SUBADDITIVE THEOREMS IN TIME-DEPENDENT ENVIRONMENTS

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Abstract. We prove time-dependent versions of Kingman's subadditive ergodic theorem, which can be used to study stochastic processes as well as propagation of solutions to PDE in time-dependent environments.

1. Introduction and Main Results

During the last half-century, Kingman's subadditive ergodic theorem [8] and its versions (in particular, by Liggett [10]) have been a crucial tool in the study of evolution processes in stationary ergodic environments, including first passage percolation and related models as well as processes modeled by partial differential equations (PDE) which satisfy the maximum principle. Typically, the theorem is used to show that propagation of such a process in each spatial direction has almost surely some deterministic asymptotic speed. This can also often be extended to existence of a deterministic asymptotic propagation shape when the propagation involves invasion of one state of the process (e.g., the region not yet affected by it) by another (e.g., the already affected region).

Kingman's theorem concerns a family $\{X_{m,n}\}$ ($n > m \geq 0$) of random variables on a probability space which satisfy the crucial subadditivity hypothesis

$$X_{m,n} \leq X_{m,k} + X_{k,n} \qquad \text{for all } k \in \{m+1, \dots, n-1\}, \tag{1.1}$$

together with $\mathbb{E}[X_{0,n}] \in [-Cn, \infty)$ for some $C \geq 0$ and each $n \in \mathbb{N}$. Also, $\{X_{m,n}\}$ is stationary in the sense that the joint distribution of $\{X_{m+n,m+n+k} \,|\, (n,k) \in \mathbb{N}_0 \times \mathbb{N}\}$ is independent of $m \in \mathbb{N}_0$. It then concludes that $X := \lim_{n\to\infty} \frac{X_{0,n}}{n}$ exists almost surely, and

$$\mathbb{E}[X] = \lim_{n\to\infty} \frac{\mathbb{E}\left[X_{0,n}\right]}{n} = \inf_{n\geq 1} \frac{\mathbb{E}\left[X_{0,n}\right]}{n}.$$

Moreover, X is a constant if $\{X_{m,n}\}$ is also ergodic, that is, any event defined in terms of $\{X_{m,n}\}$ and invariant under the shift $(m,n) \mapsto (m+1, n+1)$ has probability either 0 or 1.

A typical use of such a result in the study of PDE is described in Example 5.1 below. We let $X_{m,n}$ be the time it takes for a solution to the PDE to propagate from $me \in \mathbb{R}^d$ to $ne \in \mathbb{R}^d$ (see the example for details), with e some fixed unit vector (i.e., direction). Subadditivity is then guaranteed by the maximum principle for the PDE, and Kingman's theorem may therefore often be used to conclude existence of a deterministic propagation speed in direction e, in an appropriate sense and under some basic hypotheses.

However, this approach only works when the coefficients of the PDE are either independent of time or time-periodic. The present work is therefore motivated by our desire to apply subadditivity-based techniques to PDE with more general time dependence of coefficients (and to other non-autonomous models), in particular, those with finite temporal ranges of dependence as well as with decreasing temporal correlations. Despite this being a very natural question, we were not able to find relevant results in the existing literature. We thus prove here the following two results, and also provide applications to a *time-dependent* first passage percolation model (see Examples 5.2 and 5.3 below). In the companion paper [13] we apply these results to specific PDE models (as described in Example 5.1), namely reaction-diffusion equations and G-equations.

Our first main result in the present paper applies when the process in question (or rather the environment in which it occurs) has a finite temporal range of dependence, with $\mathcal{F}_t^\pm$ being the sigma-algebras generated by the environment up to and starting from time t, respectively. It mirrors Kingman's theorem, with a weaker stationarity hypothesis (3) below (analogous to [10]) but under the additional hypothesis (6). The latter is the natural requirement that if the process propagates from some "location" m to another location n, starting at some time t, it cannot reach n later than the same process starting from m at some later time $t+s$, at least when s is sufficiently large. In the case of PDE, maximum principle will often guarantee this if the (non-negative) time-dependent propagation times $X_{m,n}^t$ (i.e., from location m to n, starting at time $t\in[0,\infty)$) are defined appropriately (see Example 5.1). We also note that (1) below is the natural version of (1.1) in the time-dependent setting.

Theorem 1.1. *Let $(\Omega,\mathbb{P},\mathcal{F})$ be a probability space, and $\{\mathcal{F}_t^\pm\}_{t\geq 0}$ two filtrations such that*

$$\mathcal{F}_s^- \subseteq \mathcal{F}_t^- \subseteq \mathcal{F} \qquad \text{and} \qquad \mathcal{F} \supseteq \mathcal{F}_s^+ \supseteq \mathcal{F}_t^+$$

for all $t\geq s\geq 0$. For any $t\geq 0$ and integers $n>m\geq 0$, let $X_{m,n}^t:\Omega\to[0,\infty)$ be a random variable. Let there be $C\geq 0$ such that the following statements hold for all such t,m,n.

(1) $X_{m,n}^t \leq X_{m,k}^t + X_{k,n}^{t+X_{m,k}^t}$ for all $k\in\{m+1,\dots,n-1\}$;

(2) $\mathbb{E}\left[X_{0,1}^0\right]<\infty$;

(3) the joint distribution of $\{X_{m,m+1}^t, X_{m,m+2}^t,\dots\}$ is independent of (t,m);

(4) $X_{m,n}^t$ is $\mathcal{F}_t^+$-measurable, and $\{\omega\in\Omega\,|\,X_{m,n}^t(\omega)\leq s\}\in\mathcal{F}_{t+s}^-$ for any $s\geq 0$;

(5) $\mathcal{F}_t^-$ and $\mathcal{F}_{t+C}^+$ are independent;

(6) $X_{m,n}^t\leq X_{m,n}^{t+s}+s$ for all $s\in[C,C+c]$, with some $c>0$.

Then

$$\lim_{n\to\infty}\frac{X_{0,n}^0}{n}=\lim_{n\to\infty}\frac{\mathbb{E}\left[X_{0,n}^0\right]}{n}=\inf_{n\geq 1}\frac{\mathbb{E}\left[X_{0,n}^0\right]+C}{n} \qquad \textit{almost surely.} \tag{1.2}$$

Moreover, if $C\in\mathbb{N}$ and $X_{m,n}^t$ are all integer-valued, then it suffices to have $c=0$ in (6).

Remarks. 1. Of course, it suffices to assume (1) and (6) only almost surely.

2. There would be little benefit in using different C in (5) and (6) because (5) clearly holds with any larger C, while iterating (6) yields (6) for all $s\in[kC,kC+kc]$ and any $k\in\mathbb{N}$.

3. The ergodicity hypothesis in [8] is here replaced by (5) (or by (5*) below).

4. Property $\mathcal{F}_s^+\supseteq\mathcal{F}_t^+$ for $t\geq s$ makes $\{\mathcal{F}_t^+\}_{t\geq 0}$ technically a *backward* filtration.

Our second main result allows for an infinite temporal range of dependence of the environment, provided this dependence decreases with time in an appropriate sense, and we then also need a uniform bound in place of (2).

Theorem 1.2. *Assume the hypotheses of Theorem 1.1, but with (2) and (5) replaced by*

(2)* $X^0_{0,1} \leq C$*;*

(5)* $\lim_{s\to\infty} \phi(s) = 0$*, where*

$$\phi(s) := \sup\left\{ |\mathbb{P}[F|E] - \mathbb{P}[F]| \,\middle|\, t \geq 0 \;\&\; (E,F) \in \mathcal{F}^-_t \times \mathcal{F}^+_{t+s} \;\&\; \mathbb{P}[E] > 0 \right\}.$$

Then

$$\lim_{n\to\infty} \frac{X^0_{0,n}}{n} = \lim_{n\to\infty} \frac{\mathbb{E}\left[X^0_{0,n}\right]}{n} \qquad \textit{in probability,} \tag{1.3}$$

and if there is $\alpha > 0$ *such that* $\lim_{s\to\infty} s^\alpha \phi(s) = 0$*, then also*

$$\lim_{n\to\infty} \frac{X^0_{0,n}}{n} = \lim_{n\to\infty} \frac{\mathbb{E}\left[X^0_{0,n}\right]}{n} \qquad \textit{almost surely.} \tag{1.4}$$

Moreover, if $C \in \mathbb{N}$ *and* $X^t_{m,n}$ *are all integer-valued, then it suffices to have* $c = 0$ *in (6).*

Remarks. 1. Again, using different C in (2*) and (6) would not strengthen the result.

2. We will actually prove this result with $\phi(s)$ being instead the supremum of

$$\sum_{i\geq 0} |\mathbb{P}[F_i \cap E_i] - \mathbb{P}[F_i]\mathbb{P}[E_i]|$$

over all $\{(E_i, F_i) \in \mathcal{F}^-_{t_i} \times \mathcal{F}^+_{t_i+s}\}_{i\geq 0}$ with $t_0, t_1, \dots \geq 0$ and $E_0, E_1, \dots$ pairwise disjoint (which is clearly no more than $\phi(s)$ from (5*)).

3. We will also show that without assuming $\lim_{s\to\infty} s^\alpha \phi(s) = 0$, we still have

$$\liminf_{n\to\infty} \frac{X^0_{0,n}}{n} \geq \lim_{n\to\infty} \frac{\mathbb{E}\left[X^0_{0,n}\right]}{n} \qquad \text{almost surely.} \tag{1.5}$$

Organization of the Paper and Acknowledgements. We prove Theorem 1.1 in Section 2 and the claims in Theorem 1.2 in Sections 3 and 4. Section 5 contains applications of our results to two models, PDE and first passage percolation in time-dependent environments.

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2. Finite Temporal Range of Dependence

Let us first prove a version of Theorem 1.1 with $\mathbb{N}_0$-valued random variables and $C = 0$ in (5). Theorem 1.1 will then easily follow. Let us denote $\{X = s\} := \{\omega \in \Omega \,|\, X(\omega) = s\}$.

Theorem 2.1. *Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a probability space, and $\{\mathcal{F}_t^{\pm}\}_{t\in\mathbb{N}_0}$ two filtrations such that*

$$\mathcal{F}_0^- \subseteq \mathcal{F}_1^- \subseteq \cdots \subseteq \mathcal{F} \qquad \text{and} \qquad \mathcal{F} \supseteq \mathcal{F}_0^+ \supseteq \mathcal{F}_1^+ \supseteq \dots . \tag{2.1}$$

For any integers $t \geq 0$ and $n > m \geq 0$, let $T_{m,n}^t : \Omega \to \mathbb{N}_0$ be a random variable. Let there be $C, C' \in \mathbb{N}$ such that the following statements hold for all such t, m, n.

- *(1') $T_{m,n}^t \leq T_{m,k}^t + T_{k,n}^{t+T_{m,k}^t}$ for all $k \in \{m+1, \dots, n-1\}$;*
- *(2') $\mathbb{E}\left[T_{0,1}^0\right] \leq C'$;*
- *(3') the joint distribution of $\{T_{m,m+1}^t, T_{m,m+2}^t, \dots\}$ is independent of (t, m);*
- *(4') $T_{m,n}^t$ is $\mathcal{F}_t^+$-measurable, and $\{T_{m,n}^t = j\} \in \mathcal{F}_{t+j}^-$ for any $j \in \mathbb{N}_0$;*
- *(5') $\mathcal{F}_t^-$ and $\mathcal{F}_t^+$ are independent;*
- *(6') $T_{m,n}^t \leq T_{m,n}^{t+C} + C$.*

Then

$$\lim_{n\to\infty} \frac{T_{0,n}^0}{n} = \lim_{n\to\infty} \frac{\mathbb{E}\left[T_{0,n}^0\right]}{n} = \inf_{n\geq 1} \frac{\mathbb{E}\left[T_{0,n}^0\right]}{n} \qquad \text{almost surely.} \tag{2.2}$$

Proof. First, we claim that almost surely we have

$$\limsup_{n\to\infty} \frac{T_{0,n}^0}{n} \leq \lim_{n\to\infty} \frac{\mathbb{E}\left[T_{0,n}^0\right]}{n} = \inf_{n\geq 1} \frac{\mathbb{E}\left[T_{0,n}^0\right]}{n}. \tag{2.3}$$

The proof of (2.3) is similar to the proof of [2, Lemma 6.7], although there the analogs of $T_{m,n}^t$ were bounded random variables; the idea goes back to [8], where the analogs of $T_{m,n}^t$ were t-independent. For any integers $n > m > 0$, (4') shows that for any $i, j \in \mathbb{N}_0$ we have

$$\{T_{0,m}^0 = i\} \in \mathcal{F}_i^- \qquad \text{and} \qquad \{T_{m,n}^i = j\} \in \mathcal{F}_i^+.$$

Therefore (5') and (3') yield

$$\mathbb{P}\left[T_{0,m}^0 = i \;\&\; T_{m,n}^i = j\right] = \mathbb{P}\left[T_{0,m}^0 = i\right] \mathbb{P}\left[T_{m,n}^i = j\right] = \mathbb{P}\left[T_{0,m}^0 = i\right] \mathbb{P}\left[T_{0,n-m}^0 = j\right].$$

Summing this over $i \in \mathbb{N}_0$, we find that $T_{m,n}^{T_{0,m}^0}$ $(= T_{m,n}^{T_{0,m}^0(\cdot)}(\cdot))$ has the same distribution as $T_{0,n-m}^0$. Thus from (1') we obtain

$$\mathbb{E}\left[T_{0,n}^0\right] \leq \mathbb{E}\left[T_{0,m}^0\right] + \mathbb{E}\left[T_{m,n}^{T_{0,m}^0}\right] \leq \mathbb{E}\left[T_{0,m}^0\right] + \mathbb{E}\left[T_{0,n-m}^0\right].$$

Fekete's subadditive lemma thus implies that the equality in (2.3) holds.

