2^{-\gamma n}$ for a large enough γ, then w will not distort the dynamic provided by v much. First, (2.17) again gives (3.10) for times $t \in [0, T_\theta]$ for some $T_\theta \in (0,T]$, but now we obtain it with interval $[\frac{8}{9}, \frac{9}{8}]$ instead of $[\frac{1}{2}, 2]$. We then again conclude (3.13) when γ is large enough, and similarly also

$$|w(t,x) - w(t,x')| \le |x - x'| \tag{3.33}$$

for all $t \in [0, T_\theta]$ and $x, x' \in D_{t,n_0} := D_t \cup (\mathbb{R} \times [-\lambda_\beta(2^{-4n_0-3}), \lambda_\beta(2^{-4n_0-3})])$ as long as

$$\text{supp } \phi_{n_0}(\Lambda_\beta^{-1}(z^{-t,\cdot})) \subseteq E_{t,n_0} := \left\{ x \in \mathbb{R}^2 \,\middle|\, |x_2| \in \left[\frac{1}{2}(x_1 - z_1^{t,0}), 2(x_1 - z_1^{t,0})\right] \right\}. \tag{3.34}$$

Here γ, T_θ can be made uniform in n_0 because so are all the above estimates.

If now $\tau_{n_0} > 0$ is the first time when $\text{supp } \phi_{n_0}(\Lambda_\beta^{-1}(z^{-t,\cdot}))$ does not lie within E_{t,n_0} or $\text{supp } \phi_0(z^{-t,\cdot})$ does not lie within D_{t,n_0} (τ_{n_0} must be positive because $n_0 < \infty$ and (3.22) holds), then on the time interval $[0, \min\{\tau_{n_0}, T_\theta\}]$ we have (3.33) when $x, x' \in D_{t,n_0}$, which means that similarly to (2.9) we can conclude

$$\frac{d}{dt}\|\phi_0(z^{-t,\cdot})\|_{W^{1,\infty}} \le C''\|\phi_0(z^{-t,\cdot})\|_{W^{1,\infty}}\left(\|\phi_0(z^{-t,\cdot})\|_{W^{1,\infty}} + 1\right).$$

This is because $\phi_0(z^{-t,\cdot})$ is transported by $u = v + w$, where v is generated by $\phi_0(z^{-t,\cdot})$ and (3.33) holds (since ϕ_0 is Lipschitz on $\mathbb{R}^2$, we do not need to use the space X_β here). This C'' is independent of n_0, so there is n_0-independent $T'_\theta \in (0, T_\theta]$ such that (3.28) holds for $t \in [0, \min\{\tau_{n_0}, T'_\theta\}]$. But then so do (3.23) and (3.24) because (3.13) and (3.33) hold, and (3.24) means that $\tau_{n_0} \ge T'_\theta$ because $T'_\theta \le T \le \frac{1}{10C}$.

Hence the above setup holds on the time interval $[0, T'_\theta]$, and in particular (3.13), (3.28), and (3.32) hold on E_{t,n_0} for all $t \in [0, T'_\theta]$, and $\text{supp } \phi_{n_0}(\Lambda_\beta^{-1}(z^{-t,\cdot})) \subseteq E_{t,n_0}$ for all $t \in [0, T'_\theta]$. We now take

$$y := (\lambda_\beta(2^{-4n_0}), \lambda_\beta(2^{-4n_0})) \qquad \text{and} \qquad y' := \left(\lambda_\beta(2^{-4n_0}) + 2n_0^{-2}a_{n_0}, \lambda_\beta(2^{-4n_0} - a_{n_0})\right),$$

(so now we have $n_0^{-2}a_{n_0}$ in place of b_n). This means that

$$y_2 - y_2' \approx \lambda_\beta'(2^{-4n_0})a_{n_0} = (1-\beta)^{\beta/(1-\beta)}2^{-4n_0\beta/(1-\beta)}a_{n_0} \ll n_0^{-2}a_{n_0},$$

so that, for instance, (3.13) and (3.32) yield for all large n_0

$$|w_2(0,y') - w_2(0,y)| \le C'\left|2^{-4n_0\beta/(1-\beta)}a_{n_0} + 2^{-4n_0/(1-\beta)}n_0^{-2}a_{n_0} + a_{n_0}^{(3-2\alpha)/2}\right| \le C''\left|2^{-4n_0\beta/(1-\beta)}a_{n_0}\right|$$

(provided γ is large enough) as well as

$$v_2(0,y') - v_2(0,y) \ge (4C)^{-1}2^{-4n_0(1-2\alpha)/(1-\beta)}n_0^{-2}a_{n_0}.$$

Similarly to the case $\beta \in [0, 2\alpha)$, the above properties of v, w ensure that this relationship of speeds persists for a while at points $z^{t,y'}$ and $z^{t,y}$, and specifically show that at some time

$$T_{n_0} = O\left(2^{-4n_0\beta/(1-\beta)}a_{n_0}\left(2^{-4n_0(1-2\alpha)/(1-\beta)}n_0^{-2}a_{n_0}\right)^{-1}\right) = O\left(n_0^2 2^{-4n_0(\beta+2\alpha-1)/(1-\beta)}\right) \to 0$$

we will have

$$z_1^{T_{n_0},y'} - z_1^{T_{n_0},y} \in \left[n_0^{-2}a_{n_0}, 3n_0^{-2}a_{n_0}\right] \qquad \text{and} \qquad z_2^{T_{n_0},y'} = z_2^{T_{n_0},y}$$

whenever n_0 is large enough. But since $\theta_{0,n_0}(y) = n_0^{-1}a_{n_0}$ and $\theta_{0,n_0}(y') = 0$, it follows that a Lipschitz constant for $\tilde{\theta}_{n_0}(T_{n_0}, \cdot)$ on $\mathbb{R} \times \mathbb{R}^+$ is no less than $\frac{n_0}{3}$. Since $T_{n_0} < T$ for all large n_0, this contradicts our hypothesis that $\theta_{n_0} \to \theta$ in $L^\infty([0,T]; X_\beta)$ and the proof is finished.

4. Proof of Theorem 1.1(II)

The basic setup of our finite time singularity example for each $\alpha \in (0, \frac{1}{4}]$ and $\beta \in [2\alpha, 1-2\alpha]$ will be closely related to that from [21], where the corresponding patch problem on $\mathbb{R} \times \mathbb{R}^+$ was studied. As mentioned in the introduction, a relevant local regularity result in the latter setting was only obtained for $\alpha \in (0, \frac{1}{24})$, and the finite time blow-up argument worked on an only slightly larger interval. We perform here a much more careful analysis in this setting, which allows us to capture the full range $\alpha \in (0, \frac{1}{4}]$, and applies to both (1.1) and the patch problem (hence it also proves the first claim in Theorem 1.3).

Let us fix any $\alpha \in (0, \frac{1}{4}]$ and $\beta \in [2\alpha, 1-2\alpha]$ (the particular choice of β will be completely irrelevant here), and let $\varepsilon > 0$ be a small number to be determined later. Let $D^+ := \mathbb{R}^+ \times \mathbb{R}^+$, and consider any (smooth if one desires) initial datum $\theta_0 \in X_\beta$ that is odd in x_1 and satisfies

$$\chi_{\Omega'} \le \chi_{D^+}\theta_0 \le \chi_\Omega, \tag{4.1}$$

where $\Omega := (\varepsilon, 3) \times (0, 3)$ and $\Omega' := (2\varepsilon, 2) \times (0, 2)$. Assume that $T_{\theta_0} = \infty$, and extend the corresponding (unique) solution θ onto $\mathbb{R}^2$ oddly in x_2. Then θ is odd in both x_1 and x_2, and transported by u from (3.9) on the time interval $[0, \infty)$. Note also that from (2.2), (2.17), and $u_1(t, 0, s) = 0$ for all $(t, s) \in [0, \infty) \times \mathbb{R}$ (the latter due to odd symmetry in x_1), we have $D^+ = \{z^{t,x} \mid x \in D^+\}$ for all $t \ge 0$ when $z^{t,x}$ solves $\frac{d}{dt}z^{t,x} = u(t, z^{t,x})$ and $z^{0,x} := x$, and hence also $0 \le \text{sgn}(x_1 x_2)\, \theta(t, x) \le 1$ for all $(t, x) \in [0, \infty) \times \mathbb{R}^2$. Similarly, for all $t \ge 0$ we have

$$\Omega_t = \{z^{t,x} \mid x \in \Omega_0\} \qquad \text{and} \qquad D^+ \setminus \Omega_t = \{z^{t,x} \mid x \in D^+ \setminus \Omega_0\} \tag{4.2}$$

when we let

$$\Omega_t := \{x \in D^+ \mid \theta(t, x) = 1\}.$$

We will now show that if $\varepsilon > 0$ is small enough, then we obtain

$$\lim_{t \uparrow T} \|\partial_{x_1}\theta(t, \cdot)\|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} = \infty$$

for some $T < \infty$, yielding a contradiction with $\theta \in L^\infty_{\text{loc}}([0, \infty); X_\beta)$ (which follows from $T_{\theta_0} = \infty$ and Theorem 1.1(i)) and thus proving the claim. We will do this by showing that $\lim_{t \uparrow T_\varepsilon} d(0, \Omega_t) = 0$ for $T_\varepsilon := 25(3\varepsilon)^{2\alpha}$, so that blow-up of $\partial_{x_1}\theta$ no later than by time T_ε then follows from $\theta(t, 0, s) = 0$ for all $t, s > 0$. Finally, this claim will be proved by showing that for all $t \in [0, T_\varepsilon)$ we have $\Omega_t \supseteq L_t$, where

$$L_t := \{x \in D^+ \mid x_1 \in (X_t, 1)\ \&\ x_2 \in (0, x_1)\} \tag{4.3}$$
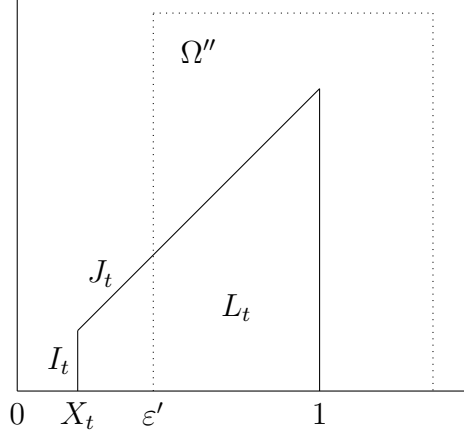
FIGURE 3. Region L_t and segments I_t and J_t.

is a growing trapezoid with $X_t := \left[(3\varepsilon)^{2\alpha} - \frac{t}{25}\right]^{1/2\alpha}$ (and so $X_{T_\varepsilon} = 0$). Note that $X_0 = 3\varepsilon$, so $\Omega_0 \supseteq \Omega' \supseteq L_0$, and for all $t \in [0, T_\varepsilon]$ we have

$$\frac{d}{dt} X_t = -(50\alpha)^{-1} X_t^{1-2\alpha}. \tag{4.4}$$

Let now $d(t) := d(D^+ \setminus \Omega_t, L_t)$, which is continuous on $[0, T_\varepsilon)$ by (2.1) and (4.2). So if $\Omega_t \supseteq L_t$ does not hold for some $t \in [0, T_\varepsilon)$, there is a first time $t' \in [0, T_\varepsilon)$ such that $d(t') = 0$. Then $t' > 0$ because $d(0) \geq \varepsilon$, as well as $L_t \subseteq \Omega_t$ for all $t \in [0, t']$.

Let now $\varepsilon' > 0$ be arbitrary. Then (2.1), (4.1), and $\lim_{\varepsilon \to 0} T_\varepsilon = 0$ show that for all small enough $\varepsilon \in (0, \frac{\varepsilon'}{3}]$ (their interval only depends on ε' and α) we have $\Omega'' := (\varepsilon', \frac{3}{2}) \times (0, \frac{3}{2}) \subseteq \Omega_t$ for all $t \in [0, T_\varepsilon)$. This and $d(t') = 0$ imply that $\overline{D^+ \setminus \Omega_{t'}} \cap \overline{L_{t'}}$ must contain a point from $I_{t'} \cup J_{t'}$, where

$$I_t := \{(X_t, s) \mid s \in [0, X_t]\} \qquad \text{and} \qquad J_t := \{(s, s) \mid s \in [X_t, \varepsilon']\}$$

(so $I_t \cup J_t$ is the part of ∂L_t lying outside Ω'', see Figure 3). We will show that if $\varepsilon' > 0$ is small enough (depending only on α) and $t \in [0, t']$, then $u(t, x)$ points (non-tangentially) outside of L_t for all $x \in J_t$, and $u_1(t, x) \leq -(45\alpha)^{-1} X_t^{1-2\alpha}$ for all $x \in I_t$. This of course makes it impossible for $D^+ \setminus \Omega_{t'}$ and $L_{t'}$ to touch at time t', in view of (4.3) and (4.4), yielding a contradiction.

The above is the basic strategy from [21], which was there applied with Ω_t being a solution patch at time t (with $\theta \equiv 1$ on Ω_t and 0 on $\mathbb{D}^+ \setminus \Omega_t$, and with a second patch where $\theta \equiv -1$ lying in $\mathbb{R}^- \times \mathbb{R}^+$, positioned symmetrically to Ω_t). The crucial estimates on $u(t, \cdot)$ on the set $I_t \cup J_t$ mentioned above were established in [21] for only small enough $\alpha > 0$, and here we will obtain them for the full local well-posedness range $\alpha \in (0, \frac{1}{4}]$.

Estimating the velocity on I_t. Let us start with $x = (X_t, x_2) \in I_t$ for some $t \in [0, t']$ (the case $x \in J_t$ will be simpler), and we will show that $L_t \subseteq \Omega_t$ guarantees

$$u_1(t, x) \leq -(45\alpha)^{-1} x_1^{1-2\alpha} \tag{4.5}$$

as long as $\varepsilon' > 0$ was chosen small enough (here of course $x_1 = X_t$).