For any $n \in \mathbb{N}$, let $t_0^n := 0$ and $\xi_0^n := T_{0,n}^0$, and then for $i \in \mathbb{N}$ define recursively

$$t_i^n := t_{i-1}^n + \xi_{i-1}^n \qquad \text{and} \qquad \xi_i^n := T_{in,(i+1)n}^{t_i^n}.$$

By iteratively applying (1'), we get for any $k \in \mathbb{N}$,

$$T_{0,kn}^0 \leq \sum_{i=0}^{k-1} \xi_i^n. \tag{2.4}$$

Similarly as above, it follows from (3')–(5') that for any $j_0, j_1, \dots, j_{k-1} \in \mathbb{N}_0$ we have

$$\begin{aligned}\mathbb{P}\left[\xi_i^n = j_i \text{ for } i = 0,\dots,k-1\right] &= \mathbb{P}\left[\xi_i^n = j_i \text{ for } i = 0,\dots,k-2 \ \& \ T_{(k-1)n,kn}^{\sum_{i=0}^{k-2} j_i} = j_{k-1}\right] \\ &= \mathbb{P}\left[\xi_i^n = j_i \text{ for } i = 0,\dots,k-2\right] \mathbb{P}\left[T_{(k-1)n,kn}^{\sum_{i=0}^{k-2} j_i} = j_{k-1}\right] \\ &= \mathbb{P}\left[\xi_i^n = j_i \text{ for } i = 0,\dots,k-2\right] \mathbb{P}\left[T_{0,n}^0 = j_{k-1}\right] \\ &= \dots = \prod_{i=0}^{k-1} \mathbb{P}\left[T_{0,n}^0 = j_i\right].\end{aligned}$$

Summing this over all indices but i shows that ξ_i^n has the same law as $T_{0,n}^0$ for each i. This, (2'), and (2.4) with $n = 1$ then show that for any $k \in \mathbb{N}$,

$$\mathbb{E}\left[T_{0,k}^0\right] \leq \sum_{i=0}^{k-1} \mathbb{E}\left[\xi_i^1\right] = k\mathbb{E}\left[T_{0,1}^0\right] \leq C'k. \tag{2.5}$$

Also, the above computation shows that $\xi_0^n, \dots, \xi_{k-1}^n$ are jointly independent random variables for all n and k, so the strong law of large numbers yields

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \xi_i^n = \mathbb{E}\left[T_{0,n}^0\right] \qquad \text{almost surely.}$$

Thus (2.4) and the equality in (2.3) yield that for any $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that

$$\limsup_{k\to\infty} \frac{T_{0,kn_\varepsilon}^0}{kn_\varepsilon} \leq \frac{\mathbb{E}\left[T_{0,n_\varepsilon}^0\right]}{n_\varepsilon} \leq (1+\varepsilon) \lim_{n\to\infty} \frac{\mathbb{E}\left[T_{0,n}^0\right]}{n} \qquad \text{almost surely.} \tag{2.6}$$

Now fix any $l \in \{0, \dots, n_\varepsilon - 1\}$ and note that (1') yields for all $k \in \mathbb{N}_0$,

$$T_{0,kn_\varepsilon+l}^0 \leq T_{0,kn_\varepsilon}^0 + T_{kn_\varepsilon,kn_\varepsilon+l}^{T_{0,kn_\varepsilon}^0}. \tag{2.7}$$

Since $T_{kn_\varepsilon,kn_\varepsilon+l}^{T_{0,kn_\varepsilon}^0}$ has the same distribution as $T_{0,l}^0$, we obtain from (2.5) that

$$\sum_{k\geq 0} \mathbb{P}\left[T_{kn_\varepsilon,kn_\varepsilon+l}^{T_{0,kn_\varepsilon}^0} > (kn_\varepsilon + l)\varepsilon\right] \leq \sum_{k\geq 0} \mathbb{P}\left[T_{0,l}^0 > k\varepsilon\right] \leq \frac{1}{\varepsilon}\mathbb{E}\left[T_{0,l}^0\right] < \infty.$$

Borel-Cantelli Lemma then implies that $\limsup_{k\to\infty} \frac{1}{kn_\varepsilon+l} T_{kn_\varepsilon,kn_\varepsilon+l}^{T_{0,kn_\varepsilon}^0} \leq \varepsilon$ almost surely. This and (2.7) for each $l \in \{0, \dots, n_\varepsilon - 1\}$, together with (2.6), now show that

$$\limsup_{n\to\infty} \frac{T_{0,n}^0}{n} \leq \varepsilon + (1+\varepsilon) \lim_{n\to\infty} \frac{\mathbb{E}\left[T_{0,n}^0\right]}{n} \qquad \text{almost surely.}$$

Taking $\varepsilon \to 0$ now yields the inequality in (2.3).

Next, for each $(t, m) \in \mathbb{N}_0^2$ let

$$Z_m^t := \liminf_{n\to\infty} \frac{T_{m,m+n}^t}{n}.$$

It follows from (6') that Z_m^{t+Ck} is non-decreasing in $k \in \mathbb{N}$. But since the law of Z_m^t is independent of (t,m) by (3'), we must almost surely have $Z_m^{t+Ck} = Z_m^t$ for all $k \in \mathbb{N}$. However, this and (3') imply that Z_m^t is independent of $\mathcal{F}_{t+Ck}^-$ for all $k \in \mathbb{N}$, while (4') shows that it is also measurable with respect to the σ-algebra generated by $\bigcup_{s\ge t} \mathcal{F}_s^-$. This shows that there is a constant $Q \in [0,\infty)$ such that $Z_m^t = Q$ almost surely for each $(t,m) \in \mathbb{N}_0^2$.

In view of (2.3), to prove (2.2) it remains to show that

$$Q \ge \lim_{n\to\infty} \frac{\mathbb{E}\left[T_{0,n}^0\right]}{n}. \tag{2.8}$$

Our proof of this is related to the approach of Levental [9] in the t-independent case, which is in turn based on [6]. However, t-dependence complicates the situation here, which is why we first needed to show that Z_m^t is in fact (t,m,ω)-independent to conclude (2.8) (in [9], it was sufficient to allow ω-dependence at first). Fix any $\varepsilon > 0$, and denote $Q_\varepsilon := Q + \varepsilon$ and

$$N_m^t := \min\left\{n \ge 1 \,\middle|\, T_{m,m+n}^t \le n\, Q_\varepsilon\right\}$$

(which also depends on ε but we suppress this in the notation). It follows from $Z_m^t = Q$ a.e. that almost surely we have $N_m^t < \infty$ for all $(t,m) \in \mathbb{N}_0^2$, and (3') yields that N_m^t has the same distribution as N_0^0. Moreover, N_m^t is $\mathcal{F}_t^+$-measurable by (4'). Next, let $M_\varepsilon \in [1,\infty)$ be a large constant such that

$$\mathbb{E}\left[T_{0,1}^0 1_{\left\{N_0^0 > M_\varepsilon\right\}}\right] \le \varepsilon. \tag{2.9}$$

Let now $t_0 := 0$ and $r_0 := 0$, and for $k \ge 0$ define recursively

$$r_{k+1} := r_k + N_{r_k}^{t_k} 1_{\left\{N_{r_k}^{t_k} \le M_\varepsilon\right\}} + 1_{\left\{N_{r_k}^{t_k} > M_\varepsilon\right\}} \qquad \text{and} \qquad t_{k+1} := t_k + T_{r_k,r_{k+1}}^{t_k}.$$

Fix any $n \in \mathbb{N}$. We will now use $\{r_k\}_{k\ge1}$ to divide the "propagation" from 0 to n into several "steps". Since this sequence is strictly increasing for each $\omega \in \Omega$, the random variable

$$K_n := \min\{k \in \mathbb{N}_0 \,|\, r_k \ge n - M_\varepsilon\}$$

is well defined, and satisfies $0 \le K_n \le n-1$ and $r_{K_n} \in [n - M_\varepsilon, n-1]$. Applying (1') iteratively K_n times yields

$$T_{0,n}^0 \le \sum_{k=0}^{K_n-1} T_{r_k,r_{k+1}}^{t_k} + T_{r_{K_n},n}^{t_{K_n}} =: S_n + T_{r_{K_n},n}^{t_{K_n}} \tag{2.10}$$

(note that, e.g., $T_{r_{K_n},n}^{t_{K_n}} = T_{r_{K_n(\cdot)}(\cdot),n}^{t_{K_n(\cdot)}(\cdot)}(\cdot)$). If $N_{r_k}^{t_k} \le M_\varepsilon$, then

$$T_{r_k,r_{k+1}}^{t_k} \le (r_{k+1} - r_k)\, Q_\varepsilon,$$

while if $N_{r_k}^{t_k} > M_\varepsilon$, then $r_{k+1} = r_k + 1$. Hence we obtain

$$S_n \le \sum_{k=0}^{K_n-1} (r_{k+1} - r_k)\, Q_\varepsilon + \sum_{k=0}^{K_n-1} T_{r_k,r_k+1}^{t_k} 1_{\left\{N_{r_k}^{t_k} > M_\varepsilon\right\}} \le r_{K_n} Q_\varepsilon + \sum_{k=0}^{n-1} T_{r_k,r_k+1}^{t_k} 1_{\left\{N_{r_k}^{t_k} > M_\varepsilon\right\}}. \tag{2.11}$$

We now want to take expectation on both sides of (2.11). From (4') we see that for any $i, j \in \mathbb{N}_0$ we have $\{r_k = i \ \&\ t_k = j\} \in \mathcal{F}_j^-$. Since $T^j_{i,i+1}$ and N^j_i are $\mathcal{F}_j^+$-measurable, from (5'), (3'), and (2.9) we obtain

$$\begin{aligned}\mathbb{E}\left[T^{t_k}_{r_k,r_k+1}1_{\left\{N^{t_k}_{r_k}>M_\varepsilon\right\}}\right] &= \sum_{i,j\geq 0}\mathbb{E}\left[T^j_{i,i+1}1_{\left\{N^j_i>M_\varepsilon\right\}}1_{\{r_k=i\ \&\ t_k=j\}}\right]\\ &= \sum_{i,j\geq 0}\mathbb{E}\left[T^j_{i,i+1}1_{\left\{N^j_i>M_\varepsilon\right\}}\right]\mathbb{P}\left[r_k=i\ \&\ t_k=j\right]\\ &= \mathbb{E}\left[T^0_{0,1}1_{\left\{N^0_0>M_\varepsilon\right\}}\right]\leq\varepsilon.\end{aligned}$$

So (2.11) and $r_{K_n} \leq n$ yield

$$\frac{\mathbb{E}\left[S_n\right]}{n} \leq \frac{\mathbb{E}[r_{K_n}]Q_\varepsilon}{n} + \varepsilon \leq Q + 2\varepsilon. \tag{2.12}$$

Finally, we claim that $\mathbb{E}\left[T^{t_{K_n}}_{r_{K_n},n}\right] \leq C'M_\varepsilon^2$; this together with (2.10) and (2.12), and then taking $\varepsilon \to 0$, will yield (2.8). To this end we note that $1 \leq n - r_{K_n} \leq M_\varepsilon$ implies

$$T^{t_{K_n}}_{r_{K_n},n} \leq \max_{l\in\{1,\dots,\min\{M_\varepsilon,n\}\}} T^{t_{K_n}}_{n-l,n}. \tag{2.13}$$