Odd symmetry of θ across both axes shows that

$$u_1(t,x) = \int_{D^+} K_1(x,y)\theta(t,y)dy, \tag{4.6}$$

where

$$K_1(x,y) := \frac{x_2 - y_2}{|x-y|^{2+2\alpha}} - \frac{x_2 - y_2}{|x-\tilde{y}|^{2+2\alpha}} - \frac{x_2 + y_2}{|x-\bar{y}|^{2+2\alpha}} + \frac{x_2 + y_2}{|x+y|^{2+2\alpha}}, \tag{4.7}$$

with $\bar{y} := (y_1, -y_2)$ and $\tilde{y} := (-y_1, y_2)$. Let also

$$A_x^{\pm} := \{y \in D^+ \mid \pm K_1(x,y) > 0\}.$$

Since $0 \leq \theta \leq 1$ on D^+, an upper bound on $u_1(t,x)$ can be obtained by replacing $\theta(t,\cdot)$ in (4.6) by 1 on $A_x^+ \cup L_x'$ and by 0 on $A_x^- \setminus L_x'$, for any $L_x' \subseteq L_t$ (because $L_t \subseteq \Omega_t$). One should think of A_x^+ being the "bad" set for an upper bound on $u_1(t,x)$ and A_x^- the "good" set, and our task is complicated by the fact that a precise identification of $A_x^{\pm}$ is not easy. We will therefore use a slightly weaker bound, obtained by replacing $\theta(t,\cdot)$ by 1 on $A_x^+ \cup L_x' \cup L_x''$ and by 0 on $A_x^- \setminus (L_x' \cup L_x'')$, as well as $K_1(x,y)$ by

$$\frac{x_2 - y_2}{|x-y|^{2+2\alpha}} - \frac{x_2 - y_2}{|x-\tilde{y}|^{2+2\alpha}} \qquad (\geq \max\{K_1(x,y),0\} \text{ when } y_2 \leq x_2)$$

for all $y \in L_x''$, with some sets $L_x' \subseteq L_t$ and $L_x'' \subseteq \mathbb{R}^+ \times (0,x_2)$. After expanding the integral from (4.6) onto $\mathbb{R}^2$, this upper bound on $u_1(t,x)$ becomes

$$\int_{S_x \cup (-S_x) \cup L_x''} \frac{x_2 - y_2}{|x-y|^{2+2\alpha}} dy - \int_{\bar{S}_x \cup \tilde{S}_x \cup \tilde{L}_x''} \frac{x_2 - y_2}{|x-y|^{2+2\alpha}} dy, \tag{4.8}$$

where $S_x := (A_x^+ \cup L_x') \setminus L_x''$. We now make the specific choice

$$L_x' := \{y \in D^+ \mid y_1 \in (x_1,1) \ \& \ y_2 \in (0, y_1 - x_1 + x_2)\} \subseteq L_t,$$
$$L_x'' := \big[((0,x_1] \cup [1,\infty)) \times (0,x_2)\big] \setminus L_x''',$$

where

$$L_x''' := \{y \in D^+ \mid y_1 \in (x_1 - x_2, x_1) \ \& \ y_2 \in (0, y_1 - x_1 + x_2)\}$$

(recall that $x_2 \in [0,x_1]$). These regions are drawn in Figure 4, with some being split into several sub-regions for later use. So, for instance, $L_x' = \bigcup_{j=1}^{4} L_{x,j}'$ and $L_x'' = L_{x,1}'' \cup L_{x,2}''$, where $L_{x,2}'' := [1,\infty) \times (0,x_2)$ lies far to the right and is not pictured.

Our first crucial observation is that

$$L_x''' \subseteq A_x^+ \subseteq L_x' \cup L_x'' \cup L_x''', \tag{4.9}$$

which means that $S_x = L_x' \cup L_x'''$ (note that L_x', L_x'', L_x''' are pairwise disjoint). The second inclusion in (4.9) holds because $L_x' \cup L_x'' \cup L_x''' \supseteq \mathbb{R}^+ \times (0,x_2)$ and for any $y \in \mathbb{R}^+ \times [x_2, \infty)$ we have

$$\frac{x_2 - y_2}{|x-y|^{2+2\alpha}} \leq \frac{x_2 - y_2}{|x-\tilde{y}|^{2+2\alpha}} \qquad \text{and} \qquad \frac{x_2 + y_2}{|x-\bar{y}|^{2+2\alpha}} \geq \frac{x_2 + y_2}{|x+y|^{2+2\alpha}}.$$

The first inclusion in (4.9) is due to the following lemma, with (x, b_1, b_2) in the lemma being $(x - y, x_1, x_2)$ here (note that for any $y \in L_x'''$ we have $0 < x_1 - y_1 \leq x_2 - y_2 \leq x_2 \leq x_1$). We also note that if $x_2 = 0$, then (4.9) holds trivially because all the sets in it are empty.
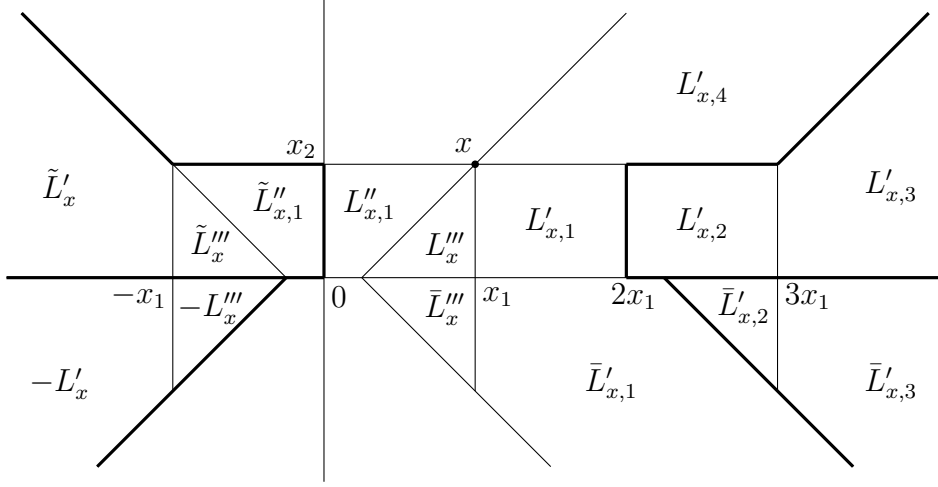
FIGURE 4. Subregions and reflections of L'_x, L''_x, L'''_x for u_1 (not pictured are $L''_{x,2} = [1, \infty) \times (0, x_2)$, $\tilde{L}''_{x,2} = (-\infty, 1] \times (0, x_2)$, and parts of $L'_x, \bar{L}'_x, \tilde{L}'_x, -L'_x$).

Lemma 4.1. *Whenever* $\alpha \geq 0$, $b_1, b_2 > 0$, *and* $0 < x_1 \leq x_2 < \min\{b_1, b_2\}$, *we have*

$$\frac{x_2}{|(x_1, x_2)|^{2+2\alpha}} - \frac{x_2}{|(2b_1 - x_1, x_2)|^{2+2\alpha}} - \frac{2b_2 - x_2}{|(x_1, 2b_2 - x_2)|^{2+2\alpha}} + \frac{2b_2 - x_2}{|(2b_1 - x_1, 2b_2 - x_2)|^{2+2\alpha}} > 0.$$

Proof. Letting $c := \frac{b_1}{b_2}$ and $y := \frac{x}{b_2}$, this claim becomes equivalent to

$$\frac{y_2}{|(y_1, y_2)|^{2+2\alpha}} - \frac{2 - y_2}{|(y_1, 2 - y_2)|^{2+2\alpha}} > \frac{y_2}{|(2c - y_1, y_2)|^{2+2\alpha}} - \frac{2 - y_2}{|(2c - y_1, 2 - y_2)|^{2+2\alpha}} \quad (4.10)$$

for $0 < y_1 \leq y_2 < \min\{c, 1\}$. If we let $f(y_1)$ be the left-hand side of (4.10), then we need to show $f(y_1) > f(2c - y_1)$. Direct differentiation of f shows that

$$\operatorname{sgn} f'(s) = \operatorname{sgn}\left((2 - y_2)(s^2 + y_2^2)^{2+\alpha} - y_2(s^2 + (2 - y_2)^2)^{2+\alpha}\right),$$

from which it easily follows that f' vanishes at a single $s' > 0$, as well as that f is decreasing on $[0, s']$ and increasing on $[s', \infty)$ (both strictly). Since $\lim_{s \to \infty} f(s) = 0$ and $2c - y_1 > y_1$, it suffices to show that $f(y_2) \geq 0$ because then $y_1 \leq y_2$ implies $f(y_1) > f(s)$ for all $s > y_1$. That is, it suffices to show that

$$y_2^{-1-2\alpha} - (2 - y_2)(y_2^2 - 2y_2 + 2)^{-1-\alpha} \geq 0$$

for all $y_2 \in (0, 1)$. Equality holds when $y_2 = 1$, and a direct computation shows that the derivative of the left-hand side is no more than

$$-(1 + 2\alpha)y_2^{-2-2\alpha} + (y_2^2 - 2y_2 + 2)^{-1-\alpha} < 0$$

for all $y_2 \in (0, 1)$ (note that $-2y_2 + 2 < 0$). Hence the claim follows. $\square$

Since $S_x = L'_x \cup L'''_x$, the bound (4.8) becomes

$$\int_{L'_x \cup (-L'_x) \cup L'''_x \cup (-L'''_x) \cup L''_x} \frac{x_2 - y_2}{|x - y|^{2+2\alpha}} dy - \int_{\bar{L}'_x \cup \tilde{L}'_x \cup \bar{L}'''_x \cup \tilde{L}'''_x \cup \tilde{L}''_x} \frac{x_2 - y_2}{|x - y|^{2+2\alpha}} dy. \quad (4.11)$$

Because the first integrand at (y_1, y_2) equals the second integrand at $(2x_1 - y_1, y_2)$, and vice versa, the integrals over the regions contained in $\mathbb{R}^- \times \mathbb{R}$ cancel with those over the images of these regions under the mapping $y \mapsto (2x_1 - y_1, y_2)$ (in the notation from Figure 4, region $\tilde{L}'_x \cup \tilde{L}'''_x \cup \tilde{L}''_{x,1} \cup \tilde{L}''_{x,2}$ cancels with $L'_{x,3} \cup L'_{x,2} \cup L''_{x,2}$ and $(-L'_x) \cup (-L'''_x)$ cancels with $\bar{L}'_{x,3} \cup \bar{L}'_{x,2}$; these regions are marked by bold lines). Each of these images is a region contained in the integration domain of the integral not containing the original region (whence the cancelation), except for the subregions $[(-1, -1+2x_1) \times (x_2, 1)] \cap \tilde{L}'_x$ and $[(-1, -1+2x_1) \times (-1, 0)] \cap (-L'_x)$, whose images lie in $(1, \infty) \times (\mathbb{R} \setminus [0, x_2])$ and so the integrals over them do not get canceled. However, since $0 < x_2 \le x_1 \le 3\varepsilon \ll 1$ and both these regions have areas $\le 2x_1$, the contribution of the integrals over them to (4.11) will be bounded above by $16x_1$ and hence negligible in the proof of (4.5) because $\alpha > 0$. After this cancellation we are left with regions

$$B := (0, 2x_1) \times (0, x_2),$$
$$G := \{y \in \mathbb{R}^2 \,|\, y_2 > x_2 \ \& \ y_1 \in (y_2 - x_2 + x_1, y_2 - x_2 + 3x_1) \ \& \ y_1 < 1\},$$
$$G^- := \{y \in \mathbb{R}^2 \,|\, y_2 < 0 \ \& \ y_1 \in (-y_2 - x_2 + x_1, -y_2 - x_2 + 3x_1) \ \& \ y_1 < 1\}$$

(so $B = L''_{x,1} \cup L'''_x \cup L'_{x,1}$, $G = L'_{x,4}$, and $G^- = \bar{L}'_{x,1} \cup \bar{L}'''_x$ in Figure 4), and this all yields

$$u_1(t, x) \le \int_{B \cup G} \frac{x_2 - y_2}{|x - y|^{2+2\alpha}} dy - \int_{G^-} \frac{x_2 - y_2}{|x - y|^{2+2\alpha}} dy + 16x_1.$$

Here the contribution of the "bad" region B is positive, while that of the "good" regions G, G^- is negative. To simplify the following computations, we can replace G, G^- by regions G_*, G_*^- that are defined identically but without the constraint $y_1 < 1$. This again changes the estimate by less than $C_\alpha x_1$ for some C_α depending only on α (and, in particular, not on x), so we obtain the bound

$$u_1(t, x) \le \int_{B \cup G_*} \frac{x_2 - y_2}{|x - y|^{2+2\alpha}} dy - \int_{G_*^-} \frac{x_2 - y_2}{|x - y|^{2+2\alpha}} dy + (C_\alpha + 16)x_1.$$

The change of variables $z := \frac{y - x}{x_1}$ turns this bound into

$$-u_1(t, x)x_1^{2\alpha - 1} \ge \int_{G_b \cup G_b^-} \frac{|z_2|}{|z|^{2+2\alpha}} dz - \int_{B_b^-} \frac{|z_2|}{|z|^{2+2\alpha}} dz - (C_\alpha + 16)x_1^{2\alpha},$$

where $b := \frac{x_2}{x_1}$ and the sets B_b^-, G_b, G_b^- (from the following lemma, also drawn in Figure 5) are the images of B, G_*, G_*^- under this change of variables. Estimate (4.5) now follows for all small enough $\varepsilon' > 0$ (recall that $x_1 \le 3\varepsilon \le \varepsilon'$) by the key Lemma 4.2, which we prove after finishing the proof of Theorem 1.1(ii).