Since $\{t_{K_n} = j\} \in \mathcal{F}_j^-$ and $T^j_{n-l,n}$ is $\mathcal{F}_j^+$-measurable, we obtain from (5'), (3'), and (2.5),

$$\mathbb{E}\left[T^{t_{K_n}}_{n-l,n}\right] = \sum_{j\geq 0}\mathbb{E}\left[T^j_{n-l,n}1_{\{t_{K_n}=j\}}\right] = \sum_{j\geq 0}\mathbb{E}\left[T^j_{n-l,n}\right]\mathbb{P}\left[t_{K_n}=j\right] = \mathbb{E}\left[T^0_{0,l}\right] \leq C'l.$$

Therefore indeed

$$\mathbb{E}\left[T^{t_{K_n}}_{r_{K_n},n}\right] \leq \sum_{l=1}^{M_\varepsilon} C'l \leq C'M_\varepsilon^2,$$

so (2.8) holds and the proof is finished. □

Proof of Theorem 1.1. Let us first assume that $c \geq 1$ and define

$$T^t_{m,n} := \lceil X^t_{m,n} + C\rceil \qquad (\in \mathbb{N}_0).$$

Let us redefine $\mathcal{F}_t^-$ to be $\mathcal{F}_{t-C}^-$ for $t \geq C$ and $\{\emptyset, \Omega\}$ for $t \in [0, C)$ (i.e., shift $\mathcal{F}_t^-$ to the right by C) and let $C' := \mathbb{E}\left[\lceil X^0_{0,1} + C\rceil\right]$. After restricting t to $\mathbb{N}_0$, it is clear that $T^t_{m,n}$ satisfies hypotheses (2')–(6') of Theorem 2.1, with $\max\{\lceil C\rceil, 1\}$ in place of C. And (1') also holds because if $n > k > m \geq 0$ are integers, then (1) and (6) with $s := \lceil X^t_{m,n} + C\rceil - X^t_{m,n}$ yield

$$T^t_{m,n} = \lceil X^t_{m,n} + C\rceil \leq \left\lceil X^{t+X^t_{m,k}}_{k,n} + X^t_{m,k} + C\right\rceil \leq \left\lceil X^{t+T^t_{m,k}}_{k,n} + T^t_{m,k} + C\right\rceil \leq T^{t+T^t_{m,k}}_{k,n} + T^t_{m,k}.$$

Hence (2.2) proves (1.2) with the last numerator being $\mathbb{E}\left[\lceil X^0_{0,n} + C\rceil\right]$. Note that this argument also applies in the setting of the last claim in Theorem 1.1 and without $\lceil\cdot\rceil$.

To get (1.2) as stated and for any $c > 0$, let $S \geq \frac{1}{c}$, $\mathcal{G}_t^\pm := \mathcal{F}^\pm_{t/S}$, and $Y^t_{m,n} := SX^{t/S}_{m,n}$. Since the above argument applies with $(\mathcal{G}_t^\pm, Y^t_{m,n}, SC, Sc)$ in place of $(\mathcal{F}_t^\pm, X^t_{m,n}, C, c)$, we obtain (1.2) with the last numerator being $\mathbb{E}\left[\frac{1}{S}\lceil S(X^0_{0,n} + C)\rceil\right]$. Taking $S \to \infty$ yields (1.2). □

3. Time-Decaying Dependence I

In this section we will prove the first claim in Theorem 1.2 and the corresponding integer-valued claim. Let us first prove a version of the latter with weaker (2*) and stronger (5*).

Theorem 3.1. *Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a probability space, and $\{\mathcal{F}_t^\pm\}_{t\in\mathbb{N}_0}$ two filtrations satisfying (2.1). For any integers $t \ge 0$ and $n > m \ge 0$, let $X_{m,n}^t : \Omega \to \mathbb{N}_0$ be a random variable. Let there be $C \in \mathbb{N}$ such that for all such t, m, n we have (1) and (3) from Theorem 1.1, and*

*(2**)* $\mathbb{E}\left[X_{0,1}^0\right] + \mathbb{E}\left[(X_{0,1}^0)^2\right] < \infty$;

*(4**)* $X_{m,n}^t$ *is* $\mathcal{F}_t^+$*-measurable, and* $\{X_{m,n}^t = j\} \in \mathcal{F}_{t+j}^-$ *for any* $j \in \mathbb{N}_0$;

*(5**)* $\lim_{s\to\infty} \phi(s) = 0$, *where*

$$\phi(s) := \sup\left\{ \left| \frac{\mathbb{P}[F|E]}{\mathbb{P}[F]} - 1 \right| \;\middle|\; t \in \mathbb{N}_0 \;\&\; (E,F) \in \mathcal{F}_t^- \times \mathcal{F}_{t+s}^+ \;\&\; \mathbb{P}[E]\mathbb{P}[F] > 0 \right\}.$$

*(6**)* $X_{m,n}^t \le X_{m,n}^{t+C} + C$.

Then (1.3) holds.

Proof. From (5**) we know that for each $\varepsilon > 0$, there is $C_\varepsilon \in \mathbb{N}$ which is a multiple of C from (6**) and

$$\phi(C_\varepsilon) \le \varepsilon. \tag{3.1}$$

Let us then define (again suppressing ε in the notation for the sake of clarity)

$$T_{m,n}^t := X_{m,n}^t + C_\varepsilon.$$

As before, one can easily check that (1), (2**), (3), (5**), and (6**) still hold with $X_{m,n}^t$ replaced by $T_{m,n}^t$, and (4**) can be replaced by

(4") $T_{m,n}^t$ is $\mathcal{F}_t^+$-measurable, and $\{T_{m,n}^t = j\} \in \mathcal{F}_{t+j-C_\varepsilon}^-$ for any $j \in \mathbb{N}_0$.

Next, let

$$\underline{X} := \liminf_{n\to\infty} \frac{\mathbb{E}\left[X_{0,n}^0\right]}{n}. \tag{3.2}$$

As before, for any $\varepsilon > 0$ and $n \in \mathbb{N}$, let $t_0^n := 0$ and $\xi_0^n := T_{0,n}^0$, and then for $i \in \mathbb{N}$ define recursively

$$t_i^n := t_{i-1}^n + \xi_{i-1}^n, \qquad \xi_i^n := T_{in,(i+1)n}^{t_i^n}, \qquad \text{and} \qquad \mu_i^n := \mathbb{E}\left[\xi_i^n\right].$$

By (1) we have $T_{0,kn}^0 \le \sum_{i=0}^{k-1} \xi_i^n$ for each $k \in \mathbb{N}$. Also, since (4") yields

$$\{t_i^n = k\} \in \mathcal{F}_{k-C_\varepsilon}^- \qquad \text{and} \qquad \{T_{in,(i+1)n}^k = j\} \in \mathcal{F}_k^+, \tag{3.3}$$

it follows from (3) and (3.1) that

$$\begin{aligned} \mu_i^n &= \sum_{k,j\ge 0} j\,\mathbb{P}\left[T_{in,(i+1)n}^k = j \;\&\; t_i^n = k\right] \le (1+\varepsilon) \sum_{k,j\ge 0} j\,\mathbb{P}\left[T_{in,(i+1)n}^k = j\right] \mathbb{P}\left[t_i^n = k\right] \\ &= (1+\varepsilon) \sum_{j\ge 0} j\,\mathbb{P}\left[T_{0,n}^0 = j\right] = (1+\varepsilon)\mathbb{E}\left[T_{0,n}^0\right] = (1+\varepsilon)\mu_0^n. \end{aligned} \tag{3.4}$$

We can similarly obtain

$$\mathbb{E}\left[\xi_i^n - C_\varepsilon\right] \leq (1+\varepsilon)(\mu_0^n - C_\varepsilon) \tag{3.5}$$

and

$$\mathbb{E}\left[\xi_i^n - C_\varepsilon\right] \geq (1-\varepsilon)(\mu_0^n - C_\varepsilon) \tag{3.6}$$

because $\xi_i^n \geq C_\varepsilon$. Invoking (1) and (3.4) with $n = 1$ yields for $C'_\varepsilon := \mathbb{E}[X_{0,1}^0] + C_\varepsilon$,

$$\mu_0^n \leq \sum_{i=0}^{n-1} \mu_i^1 \leq (1+\varepsilon)n\mu_0^1 \leq (1+\varepsilon)C'_\varepsilon n. \tag{3.7}$$

This implies that $\underline{X} \leq (1+\varepsilon)C'_\varepsilon$.

Let us now pick $n_\varepsilon \in \mathbb{N}$ such that

$$\frac{\mathbb{E}\left[T_{0,n_\varepsilon}^0\right]}{n_\varepsilon} \leq \underline{X} + \varepsilon, \tag{3.8}$$

which exists by (3.2). Then (4”) shows that for any integers $n > m > 0$ and $i, j \in \mathbb{N}_0$ we get

$$\{T_{0,m}^0 = i\} \in \mathcal{F}_{i-C_\varepsilon}^- \qquad \text{and} \qquad \{T_{m,n}^i = j\} \in \mathcal{F}_i^+.$$

Therefore (3) and (3.1) yield

$$\begin{aligned}
\mathbb{E}\left[T_{m,n}^{T_{0,m}^0}\right] &= \sum_{i,j\geq 0} j\,\mathbb{P}\left[T_{m,n}^i = j \;\&\; T_{0,m}^0 = i\right] \leq (1+\varepsilon)\sum_{i,j\geq 0} j\,\mathbb{P}\left[T_{m,n}^i = j\right]\mathbb{P}\left[T_{0,m}^0 = i\right] \\
&= (1+\varepsilon)\sum_{j\geq 0} j\,\mathbb{P}\left[T_{0,n-m}^0 = j\right] = (1+\varepsilon)\mathbb{E}\left[T_{0,n-m}^0\right].
\end{aligned} \tag{3.9}$$

For any $n \in \mathbb{N}$ write $n = kn_\varepsilon + l$, where $k \in \mathbb{N}_0$ and $l \in \{0, \dots, n_\varepsilon - 1\}$. By applying (1) and the above computations recursively, we obtain

$$\begin{aligned}
\mathbb{E}\left[T_{0,n}^0\right] &\leq \mathbb{E}\left[T_{0,(k-1)n_\varepsilon+l}^0\right] + \mathbb{E}\left[T_{(k-1)n_\varepsilon+l,n}^{T_{0,(k-1)n_\varepsilon+l}^0}\right] \leq \mathbb{E}\left[T_{0,(k-1)n_\varepsilon+l}^0\right] + (1+\varepsilon)\mathbb{E}\left[T_{0,n_\varepsilon}^0\right] \\
&\leq \cdots \leq \mathbb{E}\left[T_{0,l}^0\right] + (1+\varepsilon)k\,\mathbb{E}\left[T_{0,n_\varepsilon}^0\right].
\end{aligned}$$

Thus (3.8) yields

$$\frac{\mathbb{E}\left[T_{0,n}^0\right]}{n} \leq \frac{\mathbb{E}\left[T_{0,l}^0\right]}{n} + \frac{(1+\varepsilon)kn_\varepsilon(\underline{X}+\varepsilon)}{n}$$

which then implies

$$\limsup_{n\to\infty} \frac{\mathbb{E}\left[X_{0,n}^0\right]}{n} = \limsup_{n\to\infty} \frac{\mathbb{E}\left[T_{0,n}^0\right]}{n} \leq (1+\varepsilon)(\underline{X}+\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, this and (3.2) show that

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[X_{0,n}^0\right]}{n} = \underline{X}. \tag{3.10}$$

Next we claim that there is $C_* > 0$ such that for any $\varepsilon \in (0,1]$, $n \in \mathbb{N}$, and $i \neq j$ we have

$$\operatorname{Var}\left[\xi_i^n\right] \leq C_* n^2 \qquad \text{and} \qquad \operatorname{Cov}\left[\xi_i^n, \xi_j^n\right] \leq C_*\varepsilon n^2. \tag{3.11}$$

We postpone the proof of (3.11) to the end of the proof of (i). Since $t_k^n = \sum_{i=0}^{k-1} \xi_i^n$, we now have

$$\mathrm{Var}\left[t_k^n\right] = \sum_{i=0}^{k-1} \mathrm{Var}\left[\xi_i^n\right] + 2\sum_{i=0}^{k-1}\sum_{j=i+1}^{k-1} \mathrm{Cov}\left[\xi_i^n, \xi_j^n\right] \le (1+\varepsilon k)C_* n^2 k.$$

Chebyshev's inequality then yields

$$\mathbb{P}\left[\left|\frac{t_k^n - \mathbb{E}[t_k^n]}{k}\right| > C_\varepsilon\right] \le \frac{\mathrm{Var}\left[t_k^n\right]}{C_\varepsilon^2 k^2} \le \frac{1+\varepsilon k}{C_\varepsilon^2 k} C_* n^2.$$

Since $\mathbb{E}[t_k^n] = \sum_{i=0}^{k-1} \mu_i^n$, this and (3.4) imply

$$\mathbb{P}\left[\frac{t_k^n}{k} > (1+\varepsilon)\mu_0^n + C_\varepsilon\right] \le \frac{1+\varepsilon k}{C_\varepsilon^2 k} C_* n^2.$$

For any $N \in \mathbb{N}$ write $N = kn + l$, where $k \in \mathbb{N}_0$ and $l \in \{0, \cdots, n-1\}$. Then (1) yields

$$X_{0,N}^0 \le T_{0,N}^0 \le T_{0,kn}^0 + T_{kn,kn+l}^{T_{0,kn}^0} \le t_k^n + T_{kn,kn+l}^{T_{0,kn}^0}.$$

Denoting $\tau_N^n := T_{kn,kn+l}^{T_{0,kn}^0}$, we get $\mathbb{E}[\tau_N^n] \le (1+\varepsilon)^2 C_\varepsilon' n$ by (3.9) and (3.7), as well as

$$\mathbb{P}\left[\frac{X_{0,N}^0}{N} > (1+\varepsilon)\frac{\mu_0^n}{n} + \frac{C_\varepsilon}{n} + \frac{\tau_N^n}{kn}\right] \le \frac{1+\varepsilon k}{C_\varepsilon^2 k} C_* n^2. \tag{3.12}$$

Now, for each $\varepsilon > 0$ pick $n_\varepsilon \in \mathbb{N}$ such that

$$\lim_{\varepsilon \to 0} \frac{C_\varepsilon}{n_\varepsilon} = 0 = \lim_{\varepsilon \to 0} \frac{\varepsilon n_\varepsilon^2}{C_\varepsilon^2}. \tag{3.13}$$

If we then take $n = n_\varepsilon$ in (3.12) and then $N \to \infty$ (so that $k \to \infty$), for each $\delta > 0$ we obtain

$$\limsup_{N\to\infty} \mathbb{P}\left[\frac{X_{0,N}^0}{N} > (1+\varepsilon)\frac{\mu_0^{n_\varepsilon}}{n_\varepsilon} + \frac{C_\varepsilon}{n_\varepsilon} + \delta\right] \le \frac{C_* \varepsilon n_\varepsilon^2}{C_\varepsilon^2}.$$

Since $\mu_0^{n_\varepsilon} = \mathbb{E}[X_{0,n_\varepsilon}^0] + C_\varepsilon$ and $\lim_{\varepsilon\to 0} n_\varepsilon = \infty$ by (3.13), taking $\varepsilon \to 0$ in this estimate and using (3.10) and (3.13) shows that $\lim_{N\to\infty} \mathbb{P}\left[\frac{X_{0,N}^0}{N} > \underline{X} + 2\delta\right] = 0$ for all $\delta > 0$. That is,

$$\limsup_{N\to\infty} \frac{X_{0,N}^0}{N} \le \underline{X} \qquad \text{in probability.} \tag{3.14}$$

Let us now assume that there is $\delta > 0$ and a sequence $n_k \to \infty$ such that

$$\mathbb{P}\left[\frac{X_{0,n_k}^0}{n_k} - \underline{X} < -2\delta\right] \ge 4\delta. \tag{3.15}$$

Since for all large enough k we have $\frac{\mathbb{E}[X_{0,n_k}^0]}{n_k} \ge \underline{X} - \delta$ by (3.10), we obtain

$$\mathbb{P}\left[\frac{X_{0,n}^0}{n} - \underline{X} < -2\delta\right] \le \mathbb{P}\left[\frac{X_{0,n_k}^0 - \mathbb{E}\left[X_{0,n_k}^0\right]}{n_k} < -\delta\right] \le \frac{1}{\delta}\,\mathbb{E}\left[\left(\frac{X_{0,n_k}^0 - \mathbb{E}\left[X_{0,n_k}^0\right]}{n_k}\right)_+\right]$$

$$\leq \frac{1}{\delta}\left(\delta^2+\mathbb{P}\left[\frac{X^0_{0,n_k}-\mathbb{E}\left[X^0_{0,n_k}\right]}{n_k}>\delta^2\right]+\sum_{i\geq 1}\mathbb{P}\left[\frac{X^0_{0,n_k}-\mathbb{E}\left[X^0_{0,n_k}\right]}{n_k}>i\right]\right).$$

Since $\mathrm{Var}[X^0_{0,n_k}]\leq C_* n_k^2$ by (3.11) with $i=0$, for $M:=\lceil\frac{C_*}{\delta^2}\rceil+1$ we obtain

$$\sum_{i\geq M}\mathbb{P}\left[\frac{X^0_{0,n_k}-\mathbb{E}\left[X^0_{0,n_k}\right]}{n_k}>i\right]\leq\sum_{i\geq M}\frac{1}{n_k^2 i^2}\mathrm{Var}\left[X^0_{0,n_k}\right]\leq\delta^2.$$

But (3.10) and (3.14) also show that for all large enough k we have

$$\mathbb{P}\left[\frac{X^0_{0,n_k}-\mathbb{E}\left[X^0_{0,n_k}\right]}{n_k}>\delta^2\right]+\sum_{i=1}^{M-1}\mathbb{P}\left[\frac{X^0_{0,n_k}-\mathbb{E}\left[X^0_{0,n_k}\right]}{n_k}>i\right]\leq\delta^2.$$

Hence for all large enough k we obtain

$$\mathbb{P}\left[\frac{X^0_{0,n_k}}{n_k}-\underline{X}<-2\delta\right]\leq 3\delta,$$

which contradicts (3.15). It follows that $\lim_{n\to\infty}\mathbb{P}\left[\frac{X^0_{0,n}}{n}-\underline{X}<-\delta\right]=0$ for each $\delta>0$, so this and (3.14) yield (1.3).

It therefore remains to prove (3.11). Similarly as in (3.4), for any $(i,n)\in\mathbb{N}_0\times\mathbb{N}$ and with $\tilde{\xi}^n_i:=\xi^n_i-C_\varepsilon$, we get

$$\mathbb{E}\left[\left(\tilde{\xi}^n_i\right)^2\right]\leq(1+\varepsilon)\mathbb{E}\left[\left(T^0_{0,n}-C_\varepsilon\right)^2\right]=(1+\varepsilon)\mathbb{E}\left[\left(X^0_{0,n}\right)^2\right],$$

as well as

$$\mathbb{E}\left[\left(\xi^1_i\right)^2\right]\leq(1+\varepsilon)\mathbb{E}\left[\left(T^0_{0,1}\right)^2\right]. \tag{3.16}$$

Since $\mathrm{Var}[\xi^n_i]=\mathrm{Var}[\tilde{\xi}^n_i]$, to prove the first claim in (3.11), it suffices to show $\mathbb{E}[\left(X^0_{0,n}\right)^2]\leq\frac{C_*}{2}n^2$ for some $C_*>0$ and all $n\in\mathbb{N}$. We can use $T^0_{0,n}\leq\sum_{i=0}^{n-1}\xi^1_i$ (due to (1)) and (3.16) to obtain

$$\mathbb{E}\left[\left(X^0_{0,n}\right)^2\right]\leq\mathbb{E}\left[\left(T^0_{0,n}\right)^2\right]\leq n\mathbb{E}\left[\sum_{i=0}^{n-1}(\xi^1_i)^2\right]\leq(1+\varepsilon)n^2\mathbb{E}\left[\left(T^0_{0,1}\right)^2\right],$$

which yields this estimate with $C_*:=4\mathbb{E}\left[\left(X^0_{0,1}+C_1\right)^2\right]$.

To prove the second claim in (3.11), we apply (4*) to get that for any $i,i',j,j',k,l\in\mathbb{N}_0$ satisfying $l>k$ we have

$$\begin{aligned}&\{t^n_k=i'\}\in\mathcal{F}^-_{i'-C_\varepsilon}, &&\{T^{i'}_{kn,(k+1)n}=i\}\in\mathcal{F}^-_{i'+i-C_\varepsilon}\cap\mathcal{F}^+_{i'},\\ &\{t^n_l=j'\}\in\mathcal{F}^-_{j'-C_\varepsilon}, &&\{T^{j'}_{ln,(l+1)n}=j\}\in\mathcal{F}^+_{j'}.\end{aligned} \tag{3.17}$$

Note that $l>k$ implies $j'\geq i+i'$ whenever $\mathbb{P}[T^{i'}_{kn,(k+1)n}=i\ \&\ t^n_k=i'\ \&\ t^n_l=j']>0$ because

$$t^n_l=t^n_k+\xi^n_k+\cdots+\xi^n_{l-1}\geq t^n_k+\xi^n_k=t^n_k+T^{t^n_k}_{kn,(k+1)n}.$$

Recalling that $\tilde{\xi}_k^n = T_{kn,(k+1)n}^{t_k^n} - C_\varepsilon \geq 0$, it follows from the above, (3), (3.1), and (3.5) that

$$\begin{aligned}
\mathbb{E}\left[\tilde{\xi}_k^n \tilde{\xi}_l^n\right] &= \sum_{i,j,i',j' \geq 0} ij\, \mathbb{P}\left[T_{ln,(l+1)n}^{j'} = j + C_\varepsilon \ \&\ t_l^n = j' \ \&\ T_{kn,(k+1)n}^{i'} = i + C_\varepsilon \ \&\ t_k^n = i'\right] \\
&\leq (1+\varepsilon) \sum_{i,j,i',j' \geq 0} ij\, \mathbb{P}\left[T_{ln,(l+1)n}^{j'} = j + C_\varepsilon\right] \mathbb{P}\left[t_l^n = j' \ \&\ T_{kn,(k+1)n}^{i'} = i + C_\varepsilon \ \&\ t_k^n = i'\right] \\
&= (1+\varepsilon) \sum_{i,j,i' \geq 0} ij\, \mathbb{P}\left[T_{0,n}^{0} = j + C_\varepsilon\right] \mathbb{P}\left[T_{kn,(k+1)n}^{i'} = i + C_\varepsilon \ \&\ t_k^n = i'\right] \\
&= (1+\varepsilon) \mathbb{E}\left[\tilde{\xi}_0^n\right] \mathbb{E}\left[\tilde{\xi}_k^n\right] \leq (1+\varepsilon)^2 \mathbb{E}\left[\tilde{\xi}_0^n\right]^2,
\end{aligned}$$

where in the inequality we used that the summand in the first sum vanishes when $j' < i' + i + C_\varepsilon$. Also note that (3.6) yields

$$\mathbb{E}\left[\tilde{\xi}_k^n\right] \mathbb{E}\left[\tilde{\xi}_l^n\right] \geq (1-\varepsilon)^2 \mathbb{E}\left[\tilde{\xi}_0^n\right]^2,$$

hence

$$\operatorname{Cov}\left[\xi_k^n, \xi_l^n\right] = \operatorname{Cov}\left[\tilde{\xi}_k^n, \tilde{\xi}_l^n\right] \leq 4\varepsilon \mathbb{E}\left[\tilde{\xi}_0^n\right]^2 = 4\varepsilon \mathbb{E}\left[X_{0,n}^0\right]^2. \tag{3.18}$$

Now the second claim in (3.11) follows by (3.10), and the proof of (i) is finished. □

Next we adjust this proof to obtain the integer-valued version of the first claim in Theorem 1.2. We will use in it the following lemma.