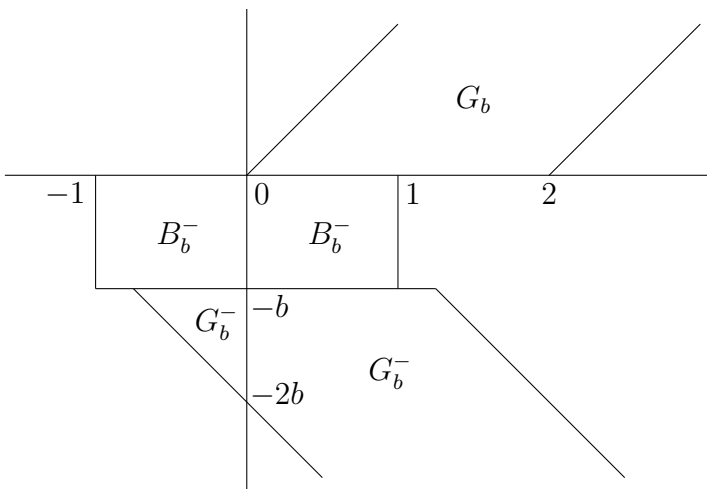**Lemma 4.2.** *If $\alpha \in (0, \frac{1}{4}]$ and for any $b \in [0, 1]$ we let*

$$B_b^- := (-1, 1) \times (-b, 0),$$
$$G_b := \{x \in \mathbb{R}^2 \,|\, x_2 > 0 \ \& \ x_1 \in (x_2, x_2 + 2)\},$$
$$G_b^- := \{x \in \mathbb{R}^2 \,|\, x_2 < -b \ \& \ x_1 \in (-x_2 - 2b, -x_2 - 2b + 2)\},$$

then

$$\inf_{b \in [0,1]} \left\{ \int_{G_b \cup G_b^-} \frac{|x_2|}{|x|^{2+2\alpha}} dx - \int_{B_b^-} \frac{|x_2|}{|x|^{2+2\alpha}} dx \right\} \ge \frac{1}{40\alpha}. \tag{4.12}$$

FIGURE 5. The regions B_b^-, G_b, G_b^- for u_1.

This concludes the case $x \in I_t$. We note that the only place where the above analysis is not optimal is the replacement of K_1 by the difference of its first two terms on $L_{x,1}''$ instead of identification of the set $A_x^+ \cap L_{x,1}''$ (the set $L_{x,2}'' \subseteq [1, \infty) \times \mathbb{R}^+$ is irrelevant because its contribution is of lower order and hence negligible here). While picking $L_x' = L_t$ for all x may seem to yield further improvement, we in fact do have $L_x' = L_t$ when $x = (X_t, X_t)$ and so $b = 1$ above, which will be the most crucial value in the proof of Lemma 4.2.

As for Lemma 4.2, if we let $V_{\alpha,b}$ be the value inside the infimum in (4.12), then we need to obtain $\inf_{b \in [0,1]} V_{\alpha,b} > 0$ for our blow-up analysis to work. The proof of Lemma 4.2 evaluates $V_{\alpha,1}$ *exactly* (we could also do it for $V_{\alpha,b}$ with $b \in [0, 1)$ but the resulting formulas are quite daunting to deal with, so we instead obtain a much more convenient lower bound) and yields

$$\frac{2\alpha}{4 - 2^{-\alpha}} V_{\alpha,1} = \int_0^1 \frac{dq}{(q^2 + 1)^\alpha} - \frac{2 + 2^{-\alpha} - 2^{1-2\alpha}}{(1 - 2\alpha)(4 - 2^{-\alpha})}.$$

This is decreasing in α, so to cover the full region $\alpha \in (0, \frac{1}{4}]$, we certainly need this number to be positive for $\alpha = \frac{1}{4}$. It indeed is, but barely so: it is approximately $0.937 - 0.903 = 0.034$. It in fact becomes negative around $\alpha = 0.257$, which illustrates why showing $L_x''' \subseteq A_x^+$ in Lemma 4.1 is also crucial for our analysis to apply to all $\alpha \in (0, \frac{1}{4}]$. If we instead included L_x''' in L_x'', this would replace $\bar{L}_x'''$ by $\bar{L}_{x,2}'$ in the region G^- and ultimately lower the upper end of the interval of α for which the resulting $V_{\alpha,1}$ is positive below $\frac{1}{4}$.

Estimating the velocity on J_t. Now consider $x = (x_*, x_*) \in J_t$ for some $t \in [0, t']$, so $x_* \in [X_t, \varepsilon']$. Since

$$\Omega_t \supseteq L_t \supseteq L_x := \{y \in L_t \,|\, y_1 > x_1\},$$

the proof from the case $x \in I_t$, with L_t replaced by L_x, again yields (4.5). Since we need to show that $u(t, x)$ points (non-tangentially) outside L_t at x, it now suffices to show that

$$u_2(t, x) \geq 0 \tag{4.13}$$

when $\varepsilon' > 0$ is small enough.

We proceed similarly as for $x \in I_t$, but the argument will this time be a little simpler. Odd symmetry of θ in both x_1 and x_2 shows that

$$u_2(t,x) = \int_{D^+} K_2(x,y)\theta(t,y)dy, \tag{4.14}$$

where

$$K_2(x,y) := \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} - \frac{y_1 - x_1}{|x-\bar{y}|^{2+2\alpha}} + \frac{y_1 + x_1}{|x-\tilde{y}|^{2+2\alpha}} - \frac{y_1 + x_1}{|x+y|^{2+2\alpha}}, \tag{4.15}$$

and now we let

$$A_x^{\pm} := \{y \in D^+ \mid \pm K_2(x,y) > 0\}.$$

Since $0 \le \theta \le 1$ on D^+, a lower bound on $u_2(t,x)$ can be obtained by replacing $\theta(t,\cdot)$ in (4.14) by 1 on $A_x^- \cup L_x$ and by 0 on $A_x^+ \setminus L_x$ (so now we just pick $L_x' = L_x$) because A_x^- is now the "bad" set for a lower bound. We will again use a slightly weaker bound, replacing $\theta(t,\cdot)$ by 1 on $A_x^- \cup L_x \cup L_x''$ and by 0 on $A_x^+ \setminus (L_x \cup L_x'')$, as well as $K_2(x,y)$ by

$$\frac{y_1 - x_1}{|x-y|^{2+2\alpha}} - \frac{y_1 - x_1}{|x-\bar{y}|^{2+2\alpha}} \qquad (\le \min\{K_2(x,y), 0\} \text{ when } y_1 \le x_1)$$

for all $y \in L_x'' := (0, x_1) \times \mathbb{R}^+$ (so this time there will be no L_x''' and no need for Lemma 4.1). After expanding the integral from (4.14) onto $\mathbb{R}^2$, this lower bound on $u_2(t,x)$ becomes

$$\int_{S_x \cup (-S_x) \cup L_x''} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy - \int_{\bar{S}_x \cup \tilde{S}_x \cup \bar{L}_x''} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy, \tag{4.16}$$

where $S_x := (A_x^- \cup L_x) \setminus L_x''$. But since now $A_x^- \subseteq L_x''$ due to

$$\frac{y_1 - x_1}{|x-y|^{2+2\alpha}} \ge \frac{y_1 - x_1}{|x-\bar{y}|^{2+2\alpha}} \qquad \text{and} \qquad \frac{y_1 + x_1}{|x-\tilde{y}|^{2+2\alpha}} \ge \frac{y_1 + x_1}{|x+y|^{2+2\alpha}}$$

for all $y \in [x_1, \infty) \times \mathbb{R}^+$, it follows from $L_x \cap L_x'' = \emptyset$ that $S_x = L_x$. The regions L_x, L_x'', and their reflections are drawn in Figure 6 (recall that $x_1 = x_2 = x_*$).