Lemma 3.2. *For $E, F \in \mathcal{F}$, let $\Psi(E,F) := \max\{s \in \mathbb{Z} \,|\, (E,F) \in \mathcal{F}_t^- \times \mathcal{F}_{t+s}^+$ for some $t \in \mathbb{N}_0\}$ (if there is no such s, then $\Phi(E,F) := -\infty$). Assume that $A_j^k, B_k \in \mathcal{F}$ ($k, j \in \mathbb{N}_0$) are such that $B_0, B_1, \dots$ are pairwise disjoint, and so are $A_0^k, A_1^k, \dots$ for each $k \in \mathbb{N}_0$. If $s := \min\{\Psi(B_k, A_j^k) \,|\, j,k \in \mathbb{N}_0\} \geq 0$ and $S := \sup\{f(j,k) \,|\, \mathbb{P}[A_j^k \cap B_k] > 0\}$ for some $f : \mathbb{N}_0^2 \to [0,\infty)$, then with ϕ from Remark 2 after Theorem 1.2 we have*

$$\sum_{j,k \geq 0} f(j,k) \left|\mathbb{P}[A_j^k \cap B_k] - \mathbb{P}[A_j^k]\mathbb{P}[B_k]\right| \leq 2S\phi(s).$$

Proof. Let

$$U^\pm := \left\{(j,k) \in \mathbb{N}_0^2 \,\middle|\, \pm\left(\mathbb{P}[A_j^k \cap B_k] - \mathbb{P}[A_j^k]\mathbb{P}[B_k]\right) > 0\right\},$$

and let $U_k^\pm := \{j \in \mathbb{N}_0 \,|\, (j,k) \in U^\pm\}$ for each $k \in \mathbb{N}_0$. Then

$$\sum_{(j,k) \in U^+} f(j,k) \left|\mathbb{P}[A_j^k \cap B_k] - \mathbb{P}[A_j^k]\mathbb{P}[B_k]\right| \leq S \sum_{k \geq 0} \left(\mathbb{P}\left[\bigcup_{j \in U_k^+} A_j^k \cap B_k\right] - \mathbb{P}\left[\bigcup_{j \in U_k^+} A_j^k\right] \mathbb{P}[B_k]\right),$$

which $\leq S\phi(s)$ because if $t_k \in \mathbb{N}_0$ is minimal such that $B_k \in \mathcal{F}_{t_k}^-$, then $\bigcup_{j \in U_k^+} A_j^k \in \mathcal{F}_{t_k+s}^+$. The same estimate holds for the sum over U^-, finishing the proof. □

Theorem 3.3. *Assume the hypotheses of Theorem 3.1, but with (2^{**}) and (5^{**}) replaced by (2^*) and $\lim_{s\to\infty} \phi(s) = 0$ for ϕ from Remark 2 after Theorem 1.2 (with $s, t_0, t_1, \dots \in \mathbb{N}_0$). Then* (1.3) *holds.*

Proof. This proof follows along the same lines as the one of Theorem 3.1, with some minor adjustments. From (1), (2*), and (3) we see that for any integers $t \geq 0$ and $n > m \geq 0$ we have

$$X_{m,n}^t \leq C(n-m). \tag{3.19}$$

With the ϕ considered here, let $C_\varepsilon \in \mathbb{N}$ be such that

$$\phi(C_\varepsilon) \leq \frac{\varepsilon}{2}, \tag{3.20}$$

and let $T_{m,n}^t, \underline{X}, t_i^n, \xi_i^n, \mu_i^n$ be defined as before. Then (3.19), Lemma 3.2, and (3) yield

$$\begin{aligned} \mu_i^n &= \sum_{k,j\geq 0} j\,\mathbb{P}\left[T_{in,(i+1)n}^k = j \ \&\ t_i^n = k\right] \\ &\leq \sum_{k,j\geq 0} j\,\mathbb{P}\left[T_{in,(i+1)n}^k = j\right]\mathbb{P}\left[t_i^n = k\right] + (Cn + C_\varepsilon)\varepsilon \\ &= \sum_{k,j\geq 0} j\,\mathbb{P}\left[T_{0,n}^0 = j\right]\mathbb{P}\left[t_i^n = k\right] + (Cn + C_\varepsilon)\varepsilon \\ &= \mu_0^n + (Cn + C_\varepsilon)\varepsilon \end{aligned} \tag{3.21}$$

instead of (3.4). Similarly, we obtain

$$\mathbb{E}\left[\xi_i^n - C_\varepsilon\right] \leq \mu_0^n - C_\varepsilon + Cn\varepsilon. \tag{3.22}$$

and

$$\mathbb{E}\left[\xi_i^n - C_\varepsilon\right] \geq \mu_0^n - C_\varepsilon - Cn\varepsilon. \tag{3.23}$$

Using (1) and (3.21) in place of (3.4), we now get

$$\mu_0^n \leq C_\varepsilon' n \tag{3.24}$$

in place of (3.7), with $C_\varepsilon' := \mathbb{E}\left[X_{0,1}^0\right] + C_\varepsilon + C\varepsilon$.

Next, similarly to (3.21) and using Lemma 3.2 and (3.19), we can replace (3.9) by

$$\begin{aligned} \mathbb{E}\left[T_{m,n}^{T_{0,m}^0}\right] &= \sum_{i,j\geq 0} j\,\mathbb{P}\left[T_{m,n}^i = j \ \&\ T_{0,m}^0 = i\right] \\ &\leq \sum_{i,j\geq 0} j\,\mathbb{P}\left[T_{m,n}^i = j\right]\mathbb{P}\left[T_{0,m}^0 = i\right] + (C(n-m) + C_\varepsilon)\varepsilon \\ &= \mathbb{E}\left[T_{0,n-m}^0\right] + (C(n-m) + C_\varepsilon)\varepsilon. \end{aligned} \tag{3.25}$$

With this, we again obtain (3.10).

The proof of (3.11) is also adjusted similarly to (3.21). We now obtain

$$\mathbb{E}\left[\left(\tilde{\xi}_i^n\right)^2\right] \leq \mathbb{E}\left[\left(T_{0,n}^0 - C_\varepsilon\right)^2\right] + (Cn)^2\varepsilon = \mathbb{E}\left[\left(X_{0,n}^0\right)^2\right] + (Cn)^2\varepsilon$$

and

$$\mathbb{E}\left[(\xi_i^1)^2\right] \leq \mathbb{E}\left[(T_{0,1}^0)^2\right] + (C + C_\varepsilon)^2\varepsilon,$$

which yields the first claim in (3.11) as before (with a different C_*). In the proof of the second claim, we use (3.22) in place of (3.5), as well as $\tilde{\xi}_k^n \leq Cn$ (due to (3.19)). We also use the same adjustment as in (3.21), but now replacing the sum over k by the sum over (i,i',j') (with $A_j^{(i,i',j')} := \{T^{j'}_{ln,(l+1)n} = j\}$ when we use Lemma 3.2). This and (3.19) show that

$$\begin{aligned}\mathbb{E}\left[\tilde\xi_k^n\tilde\xi_l^n\right] &= \sum_{i,j,i',j'\ge 0} ij\,\mathbb{P}\left[T^{j'}_{ln,(l+1)n} = j + C_\varepsilon \;\&\; t_l^n = j' \;\&\; T^{i'}_{kn,(k+1)n} = i + C_\varepsilon \;\&\; t_k^n = i'\right]\\ &\le \sum_{i,j,i',j'\ge 0} ij\,\mathbb{P}\left[T^{j'}_{ln,(l+1)n} = j + C_\varepsilon\right]\mathbb{P}\left[t_l^n = j' \;\&\; T^{i'}_{kn,(k+1)n} = i + C_\varepsilon \;\&\; t_k^n = i'\right] + (Cn)^2\varepsilon\\ &= \sum_{i,j,i'\ge 0} ij\,\mathbb{P}\left[T^0_{0,n} = j + C_\varepsilon\right]\mathbb{P}\left[T^{i'}_{kn,(k+1)n} = i + C_\varepsilon \;\&\; t_k^n = i'\right] + (Cn)^2\varepsilon\\ &= \mathbb{E}\left[\tilde\xi_0^n\right]\mathbb{E}\left[\tilde\xi_k^n\right] + (Cn)^2\varepsilon \le \mathbb{E}\left[\tilde\xi_0^n\right]^2 + Cn\varepsilon\mathbb{E}\left[\tilde\xi_0^n\right] + (Cn)^2\varepsilon.\end{aligned}$$

This, (3.23) applied with $i = k, l$, and $\tilde\xi_0^n \le Cn$ then yield the second claim in (3.11) with $C_* := 4C^2$.

Now, the proof of (3.12), but with (3.4), (3.7), and (3.9) replaced by (3.21), (3.24), and (3.25), shows that

$$\mathbb{P}\left[\frac{X^0_{0,N}}{N} > \frac{\mu_0^n}{n} + C\varepsilon + \frac{(1+\varepsilon)C_\varepsilon}{n} + \frac{\tau_N^n}{kn}\right] \le \frac{1+\varepsilon k}{C_\varepsilon^2 k} C_* n^2,$$

where $\tau_N^n := T^{T^0_{0,kn}}_{kn,kn+l}$ satisfies $\mathbb{E}[\tau_N^n] \le (C'_\varepsilon + (C + C_\varepsilon)\varepsilon)n$. This then implies (3.14) as before, and the rest of the proof is identical to the proof of Theorem 3.1. □

We can now prove the first claim in Theorem 1.2 similarly to the proof of Theorem 1.1.

Proof of the first claim in Theorem 1.2. Let us first assume that $c \ge 1$. Let

$$T^t_{m,n} := \lceil X^t_{m,n} + C\rceil \quad (\in \mathbb{N}_0)$$

and restrict t to $\mathbb{N}_0$. Similarly to the proof of Theorem 1.1, we find that $T^t_{m,n}$ satisfies hypotheses (1), (3), (4**), (6**) of Theorem 3.3 (with $X^t_{m,n}$ replaced by $T^t_{m,n}$), but with $\max\{\lceil C\rceil, 1\}$ in place of C in (6**). Hence iteration of (6**) shows that it also holds for $T^t_{m,n}$ and $C' := 2\max\{\lceil C\rceil, 1\}$ in place of C. From (2*) for $X^t_{m,n}$ we see that $T^t_{m,n}$ also satisfies (2*) with C' in place of C.

Let now ϕ be as in Remark 2 after Theorem 1.2. Note that if we define $\tilde\phi(s)$ as in that remark but only with $s, t_0, t_1, \dots \in \mathbb{N}_0$, then $\tilde\phi \le \phi$. Therefore our hypothesis $\lim_{s\to\infty}\phi(s) = 0$ implies the last hypothesis in Theorem 3.3 as well. That theorem for $T^t_{m,n}$ now yields (1.3).

For $c \in (0,1)$, we let $\mathcal{G}_t^\pm$ and $Y^t_{m,n}$ be as in the proof of Theorem 1.1. The above argument with $(\mathcal{G}_t^\pm, Y^t_{m,n}, SC, Sc)$ in place of $(\mathcal{F}_t^\pm, X^t_{m,n}, C, c)$ then again concludes (1.3).

Finally, in the setting of the last claim in Theorem 1.2 we can just apply Theorem 3.3 directly to $X^t_{m,n}$ (with $\tilde\phi$ above). □

4. Time-Decaying Dependence II

In this section we will prove the second claim in Theorem 1.2, as well as the corresponding integer-valued claim.

Proof of the second claim in Theorem 1.2. Similarly to the proof of the first claim in Theorem 1.2, this again follows from the corresponding integer-valued claim. Hence, without loss, we can restrict t to $\mathbb{N}_0$ and assume that $X^t_{m,n}$ only takes values in $\mathbb{N}_0$.

The first claim in Theorem 1.2 yields

$$\lim_{n\to\infty} \frac{X^0_{0,n}}{n} = \lim_{n\to\infty} \frac{\mathbb{E}\left[X^0_{0,n}\right]}{n} =: \underline{X} \geq 0 \qquad \text{in probability.} \tag{4.1}$$

Let us now prove

$$\limsup_{n\to\infty} \frac{X^0_{0,n}}{n} \leq \underline{X} \qquad \text{almost surely.} \tag{4.2}$$

As in the proof of Theorem 3.1, let $T^t_{m,n} := X^t_{m,n} + C_\varepsilon$ some $C_\varepsilon \in \mathbb{N}$ that is a multiple of C and (3.20) also holds. Then (1'), (3'), (6') from Theorem 2.1 hold and so does (4") from the proof of Theorem 3.1, while (2') is replaced by $T^0_{0,1} \leq C + C_\varepsilon$, and (5*) also holds.

For any $n \in \mathbb{N}$, define t^n_i and ξ^n_i as at the start of the proof of Theorem 3.1. From (4") we again get (3.17) for any $i, i', j, j', k, l \in \mathbb{N}_0$, and the argument after (3.17) again shows that if $l > k$, then $j' \geq i + i'$ whenever $\mathbb{P}[T^{i'}_{kn,(k+1)n} = i \;\&\; t^n_k = i' \;\&\; t^n_l = j'] > 0$. Then the argument from the proof of the second claim in (3.11) in the proof of Theorem 3.3 (which uses Lemma 3.2) shows that for any $\nu, \nu' \in \mathbb{N}$ we have

$$\begin{aligned}
\mathbb{P}\left[\xi^n_k \geq \nu \;\&\; \xi^n_l \geq \nu'\right] &= \sum_{i-\nu, j-\nu', i', j' \geq 0} \mathbb{P}\left[T^{j'}_{ln,(l+1)n} = j \;\&\; t^n_l = j' \;\&\; T^{i'}_{kn,(k+1)n} = i \;\&\; t^n_k = i'\right] \\
&\leq \sum_{i-\nu, j-\nu', i', j' \geq 0} \mathbb{P}\left[T^{j'}_{ln,(l+1)n} = j\right] \mathbb{P}\left[t^n_l = j' \;\&\; T^{i'}_{kn,(k+1)n} = i \;\&\; t^n_k = i'\right] + \varepsilon \\
&= \sum_{i-\nu, j-\nu', i' \geq 0} \mathbb{P}\left[T^0_{0,n} = j\right] \mathbb{P}\left[T^{i'}_{kn,(k+1)n} = i \;\&\; t^n_k = i'\right] + \varepsilon \\
&\leq \mathbb{P}\left[T^0_{0,n} \geq \nu'\right] \sum_{i-\nu, i' \geq 0} \mathbb{P}\left[T^{i'}_{kn,(k+1)n} = i\right] \mathbb{P}\left[t^n_k = i'\right] + 2\varepsilon \\
&= \mathbb{P}\left[T^0_{0,n} \geq \nu'\right] \mathbb{P}[T^0_{0,n} \geq \nu] + 2\varepsilon.
\end{aligned} \tag{4.3}$$

Now fix some $K \in \mathbb{N}$. From (1') we see that for any $n \in \mathbb{N}$ we have

$$T^0_{0,Kn} \leq \sum_{i=0}^{K-1} T^{t^n_i}_{in,(i+1)n}. \tag{4.4}$$

From (4.1) we see that there is ε-independent $n_K \in \mathbb{N}$ such that for all $n \geq \max\{C_\varepsilon K, n_K\}$,

$$\mathbb{P}\left[\frac{T^0_{0,n}}{n} - \underline{X} \geq \frac{2}{K}\right] \leq \mathbb{P}\left[\frac{X^0_{0,n}}{n} - \underline{X} \geq \frac{1}{K}\right] \leq \frac{1}{2K^2}. \tag{4.5}$$

From (1), (2*), and (3) we get $T^{t_i^n}_{in,(i+1)n} \le C_\varepsilon + Cn \le (C+1)n$ for these n and all $i \in \mathbb{N}_0$. This means that if only one of the numbers

$$g_i^n := \frac{T^{t_i^n}_{in,(i+1)n}}{n} - \underline{X} - \frac{2}{K} \qquad (i \in \{0,\dots,K-1\})$$

is positive, then (4.4) yields

$$\frac{T^0_{0,Kn}}{Kn} - \underline{X} \le \sum_{i=0}^{K-1} \left(\frac{g_i^n}{K} + \frac{2}{K^2} \right) < \frac{C+1}{K} + \frac{2}{K} = \frac{C+3}{K}.$$

The same estimate holds if each of these numbers is less than $\frac{C+1}{K}$. These facts, (4.3), (3'), and (4.5) now imply that for any $n \ge \max\{C_\varepsilon K, n_K\}$ we have

$$\begin{aligned}\mathbb{P}\left[\frac{T^0_{0,Kn}}{Kn} - \underline{X} \ge \frac{C+3}{K} \right] &\le \sum_{0\le i,j<K \,\&\, i\ne j} \mathbb{P}\left[g_i^n \ge \frac{C+1}{K} \;\&\; g_j^n \ge 0 \right] \\ &\le K^2 \mathbb{P}\left[g_0^n \ge \frac{C+1}{K} \right] \mathbb{P}\left[g_0^n \ge 0 \right] + 2K^2\varepsilon \\ &\le \frac{1}{2} \mathbb{P}\left[\frac{T^0_{0,n}}{n} - \underline{X} \ge \frac{C+3}{K} \right] + 2K^2\varepsilon.\end{aligned}$$

We can now apply this estimate iteratively with $Kn, K^2n, \dots$ in place of n and obtain for any $n \ge \max\{C_\varepsilon K, n_K\}$ and $q \in \mathbb{N}$,

$$\mathbb{P}\left[\frac{T^0_{0,K^q n}}{K^q n} - \underline{X} \ge \frac{C+3}{K} \right] \le 2^{-q}\, \mathbb{P}\left[\frac{T^0_{0,n}}{n} - \underline{X} \ge \frac{C+3}{K} \right] + 4K^2\varepsilon \le 2^{-q} + 4K^2\varepsilon.$$

This of course also yields

$$\mathbb{P}\left[\frac{X^0_{0,K^q n}}{K^q n} - \underline{X} \ge \frac{C+3}{K} \right] \le 2^{-q} + 4K^2\varepsilon. \tag{4.6}$$

The hypothesis shows that there is $A \in \mathbb{N}$ such that $C_\varepsilon \le A\varepsilon^{-A}$ for all $\varepsilon \in (0,1)$. Let

$$C' \ge C_K := AKn_K \qquad \text{and} \qquad M_K := 2^A K,$$

with $C' \in \mathbb{N}$. Then for any $q \in \mathbb{N}$, (4.6) with $\varepsilon := 2^{-q}$ and $n := 2^{Aq}C'$ $(\ge \max\{C_\varepsilon K, n_K\})$ yields

$$\mathbb{P}\left[\frac{X^0_{0,C'M_K^q}}{C'M_K^q} - \underline{X} \ge \frac{C+3}{K} \right] \le 5K^2 2^{-q}. \tag{4.7}$$

By the Borel-Cantelli Lemma we then obtain

$$\limsup_{q\to\infty} \frac{X^0_{0,C'M_K^q}}{C'M_K^q} \le \underline{X} + \frac{C+3}{K} \qquad \text{almost surely.} \tag{4.8}$$

Now apply (4.8) with C' taking all the values in

$$U_K := \{C_K, C_K+1, \dots, C_K M_K\}.$$

Then for any large n, there is $(C', q) \in U_K \times \mathbb{N}$ such that $C'M_K^q \leq n \leq C'M_K^q + C_K^{-1}n$ and

$$\frac{X^0_{0,C'M_K^q}}{C'M_K^q} \leq \underline{X} + \frac{C+4}{K}.$$

So by (1) and (2*) we have $X^0_{0,n} \leq X^0_{0,C'M_K^q} + CC_K^{-1}n$, which yields

$$\limsup_{n\to\infty} \frac{X^0_{0,n}}{n} \leq \underline{X} + \frac{C+4}{K} + CC_K^{-1} \qquad \text{almost surely.}$$

By taking $K \to \infty$, we conclude (4.2).

It remains to prove

$$\underline{X} \leq \liminf_{n\to\infty} \frac{X^0_{0,n}}{n} \qquad \text{almost surely.} \tag{4.9}$$

We will do this with only assuming $\lim_{s\to\infty} \phi(s) = 0$ (rather than $\lim_{s\to\infty} s^\alpha \phi(s) = 0$), and without the use of the proof of (4.2). This will then also prove Remark 3 after Theorem 1.2.

For any $t, m, n, j \in \mathbb{N}_0$ with $j \geq n$, let

$$Y^t_{m;n,j} := \min_{n\leq i\leq j} \frac{X^t_{m,m+i}}{i} \qquad \text{and} \qquad Z^t_m := \lim_{n\to\infty}\lim_{j\to\infty} Y^t_{m;n,j} = \liminf_{n\to\infty} \frac{X^t_{m,m+n}}{n}.$$

Z^{t+Ck}_m is non-decreasing in $k \in \mathbb{N}$ by (6) (with $c = 0$), and since the law of Z^t_m is independent of (t, m) by (3), we almost surely have $Z^{t+Ck}_m = Z^t_m$ for all $k \in \mathbb{N}$. Moreover we claim that Z^0_0 is almost everywhere constant (which implies that Z^t_m is a.e. equal to the same constant for each $(t, m) \in \mathbb{N}_0$).

If this is not the case, let $c := \mathrm{Var}[Z^0_0] > 0$. From (1), (2*), and (3) we have

$$\max\left\{Z^0_0,\, Y^t_{0;n,j},\, \frac{X^t_{0,n}}{n}\right\} \leq C \qquad \text{for all } t, n, j \in \mathbb{N}_0 \text{ with } j \geq n \geq 1. \tag{4.10}$$

Let $\delta := \frac{c}{4C(C+1)}$. By (3) and Ergorov's Theorem, there δ-dependent $n, j \in \mathbb{N}^2$ with $j \geq n$ such that for any $t \in \mathbb{N}_0$ we have

$$|Y^t_{0;n,j} - Z^t_0| \leq \delta \qquad \text{on some } \Omega^t_\delta \subseteq \Omega \text{ with } \mathbb{P}[\Omega^t_\delta] \geq 1 - \delta.$$

Also since $Z^0_0 = Z^{Ck}_0$ a.e. for all $k \in \mathbb{N}$, (4.10) and $\mathrm{Var}[Z^0_0] = c$ imply

$$\mathrm{Cov}\left[Y^0_{0;n,j}, Y^{Ck}_{0;n,j}\right] \geq \mathrm{Cov}\left[Z^0_0, Y^{Ck}_{0;n,j}\right] - (C + C^2)\delta \geq \mathrm{Cov}\left[Z^0_0, Z^{Ck}_0\right] - 2(C + C^2)\delta \geq \frac{c}{2}. \tag{4.11}$$

Next note that by (4), (4.10), and the definition of $Y^0_{0;n,j}$ we have

$$Y^0_{0;n,j} \text{ is } \mathcal{F}^-_{Cj}\text{-measurable} \qquad \text{and} \qquad Y^{Ck}_{0;n,j} \text{ is } \mathcal{F}^+_{Ck}\text{-measurable.}$$

This, Lemma 3.2, (4.10), and $Y^t_{0;n,j}$ only taking rational values show for any $k \geq j \geq n \geq 1$,

$$\begin{aligned} \mathbb{E}\left[Y^0_{0;n,j} Y^{Ck}_{0;n,j}\right] &= \sum_{p,q\in\mathbb{Q}} pq\,\mathbb{P}\left[Y^{Ck}_{0;n,j} = p \ \&\ Y^0_{0;n,j} = q\right] \\ &\leq \sum_{p,q\in\mathbb{Q}} pq\,\mathbb{P}\left[Y^{Ck}_{0;n,j} = p\right]\mathbb{P}\left[Y^0_{0;n,j} = q\right] + C^2\phi(C(k-j)) \\ &= \mathbb{E}\left[Y^0_{0;n,j}\right]\mathbb{E}\left[Y^{Ck}_{0;n,j}\right] + C^2\phi(C(k-j)), \end{aligned} \tag{4.12}$$

Hence $\mathrm{Cov}\left[Y^0_{0;n,j}, Y^{Ck}_{0;n,j}\right] \leq C^2\phi(C(k-j))$, which contradicts with (4.11) if we take k large enough (because (5*) holds).

Therefore Z^0_0 is indeed almost everywhere equal to some constant $Q \in [0, \underline{X}]$. Then (4.9) is just $\underline{X} \leq Q$, so we only need to prove this. For any $\varepsilon > 0$ and $K \in \mathbb{N}$, let us define

$$T^t_{m,n} := X^t_{Km,Kn} + C_\varepsilon \tag{4.13}$$

(which depends on ε, K but we suppress this in the notation). Then again (1'), (3'), (6') from Theorem 2.1 hold and so does (4") from the proof of Theorem 3.1, while (2') is replaced by $T^0_{0,1} \leq CK + C_\varepsilon$, and (5*) also holds. From (1'), (3'), and $T^0_{0,1} \leq CK + C_\varepsilon$ we obtain for any $t, m, n \in \mathbb{N}_0$,

$$T^t_{m,m+n} \leq CKn + C_\varepsilon. \tag{4.14}$$

Note that to prove $\underline{X} \leq Q$, it suffices to show that

$$\frac{\mathbb{E}[T^0_{0,n}]}{n} \leq KQ + KC\varepsilon + \varepsilon(C_\varepsilon + 2) + \frac{M'_{K,\varepsilon}}{n} \tag{4.15}$$

holds for each $\varepsilon > 0$ and $K \in \mathbb{N}$, with some n-independent $M'_{K,\varepsilon}$. This is because after dividing (4.15) by K and taking $n \to \infty$, we obtain from (4.1),

$$\underline{X} = \lim_{n\to\infty} \frac{\mathbb{E}[X^0_{0,Kn}]}{Kn} \leq Q + C\varepsilon + \frac{\varepsilon(C_\varepsilon + 2)}{K}.$$

Taking $K \to \infty$ and then $\varepsilon \to 0$ now yields $\underline{X} \leq Q$, so we are indeed left with proving (4.15).