So (4.16) becomes

$$\int_{L_x \cup (-L_x) \cup L_x''} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy - \int_{\bar{L}_x \cup \tilde{L}_x \cup \bar{L}_x''} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy, \tag{4.17}$$

Because the first integrand at (y_1, y_2) equals the second integrand at $(y_1, 2x_2 - y_2)$, and vice versa, the integrals over the regions contained in $\mathbb{R} \times \mathbb{R}^-$ cancel with those over the images of these regions under the mapping $y \mapsto (y_1, 2x_2 - y_2)$ (in the notation from Figure 6, region $\bar{L}_x'' \cup \bar{L}_x$ cancels with $L_{x,2}'' \cup L_{x,2}$ and $-L_x'$ cancels with $\tilde{L}_{x,1} \cup \tilde{L}_{x,2}$). However, this cancellation requires us to add the regions

$$B_* := \{y \in \mathbb{R}^2 \mid y_1 > x^* \ \& \ y_2 \in (\min\{y_1, 2x_*\}, y_1 + 2x_*)\}$$

and $\tilde{B}_*$ to L_x and $\tilde{L}_x$, respectively (in fact, then $B_* \cap ([1,\infty) \times \mathbb{R})$ and $\tilde{B}_* \cap ((-\infty, -1] \times \mathbb{R})$ will not get cancelled, but their contribution is again only $O(x_*)$ and hence negligible). So we have to subtract their contribution, meaning that (4.13) will follow if we can show

$$\int_{G_* \cup B \cup \tilde{B}_*} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy - \int_{\tilde{G}_* \cup B_*} \frac{y_1 - x_1}{|x-y|^{2+2\alpha}} dy \ge Cx_* \tag{4.18}$$
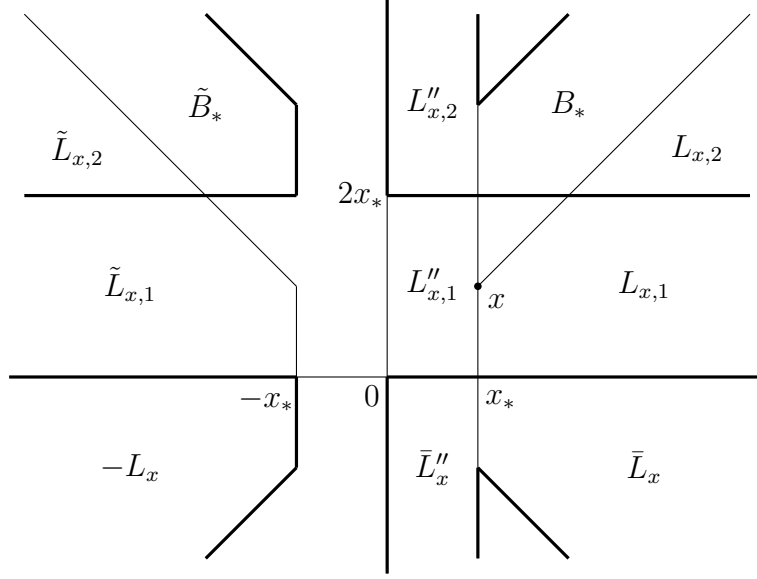
FIGURE 6. Subregions and reflections of L_x, L''_x for u_2.

for all $x_* \in (0, \varepsilon']$, for any constant $C < \infty$ and small enough (C-dependent) $\varepsilon' > 0$, where

$$G_* := \{y \in \mathbb{R}^2 \,|\, y_1 > x^* \ \& \ y_2 \in (0, \min\{y_1, 2x_*\})\}$$

and $B := (0, x_*) \times (0, 2x_*)$ (note that the difference between $G_* \cup \tilde{G}_*$ and $L_{x,1} \cup \tilde{L}_{x,1}$ from Figure 6 is a subset of $[\mathbb{R} \setminus (-1, 1)] \times (0, 2x_*)]$ and hence only contributes $O(x_*)$ to (4.18)).

We again apply to (4.18) the change of variables $z := \frac{y-x}{x_*}$, but first we note that the mapping $y \mapsto (2x_1 - y_1, y_2)$ changes the sign of the integrand, so the integral over B cancels with that over $(x_*, 2x_*) \times (0, 2x_*)$, which means that we can remove domains B and $G_* \cap [(x_*, 2x_*) \times (0, 2x_*)]$ from the first integral if we also add the triangle $[(x_*, 2x_*) \times (0, 2x_*)] \setminus G_*$ to the second. This and the change of variables now show that (4.18) is equivalent to

$$\int_{G_0 \cup G_0^-} \frac{|z_1|}{|z|^{2+2\alpha}} dz - \int_{B_0 \cup B_0^-} \frac{|z_1|}{|z|^{2+2\alpha}} dz \geq C x_*^{2\alpha},$$

where the sets G_0, G_0^-, B_0, B_0^- are from the following lemma (also drawn in Figure 7). The last estimate now follows for all small enough $\varepsilon' > 0$ and all $x_* \in (0, \varepsilon]$ by our last key Lemma 4.3, which we prove after the proof of Lemma 4.2.

Lemma 4.3. *If $\alpha \in (0, \frac{1}{4}]$ and we let*
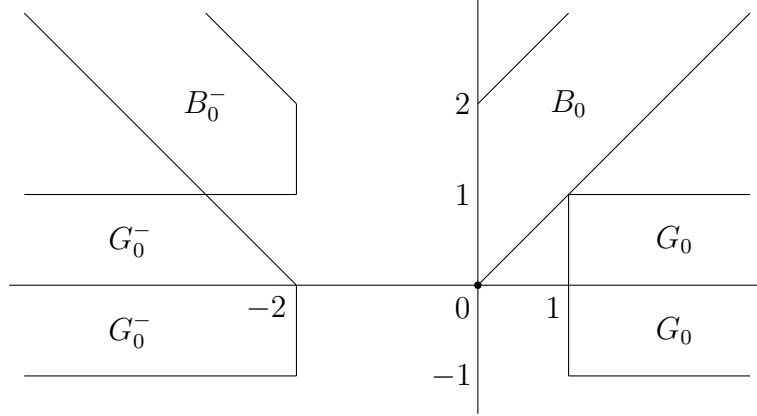
$$G_0 := (1, \infty) \times (-1, 1)$$
$$G_0^- := \{y \in \mathbb{R}^2 \,|\, y_1 < -2 \ \& \ y_2 \in (-1, \min\{-y_1 - 2, 1\})\},$$
$$B_0 := \{y \in \mathbb{R}^2 \,|\, y_1 > 0 \ \& \ y_2 \in (y_1, y_1 + 2)\},$$
$$B_0^- := \{y \in \mathbb{R}^2 \,|\, y_1 < -2 \ \& \ y_2 \in (\max\{-y_1 - 2, 1\}, -y_1)\},$$

then

$$\int_{G_0 \cup G_0^-} \frac{|x_1|}{|x|^{2+2\alpha}} dx - \int_{B_0 \cup B_0^-} \frac{|x_1|}{|x|^{2+2\alpha}} dx > 0. \qquad (4.19)$$

FIGURE 7. The regions G_0, G_0^-, B_0, B_0^- for u_2.