This is done similarly to the argument in the proof of (2.8), with KQ in place of Q. Fix $\varepsilon > 0$ and $K \in \mathbb{N}$, let $Q_\varepsilon := KQ + \varepsilon$ (as at the start of that proof), and let $T^t_{m,n}$ be from (4.13). Note that for any $t, m \in \mathbb{N}_0$ we have $\liminf_{n\to\infty} \frac{T^t_{m,m+n}}{n} = KQ$ almost surely because $Z^t_m = Q$ almost everywhere. Define

$$N^t_m, M_\varepsilon, t_k, r_k, S_n$$

as in the proof of (2.8), and follow that proof, with two adjustments near the end where (5') was used. The first is the estimate on

$$\mathbb{E}\left[T^{t_k}_{r_k,r_k+1} 1_{\{N^{t_k}_{r_k} > M_\varepsilon\}}\right].$$

From (4") we have for any $i, j \in \mathbb{N}_0$ that $\{r_k = i \ \&\ t_k = j\} \in \mathcal{F}^-_{j-C_\varepsilon}$, and $T^j_{i,i+1}$ and N^j_i are $\mathcal{F}^+_j$-measurable. Hence we can use (5*), (4.14), and Lemma 3.2 instead of (5') (as well as (3') and (2.9) as before) to obtain

$$\begin{aligned}\mathbb{E}\left[T^{t_k}_{r_k,r_k+1}1_{\left\{N^{t_k}_{r_k}>M_\varepsilon\right\}}\right] &= \sum_{i,j\geq 0}\mathbb{E}\left[T^j_{i,i+1}1_{\left\{N^j_i>M_\varepsilon\right\}}1_{\{r_k=i\ \&\ t_k=j\}}\right] \\ &\leq \sum_{i,j,l\geq 0} l\,\mathbb{P}\left[T^j_{i,i+1}=l\ \&\ N^j_i>M_\varepsilon\right]\mathbb{P}\left[r_k=i\ \&\ t_k=j\right]+(CK+C_\varepsilon)\varepsilon \qquad (4.16) \\ &= \mathbb{E}\left[T^0_{0,1}1_{\left\{N^0_0>M_\varepsilon\right\}}\right]+(CK+C_\varepsilon)\varepsilon \leq (CK+C_\varepsilon+1)\varepsilon.\end{aligned}$$

This then yields

$$\frac{\mathbb{E}[S_n]}{n} \leq KQ+(CK+C_\varepsilon+2)\varepsilon \qquad (4.17)$$

in place of (2.12). The second place is the estimate on $\mathbb{E}[T^{t_{K_n}}_{n-l,n}]$, but here we can simply use (4.14) to obtain

$$\mathbb{E}[T^{t_{K_n}}_{n-l,n}] \leq CKl + C_\varepsilon.$$

This and (2.13) yield

$$\mathbb{E}[T^{t_{K_n}}_{r_{K_n},n}] \leq \sum_{l=1}^{M_\varepsilon}\mathbb{E}\left[T^{t_{K_n}}_{n-l,n}\right] \leq M_\varepsilon(CKM_\varepsilon+C_\varepsilon) =: M'_{K,\varepsilon}. \qquad (4.18)$$

This, (4.17), and (2.10) now show (4.15), and the proof is finished. □

5. PDE and First Passage Percolation in Time-Dependent Environments

Our main motivation for this work was its application in the proof of homogenization for reaction-diffusion equations with time-dependent coefficients in several spatial dimensions in our companion paper [13]. The first crucial step is a proof of existence of asymptotic shapes of propagation (called *Wulff shapes*) for these PDE. These were previously proved to exist for reaction-diffusion equations with time-independent spatially periodic reactions under appropriate hypotheses, first in 1979 by Gärtner and Freidlin [4], but analogous results for (still time-independent or time-periodic) spatially stationary ergodic reactions, by the second author and Lin [11, 14], are much more recent. The latter results employ Kingman's subadditive ergodic theorem, which raises the question of their extension to the case of time-dependent reactions. In [13] we obtained such extensions, using Theorems 1.1 and 1.2.

Moreover, these theorems can also be used to study propagation of solutions to other PDE with time-dependent coefficients, as the following example shows. In [13] this was used to prove homogenization for Hamilton-Jacobi PDE called G-equations (an earlier result for environments with finite temporal ranges of dependence was obtained by Burago, Ivanov, and Novikov in [2]), and we refer the reader to that paper for further discussion and references concerning homogenization for reaction-diffusion and Hamilton-Jacobi PDE.

Example 5.1. Consider some PDE on $[0,\infty)\times\mathbb{R}^d$ with space-time stationary coefficients, for which the maximum principle holds. Assume that (5) resp. (5^*) holds when $\mathcal{F}^\pm_t$ are σ-algebras generated by the coefficients restricted to $[0,t]\times\mathbb{R}^d$ and $[t,\infty)\times\mathbb{R}^d$, respectively.

Fix some compactly supported "bump" function $u_0 : \mathbb{R}^d \to [0,\infty)$, and for any $(t', x') \in \mathbb{R}^d$ let $u^{t',x'}$ solve the PDE with initial value $u^{t',x'}(t',\cdot) := u_0(\cdot - x')$. Then for any $y \in \mathbb{R}^d$ let

$$X^{t'}(x', y) := \inf \left\{ t \geq 0 \,\middle|\, u^{t',x'}(t + t', \cdot) \geq u_0(\cdot - y) \right\},$$

so that $X^{t'}(x', y)$ can be thought of as the time it takes for $u^{t',x'}$ to propagate from x' to y, starting at time t' (this of course depends on the random parameter ω). Let us also assume that u_0 was chosen so that for some $C \geq 0$ and all $t' \geq C$ we have $u^{0,0}(t', \cdot) \geq u_0$.

Fix any $t \in [0,\infty)$ and unit vector $e \in \mathbb{S}^{d-1}$, and let $X^{t,e}_{m,n} := X^t(me, ne)$. Then (4) is obvious from the definition of $X^{t,e}_{m,n}$, while maximum principle, space-time stationarity of coefficients, and $u^{0,0}(t', \cdot) \geq u_0$ for all $t' \geq C$ yield (1), (3), and (6). Hence if (2) resp. (2*) holds, Theorem 1.1 resp. 1.2 can be used to show that the limit

$$\lim_{n\to\infty} \frac{X^{0,e}_{0,n}}{n} \tag{5.1}$$

exists and equals a constant function of ω (almost surely or in probability). Of course, its reciprocal then represents the deterministic asymptotic speed of propagation in direction e for this PDE.

In fact, if $\frac{X^{t'}(x',y)}{|x'-y|}$ is bounded below and above by positive constants $c_0 \leq c_1$ whenever $|x' - y| \geq 1$, then (2) and (2*) clearly hold, asymptotic propagation speeds in all directions are between $\frac{1}{c_1}$ and $\frac{1}{c_0}$, and the PDE even has a *deterministic* asymptotic shape of propagation (i.e., a Wulff shape). Indeed, a version of a standard argument going back to [3,12] (see [13]) can typically be used to show that there is a convex open set $S \subseteq \mathbb{R}^d$, containing and contained in the balls centered at the origin with radii $\frac{1}{c_1}$ and $\frac{1}{c_0}$, respectively, such that if $S_t(\omega) := \{x \in \mathbb{R}^d \,|\, X^0(0, x) \leq t\}$, then for any $\delta > 0$ we have

$$(1 - \delta)tS \subseteq S_t(\omega) \subseteq (1 + \delta)tS,$$

either for almost every $\omega \in \Omega$ and all large-enough $t \geq 0$ (depending on ω and δ) or on sets (of ω) whose measures converge to 1 as $t \to \infty$.

The next two examples concern an application of our results to a different model, *first passage percolation in time-dependent environments*, which we introduce next. Let V_d be the set of edges of the lattice $\mathbb{Z}^d$, that is, each $v \in V_d$ connects two points $A, B \in \mathbb{Z}^d$ which share $d-1$ of their d coordinates and differ by 1 in the last coordinate (these can be either directed edges or not). Let us consider a traveler moving on the lattice $\mathbb{Z}^d$ from point A to B. The traveler can move along any path γ made of a sequence of edges $v^\gamma_1, v^\gamma_2, \ldots, v^\gamma_{n_\gamma}$, where each v^γ_i connects some points A_{i-1} and A_i, with $A = A_0$ and $B = A_{n_\gamma}$. Let us denote by $\Gamma_{A,B}$ the set of all such paths. Let us assume that the travel time for any edge v, if it is reached by the traveler at time t, is some number $\tau^t_v \geq 0$. For any $\gamma \in \Gamma_{A,B}$ and any time t_0, define recursively (for $i = 1, 2, \ldots, n_\gamma$) the times

$$t_i := t_{i-1} + \tau^{t_{i-1}}_{v^\gamma_i} \qquad \text{and} \qquad T^{t_0}_\gamma := t_{n_\gamma} - t_0.$$

That is, t_i is the time of arrival at the point A_i, and $T_\gamma^{t_0}$ is the travel time along γ when the starting time is t_0. Finally, let

$$X^t(A,B) := \inf\left\{T_\gamma^t \,\middle|\, \gamma \in \Gamma_{A,B}\right\} \tag{5.2}$$

be the shortest travel time from A to B when starting at time t.

When the travel times are independent of t, this is of course the standard first passage percolation model, introduced by Hammersley and Welsh [5]. Their work was extended by Kingman [8], whose subadditive ergodic theorem was in turn employed by Richardson [12] in the proof of a (Wulff) shape theorem for this model. Further improvements and extensions, including those by Cox and Durrett [3] and Kesten [7], were obtained by many authors in the last five decades, and we refer the reader to the review [1] for a comprehensive discussion and an extensive list of references.

Let us consider one of the following two setups when time-dependence is included in the model above. Let $\xi_v^t \ge 0$ be some number, and let τ_v^t be either the first time such that

$$\int_0^{\tau_v^t} \xi_v^{t+s} ds = 1, \tag{5.3}$$

or let

$$\tau_v^t := \inf\left\{s + \left(\xi_v^{t+s}\right)^{-1} \,\middle|\, s \ge 0\right\}. \tag{5.4}$$

In the first case, one can think of ξ_v^{t+s} as the instantaneous travel speed along v at time $t+s$, which changes due to changing road conditions (so $\int_0^\tau \xi_v^{t+s} ds$ is distance traveled in time τ). In the second case, one can think of ξ_v^{t+s} as the (constant) speed of a train leaving one end of v at time $t+s$ (which could be zero if there is no such train), and the traveller chooses the one that brings him to the other end at the earliest time.

Now for any $e \in \mathbb{Z}^d$ we can define $X_{m,n}^{t,e} := X^t(me, ne)$, so that asymptotic speed of travel in direction e is $|e|$ divided by the reciprocal of (5.1), provided that limit exists (and preferably is also deterministic). Theorems 1.1 and 1.2 can again be used to show this, either almost surely or in probability, if the speeds ξ_v^t are random variables satisfying appropriate hypotheses.

Note that hypotheses (1) and (6) in these theorems will always be satisfied (the latter with $c := \infty$ and any $C \ge 0$) for both models (5.3) and (5.4). If there is $L < \infty$ such that for all $(t,v) \in [0,\infty) \times V_d$ we have $\int_t^{t+L} \xi_v^s ds \ge 1$ or $\sup\{\xi_v^s \,|\, s \in [t, t+L]\} \ge \frac{1}{L}$ when we define τ_v^t via (5.3) or via (5.4), respectively, this will also guarantee (2*) (and so (2) as well). Finally, we will let $\mathcal{F}_t^-$ be the σ-algebra generated by the family of random variables

$$\{\xi_v^s \,|\, s \in [0,t] \,\&\, v \in V_d\}, \tag{5.5}$$

and $\mathcal{F}_t^+$ the σ-algebra generated by the family of random variables

$$\{\xi_v^s \,|\, s \ge t \,\&\, v \in V_d\}, \tag{5.6}$$

which will guarantee (4).

We note that (3) follows from space-time stationarity of ξ_v^t. For any $y \in \mathbb{Z}^d$, the translation $\sigma_y(x) := x + y$ on $\mathbb{Z}^d$ induces a translation map on V_d, which we also call σ_y. If $(\Omega, \mathcal{F}, \mathbb{P})$ is the involved probability space, then the speeds ξ_v^t are space-time stationary provided there is a semigroup of measure-preserving bijections $\{\Upsilon_{(s,y)} : \Omega \to \Omega\}_{(s,y)\in[0,\infty)\times\mathbb{Z}^d}$ such that $\Upsilon_{(0,0)} = \mathrm{Id}_\Omega$, for any $(s, y), (r, z) \in [0, \infty) \times \mathbb{Z}^d$ we have

$$\Upsilon_{(s,y)} \circ \Upsilon_{(r,z)} = \Upsilon_{(s+r,y+z)},$$

and for any $(t, s, v, y, \omega) \in [0, \infty)^2 \times V_d \times \mathbb{Z}^d \times \Omega$ we have

$$\xi_v^t(\Upsilon_{(s,y)}\omega) = \xi_{\sigma_y(v)}^{t+s}(\omega).$$

Hence if the speeds ξ_v^t are also space-time stationary, we will only need to check (5) or (5*).

We can construct space-time stationary environments with appropriately time-decreasing correlations by sampling space stationary environments, and below we provide two examples of this. Let $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ be a probability space and let σ_y be as above. We say that a random field $\eta : V_d \times \Omega_0 \to \mathbb{R}$ is space stationary, if there is a semigroup of measure-preserving bijections $\{\Upsilon_y : \Omega_0 \to \Omega_0\}_{y\in\mathbb{Z}^d}$ such that $\Upsilon_0 = \mathrm{Id}_{\Omega_0}$, for any $y, z \in \mathbb{Z}^d$ we have $\Upsilon_y \circ \Upsilon_z = \Upsilon_{y+z}$, and for any $(v, y, \omega) \in V_d \times \mathbb{Z}^d \times \Omega_0$ we have

$$\eta(v, \Upsilon_y\omega) = \eta(\sigma_y(v), \omega).$$

Let us assume below that η satisfies this as well as $\frac{1}{L} \le \eta \le L$ for some $L \ge 1$.

Example 5.2. Let $\Omega := [0, C) \times \Omega_0^{\mathbb{N}_0}$ have the product probability measure (with some $C > 0$ and the uniform measure on $[0, C)$). Consider the above setting, with

$$\xi_v^t(\omega) := \eta(v, \omega_{\lfloor (t+a)/C \rfloor}) \tag{5.7}$$

for $\omega = (a, \omega_0, \omega_1, \dots) \in \Omega$. That is, *the speeds ξ_v^t always change after time interval C*, starting from some time $a \in [0, C)$. Then they are clearly space-time stationary. Moreover, if $\mathcal{F}_t^\pm$ are defined via (5.5) and (5.6), then $\mathcal{F}_t^-$ and $\mathcal{F}_{t+C}^+$ are independent for each $t \ge 0$ because random variables $\alpha(\omega) := \eta(v_1, \omega_i)$ and $\beta(\omega) := \eta(v_2, \omega_j)$ are independent for any $v_1, v_2 \in V_d$ and any distinct $i, j \in \mathbb{N}_0$. The above discussion now shows that Theorem 1.1 applies to $X_{m,n}^{t,e}$ above for any $e \in \mathbb{Z}^d$, so $\frac{1}{n} X_{0,n}^{t,e}$ converges to some ω-independent constant almost surely.

Moreover, for any $(A, B, t) \in \mathbb{Z}^{2d} \times [0, \infty)$ (and with L above) we clearly have

$$L^{-1}|A - B|_1 \le X^t(A, B) \le L|A - B|_1, \tag{5.8}$$

where $|e|_1 := |e_1| + \dots + |e_d|$ is the L^1 norm, so the deterministic limit (5.1) is from $[\frac{1}{L}|e|_1, L|e|_1]$. Let us denote by $B_r^1(0)$ the ball in $\mathbb{R}^d$ with respect to the L^1 norm, with radius r and centered at the origin. Then as in Example 5.1, we can show that there is convex open $S \subseteq \mathbb{R}^d$, containing $B_{1/L}^1(0)$ and contained in $B_L^1(0)$, such that if $S_t(\omega)$ is the set of all $A \in \mathbb{Z}^d$ with $X^0(0, A) \le t$ (for $t \ge 0$ and ξ_v^s from (5.7)), then for almost every $\omega \in \Omega$ we have that for any $\delta > 0$ and all large-enough $t \ge 0$ (depending on ω and δ),

$$(1 - \delta)tS \cap \mathbb{Z}^d \subseteq S_t(\omega) \subseteq (1 + \delta)tS \cap \mathbb{Z}^d. \tag{5.9}$$

That is, S is again the deterministic asymptotic shape of all points reachable from the origin in large times (after scaling by t).

Example 5.3. Consider a Poisson point process with parameter $\lambda > 0$ on $\mathbb{R}$, defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and let N_t be the corresponding counting process (i.e., N_t is the number of points in the interval $(0, t]$). We now let $\Omega := \Omega' \times \Omega_0^{\mathbb{N}_0}$ have the product probability measure, and for $\omega = (\omega', \omega_0, \omega_1, \dots) \in \Omega$ we let

$$\xi_v^t(\omega) := \eta(v, \omega_{N_t})$$

(again considering the setup described before Example 5.2). That is, now *the interval after which the speeds ξ_v^t change has an exponential distribution.* The speeds are again space-time stationary, and (5*) holds with $\phi(s) := e^{-\lambda s}$ when $\mathcal{F}_t^{\pm}$ are defined via (5.5) and (5.6). Indeed, if $G_{t,s} := \{N_{t+s} = N_t\}$ for $t, s \geq 0$, then $\mathbb{P}[G_{t,s}] = e^{-\lambda s}$ and events E and $F \cap G_{t,s}^c$ are independent whenever $E \in \mathcal{F}_t^-$ and $F \in \mathcal{F}_{t+s}^+$ (see below). This includes $F = \Omega$, which yields for general $E \in \mathcal{F}_t^-$ and $F \in \mathcal{F}_{t+s}^+$,

$$0 \leq \mathbb{P}[F \cap G_{t,s} \cap E] \leq \mathbb{P}[G_{t,s} \cap E] = \mathbb{P}[G_{t,s}]\mathbb{P}[E].$$

Therefore $|\mathbb{P}[F \cap G_{t,s}|E] - P[F \cap G_{t,s}]| \leq \mathbb{P}[G_{t,s}]$ and so

$$|\mathbb{P}[F|E] - P[F]| \leq \left|\mathbb{P}[F \cap G_{t,s}^c|E] - P[F \cap G_{t,s}^c]\right| + \mathbb{P}[G_{t,s}] = e^{-\lambda s}.$$

The discussion before Example 5.2 therefore shows that Theorem 1.2 applies to $X_{m,n}^{t,e}$ above for any $e \in \mathbb{Z}^d$, so $\frac{1}{n} X_{0,n}^{0,e}$ converges to some ω-independent constant almost surely. And just as before, we can again also conclude (5.8) and (5.9).

It remains to prove independence of E and $F \cap G_{t,s}^c$ for any $E \in \mathcal{F}_t^-$ and $F \in \mathcal{F}_{t+s}^+$. Let us denote $v_0, v_1, \dots$ all the edges in V_d and for $m, J \in \mathbb{N}_0$ let $Y_m^J(\omega) := (\eta(v_0, \omega_m), \dots, \eta(v_J, \omega_m))$. By Dynkin's π-λ Theorem, it suffices to show that $\mathbb{P}[E \cap F \cap G_{t,s}^c] = \mathbb{P}[E]\mathbb{P}[F \cap G_{t,s}^c]$ for

$$E = \left\{Y_{N_{t_i}}^J \in A_i \text{ for } i = 1, \dots, n\right\} \qquad \text{and} \qquad F = \left\{Y_{N_{t_i}}^J \in A_i \text{ for } i = n+1, \dots, 2n\right\},$$

with arbitrary $J \in \mathbb{N}_0$, Borel sets $A_1, \dots, A_{2n} \subseteq \mathbb{R}^J$, and times

$$0 \leq t_1 < \cdots < t_n = t < t + s = t_{n+1} < \cdots < t_{2n}.$$

Note that $N_{t_i} \geq N_{t_{i-1}}$ for all i (let $t_0 := 0$, so $N_{t_0} \equiv 0$), and for any $k_1, \dots, k_{2n} \in \mathbb{N}_0$ we have

$$\mathbb{P}\left[N_{t_i} - N_{t_{i-1}} = k_i \text{ for } i = 1, \dots, 2n\right] = \prod_{i=1}^{2n} \frac{(\lambda(t_i - t_{i-1}))^{k_i}}{k_i!} e^{-\lambda(t_i - t_{i-1})} =: \prod_{i=1}^{2n} p_{i,k_i}$$

(clearly $\sum_{k \in \mathbb{N}_0} p_{i,k} = 1$). Since $G_{t,s}^c = \{N_{t_{n+1}} > N_{t_n}\}$, with $K_2 := (k_{n+1}, \dots, k_{2n})$ we obtain

$$\begin{aligned}
\mathbb{P}\left[F \cap G_{t,s}^c\right] &= \sum_{K_2 \in \mathbb{N} \times \mathbb{N}_0^{n-1}} \mathbb{P}\left[Y_{N_{t_n} + \sum_{j=n+1}^i k_i}^J \in A_i \,\&\, N_{t_i} - N_{t_{i-1}} = k_i \text{ for } i = n+1, \dots, 2n\right] \\
&= \sum_{K_2 \in \mathbb{N} \times \mathbb{N}_0^{n-1}} \left(\prod_{i=n+1}^{2n} p_{i,k_i}\right) \mathbb{P}\left[Y_{\sum_{j=n+1}^i k_i}^J \in A_i \text{ for } i = n+1, \dots, 2n\right]
\end{aligned}$$

because the σ-algebras $\mathcal{F}' \times \{\emptyset, \Omega_0^{\mathbb{N}_0}\}$ and $\{\emptyset, \Omega'\} \times \mathcal{F}_0^{\mathbb{N}_0}$ are independent, random variables $\{N_{t_i} - N_{t_{i-1}}\}_{i=1,\dots,2n}$ are jointly independent, and the joint distribution of $\{Y_m^J, Y_{m+1}^J, \dots\}$ is independent of m. But then with $K_1 := (k_1, \dots, k_n)$ we similarly obtain the desired claim

$$\begin{aligned}
\mathbb{P}\left[E \cap F \cap G_{t,s}^c\right] &= \sum_{(K_2,K_1)\in\mathbb{N}\times\mathbb{N}_0^{2n-1}} \mathbb{P}\left[Y^J_{\sum_{j=1}^i k_i} \in A_i \,\&\, N_{t_i} - N_{t_{i-1}} = k_i \text{ for } i = 1, \dots, 2n\right] \\
&= \sum_{(K_2,K_1)\in\mathbb{N}\times\mathbb{N}_0^{2n-1}} \left(\prod_{i=1}^{2n} p_{i,k_i}\right) \mathbb{P}\left[Y^J_{\sum_{j=1}^i k_i} \in A_i \text{ for } i = 1, \dots, 2n\right] \\
&= \sum_{K_1\in\mathbb{N}_0^n} \left(\prod_{i=1}^{n} p_{i,k_i}\right) \mathbb{P}\left[Y^J_{\sum_{j=1}^i k_i} \in A_i \text{ for } i = 1, \dots, n\right] \\
&\qquad \sum_{K_2\in\mathbb{N}\times\mathbb{N}_0^{n-1}} \left(\prod_{i=n+1}^{2n} p_{i,k_i}\right) \mathbb{P}\left[Y^J_{\sum_{j=1}^i k_i} \in A_i \text{ for } i = 1, \dots, 2n\right] \\
&= \sum_{K_1\in\mathbb{N}_0^n} \left(\prod_{i=1}^{n} p_{i,k_i}\right) \mathbb{P}\left[Y^J_{\sum_{j=1}^i k_i} \in A_i \text{ for } i = 1, \dots, n\right] \\
&\qquad \sum_{K_2\in\mathbb{N}\times\mathbb{N}_0^{n-1}} \left(\prod_{i=n+1}^{2n} p_{i,k_i}\right) \mathbb{P}\left[Y^J_{\sum_{j=n+1}^i k_i} \in A_i \text{ for } i = n+1, \dots, 2n\right] \\
&= \mathbb{P}[E]\,\mathbb{P}\left[F \cap G_{t,s}^c\right],
\end{aligned}$$

where we also used $k_{n+1} \geq 1$ in the third equality.

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