So (4.5) and (4.13) hold whenever $\varepsilon' > 0$ is small enough, which concludes the proof of Theorem 1.1(ii) as explained before (4.5).

Proofs of the two velocity estimates. It remains to prove Lemmas 4.2 and 4.3.

Proof of Lemma 4.2. Let $I(b)$ be 2α times the value inside the infimum in (4.12) when G_b^- is replaced by $G_b^* := G_b^- \cap \{x \in \mathbb{R}^2 \,|\, x_1 < -x_2\}$, which turns out to simplify the proof significantly. This does not change the integration region when $b = 1$, while for $b < 1$ it decreases the value inside the infimum (after being multiplied by 2α, which will also be convenient). Note that since I is continuous on $[0,1]$, it suffices to consider $b \in (0,1]$.

If now G_b^+, B_b^+ are the reflections of G_b^*, B_b^- across the x_1 axis, then $G_b, G_b^+, B_b^+ \subseteq \mathbb{R} \times \mathbb{R}^+$ and

$$I(b) = 2\alpha \int_{G_b \cup G_b^+} \frac{x_2}{|x|^{2+2\alpha}} dx - 2\alpha \int_{B_b^+} \frac{x_2}{|x|^{2+2\alpha}} dx.$$

When $0 \leq a \leq c$, then direct integration gives

$$2\alpha \int_a^c \frac{x_2}{|(x_1, x_2)|^{2+2\alpha}} dx_2 = \frac{1}{(x_1^2 + a^2)^\alpha} - \frac{1}{(x_1^2 + c^2)^\alpha}, \tag{4.20}$$

and for $0 \leq r \leq s$ we also have

$$\int_r^s \frac{dx_1}{(x_1^2 + a^2)^\alpha} = \begin{cases} a^{1-2\alpha} \left(f\left(\frac{s}{a}\right) - f\left(\frac{r}{a}\right) \right) & a > 0, \\ \frac{1}{1-2\alpha} \left(s^{1-2\alpha} - r^{1-2\alpha} \right) & a = 0, \end{cases} \tag{4.21}$$

where

$$f(s) := \int_0^s \frac{dq}{(q^2 + 1)^\alpha}. \tag{4.22}$$

We will use (4.20) and (4.21) repeatedly below, without saying so explicitly. We also note that from $0 \leq (q^2)^{-\alpha} - (q^2 + 1)^{-\alpha} \leq \alpha q^{-2-2\alpha}$ we find that there exists

$$\mu_\alpha := \lim_{s \to \infty} \left(\frac{s^{1-2\alpha}}{1 - 2\alpha} - f(s) \right) > 0, \tag{4.23}$$

which then satisfies

$$\mu_\alpha \leq \frac{1}{1-2\alpha} - f(1) + \int_1^\infty \frac{\alpha}{q^{2+2\alpha}} dq \leq \frac{1}{1-2\alpha} - f(1) + \frac{\alpha}{1+2\alpha}. \tag{4.24}$$

as well as

$$\lim_{s\to\infty} \left(\gamma^{2\alpha-1} f(\gamma s) - \frac{s^{1-2\alpha}}{1-2\alpha} \right) = -\gamma^{2\alpha-1} \mu_\alpha \tag{4.25}$$

for any $\gamma > 0$.

First, we easily get

$$2\alpha \int_{B_b^+} \frac{x_2}{|x|^{2+2\alpha}} dx = 2 \int_0^1 \left(\frac{1}{x_1^{2\alpha}} - \frac{1}{(x_1^2+b^2)^\alpha} \right) dx_1 = \frac{2}{1-2\alpha} - 2b^{1-2\alpha} f\left(\frac{1}{b}\right). \tag{4.26}$$

Next, $x_1^2 + (x_1-2)^2 = 2((x_1-1)^2+1)$ and (4.25) yield

$$2\alpha \int_{G_b} \frac{x_2}{|x|^{2+2\alpha}} dx = \int_0^2 \left(\frac{1}{x_1^{2\alpha}} - \frac{2^{-\alpha}}{x_1^{2\alpha}} \right) dx_1 + \int_2^\infty \left(\frac{2^{-\alpha}}{((x_1-1)^2+1)^\alpha} - \frac{2^{-\alpha}}{x_1^{2\alpha}} \right) dx_1$$
$$= \frac{2^{1-2\alpha}}{1-2\alpha} - 2^{-\alpha} f(1) - 2^{-\alpha} \mu_\alpha \tag{4.27}$$

(here we also used that $\lim_{s\to\infty} f'(s) = 0$). Finally, $x_1^2 + (x_1+2b)^2 = 2((x_1+b)^2+b^2)$ yields

$$2\alpha \int_{G_b^+} \frac{x_2}{|x|^{2+2\alpha}} dx = \int_{-b}^b \left(\frac{1}{(x_1^2+b^2)^\alpha} - \frac{2^{-\alpha}}{((x_1+b)^2+b^2)^\alpha} \right) dx_1 + \int_b^\infty \left(\frac{2^{-\alpha}}{x_1^{2\alpha}} - \frac{2^{-\alpha}}{((x_1+b)^2+b^2)^\alpha} \right) dx_1$$
$$= 2b^{1-2\alpha} f(1) - \frac{2^{-\alpha} b^{1-2\alpha}}{1-2\alpha} + 2^{-\alpha} b^{1-2\alpha} \mu_\alpha,$$

where we also used (4.25). Hence we see that

$$I(b) = \frac{2^{1-2\alpha}-2}{1-2\alpha} - 2^{-\alpha}(f(1)+\mu_\alpha) + 2b^{1-2\alpha}\left(f\left(\frac{1}{b}\right) + f(1) - \frac{2^{-1-\alpha}}{1-2\alpha} + 2^{-1-\alpha}\mu_\alpha \right)$$

for all $b \in (0,1]$, and so also

$$I(1) = (4-2^{-\alpha})f(1) - \frac{2+2^{-\alpha}-2^{1-2\alpha}}{1-2\alpha}. \tag{4.28}$$

If we let $J(c) := \frac{1}{2} I(\frac{1}{c})$ for $c \in [1,\infty)$, then

$$J'(c) = (1-2\alpha)c^{2\alpha-2}\left(\frac{c}{(1-2\alpha)(c^2+1)^\alpha} - f(c) - f(1) + \frac{2^{-1-\alpha}}{1-2\alpha} - 2^{-1-\alpha}\mu_\alpha \right) =: (1-2\alpha)c^{2\alpha-2}g(c).$$

Since

$$g'(c) = \frac{1}{(1-2\alpha)(c^2+1)^\alpha}\left(1 - \frac{2\alpha c^2}{c^2+1} - (1-2\alpha) \right) = \frac{2\alpha}{(1-2\alpha)(c^2+1)^{1+\alpha}} \geq 0,$$

we see that

$$\min_{c\in[1,\infty)} g(c) = g(1) = \frac{3 \cdot 2^{-1-\alpha}}{1-2\alpha} - 2f(1) - 2^{-1-\alpha}\mu_\alpha. \tag{4.29}$$

When $g(1) \geq 0$ (which happens for α not too small), then J attains its minimum at $c = 1$, so we also have $\min_{b \in (0,1]} I(b) = I(1)$. If instead $g(1) < 0$, then

$$\inf_{c \in [1,\infty)} J(c) \geq J(1) + \int_1^\infty (1-2\alpha)c^{2\alpha-2}g(1)dc = J(1) + g(1),$$

and so $\inf_{b \in (0,1]} I(b) \geq I(1) + 2g(1)$. This means that

$$\inf_{b \in (0,1]} I(b) \geq \min\{I(1), I(1) + 2g(1)\}.$$

So it remains to estimate these two values. From (4.28), (4.29), and (4.24) we obtain

$$2^\alpha(I(1) + 2g(1)) = \frac{2 + 2^{1-\alpha} - 2^{1+\alpha}}{1 - 2\alpha} - \mu_\alpha - f(1) \geq \frac{1 + 2^{1-\alpha} - 2^{1+\alpha}}{1 - 2\alpha} - \frac{\alpha}{1 + 2\alpha}.$$

Since $\frac{\alpha}{1+2\alpha} \leq \frac{1}{6}$ and $1 + 2^{1-\alpha} - 2^{1+\alpha} \geq \frac{3}{10}$ when $\alpha \in [0, \frac{1}{4}]$, it follows that

$$I(1) + 2g(1) \geq \frac{2^{3/4}}{15} \geq \frac{1}{10}.$$

Hence it suffices to show that $I(1) \geq \frac{1}{20}$, which thanks to (4.28) will follow from

$$f(1) \geq \frac{2 + 2^{-\alpha} - 2^{1-2\alpha}}{(1 - 2\alpha)(4 - 2^{-\alpha})} + \frac{1}{60}.$$

Since the numerator and denominator of the first fraction on the right-hand side are increasing and decreasing in α, respectively, and $f(1)$ is decreasing in α, it suffices to prove this for $\alpha = \frac{1}{4}$, when the inequality becomes

$$f(1) \geq \frac{4 + 2^{3/4} - 2^{3/2}}{4 - 2^{-1/4}} + \frac{1}{60}. \tag{4.30}$$

The first fraction on the right is less than 0.9033 and $f(1)$ for $\alpha = \frac{1}{4}$ can be numerically bounded below by 0.9374, which yields (4.30). Alternatively, since $(q^2 + 1)^{-1/4}$ is concave on $[0, \sqrt{2/3}] \ni \frac{4}{5}$, and we have $((\frac{4}{5})^2 + 1)^{-1/4} \geq \frac{22}{25}$ and $2^{-1/4} \geq \frac{21}{25}$, it follows that

$$f(1) \geq \frac{2}{5}\left(1 + \frac{22}{25}\right) + \frac{1}{5}\frac{21}{25} = 0.92,$$

and we conclude by $0.92 \geq 0.9033 + \frac{1}{60}$. $\qquad\square$

Proof of Lemma 4.3. If G_0^+, B_0^+ are reflections of G_0^-, B_0^- across the x_2 axis, then the left-hand side of (4.19) times 2α is

$$I := 2\alpha \int_{G_0 \cup G_0^+} \frac{x_1}{|x|^{2+2\alpha}}dx - 2\alpha \int_{B_0 \cup B_0^+} \frac{x_1}{|x|^{2+2\alpha}}dx$$

and all 4 sets are contained in $\mathbb{R}^+ \times \mathbb{R}$. When $0 \leq a \leq c$, then direct integration gives

$$2\alpha \int_a^c \frac{x_1}{|(x_1, x_2)|^{2+2\alpha}}dx_1 = \frac{1}{(x_2^2 + a^2)^\alpha} - \frac{1}{(x_2^2 + c^2)^\alpha}, \tag{4.31}$$

and with f from (4.22) we again have

$$\int_r^s \frac{dx_2}{(x_2^2 + a^2)^\alpha} = \begin{cases} a^{1-2\alpha}\left(f\left(\frac{s}{a}\right) - f\left(\frac{r}{a}\right)\right) & a > 0, \\ \frac{1}{1-2\alpha}\left(s^{1-2\alpha} - r^{1-2\alpha}\right) & a = 0 \end{cases} \tag{4.32}$$

(we will again use these repeatedly below).

We now see that

$$2\alpha \int_{B_0} \frac{x_1}{|x|^{2+2\alpha}} dx = \frac{2^{1-2\alpha}}{1-2\alpha} - 2^{-\alpha} f(1) - 2^{-\alpha}\mu_\alpha$$

because this is the same integral as for the region G_b in the previous proof. Next,

$$2\alpha \int_{B_0^+} \frac{x_1}{|x|^{2+2\alpha}} dx = \int_1^2 \left(\frac{1}{(x_2^2+2^2)^\alpha} - \frac{2^{-\alpha}}{((x_2+1)^2+1)^\alpha} \right) dx_2 + \int_2^\infty \left(\frac{2^{-\alpha}}{x_2^{2\alpha}} - \frac{2^{-\alpha}}{((x_2+1)^2+1)^\alpha} \right) dx_2$$

$$= 2^{1-2\alpha} \left(f(1) - f\left(\frac{1}{2}\right) \right) + 2^{-\alpha} f(2) - \frac{2^{1-3\alpha}}{1-2\alpha} + 2^{-\alpha}\mu_\alpha$$

and

$$2\alpha \int_{G_0} \frac{x_1}{|x|^{2+2\alpha}} dx = 2 \int_0^1 \frac{dx_2}{(x_2^2+1)^\alpha} = 2f(1).$$

Finally, from $x_2^2 + (x_2+2)^2 = 2((x_2+1)^2+1)^2$ we obtain

$$2\alpha \int_{G_0^+} \frac{x_1}{|x|^{2+2\alpha}} dx = \int_0^1 \frac{dx_2}{(x_2^2+2^2)^\alpha} + \int_0^1 \frac{2^{-\alpha}}{((x_2+1)^2+1)^\alpha} dx_2 = 2^{1-2\alpha} f\left(\frac{1}{2}\right) + 2^{-\alpha}(f(2)-f(1)).$$

Hence

$$I = (2 - 2^{1-2\alpha})f(1) + 2^{2-2\alpha} f\left(\frac{1}{2}\right) - \frac{2^{1-2\alpha}(1-2^{-\alpha})}{1-2\alpha}.$$

Since

$$2^{2-2\alpha} f\left(\frac{1}{2}\right) > 2^{3/2} \frac{1}{2} \left(\frac{4}{5}\right)^{1/4} > 1 > 4(1 - 2^{-1/4}) > \frac{2^{1-2\alpha}(1-2^{-\alpha})}{1-2\alpha}$$

for $\alpha \in (0, \frac{1}{4}]$, (4.19) follows. $\qquad\square$

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