UNIQUENESS OF POSITIVE VORTICITY SOLUTIONS TO THE 2D EULER EQUATIONS ON SINGULAR DOMAINS

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ABSTRACT. We show that particle trajectories for positive vorticity solutions to the 2D Euler equations on fairly general bounded simply connected domains cannot reach the boundary in finite time. This includes domains with possibly nowhere C^1 boundaries and having corners with arbitrary angles, and can fail without the sign hypothesis when the domain has large angle corners. Hence positive vorticity solutions on such domains are Lagrangian, and we also obtain their uniqueness if the vorticity is initially constant near the boundary.

1. Introduction and Main Results

In this paper we study the Euler equations

$$\partial_t u + (u \cdot \nabla)\, u = -\nabla p, \tag{1.1}$$

$$\nabla \cdot u = 0 \tag{1.2}$$

on simply connected bounded open domains $\Omega \subseteq \mathbb{R}^2$ with singular boundaries and at times $t > 0$, with u the fluid velocity and p its pressure. These PDE model the motion of two-dimensional ideal fluids and it is standard to assume the *no-flow* (or *slip*) boundary condition

$$u \cdot n = 0 \tag{1.3}$$

on $(0,\infty) \times \partial\Omega$ (pointwise when $\partial\Omega$ is C^1), with n being the unit outer normal to Ω. These PDE can be equivalently reformulated as the active scalar equation

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{1.4}$$

on $(0,\infty) \times \Omega$, with

$$\omega := \nabla \times u = \partial_{x_1} u_2 - \partial_{x_2} u_1$$

being the *vorticity* of u. This of course means that the velocity in (1.4) is uniquely determined from the vorticity via $u = \nabla^\perp \Delta^{-1} \omega$, and we can then call ω a solution rather than u.

A natural class of solutions are those with bounded ω [16], and we provide the definition of weak solutions from this *Yudovich class* at the start of the next section. We will only consider such solutions on the time interval $(0,\infty)$, when they are called *global weak solutions*, because they exist for all initial values $\omega_0 \in L^\infty(\Omega)$ on very general domains Ω [5] (nevertheless, our results equally apply to solutions on finite time intervals $(0,T)$).

It is well known that the velocity u is spatially log-Lipschitz on each compact $K \subseteq \Omega$ when ω is bounded (uniformly in time, see (2.1) below). Hence for each $x \in \Omega$ there is a unique solution to the ODE

$$\frac{d}{dt} X_t^x = u(t, X_t^x) \qquad \text{and} \qquad X_0^x = x \tag{1.5}$$

on an interval $(0, t_x)$ such that

$$t_x := \sup\{t > 0 \,|\, X_s^x \in \Omega \text{ for all } s \in (0,t)\}$$

(so if X_t^x reaches $\partial\Omega$, then t_x is the first such time). That is, $\{X_t^x\}_{t\in[0,t_x)}$ is the Euler particle trajectory for the particle starting at $x \in \Omega$. We note that while a priori the ODE only holds for almost all $t \in (0, t_x)$ (with X_t^x being continuous in time), u can be shown to be continuous when ω is bounded, so that (1.5) in fact holds for each $t \in [0, t_x)$ (see Subsection 2.1 below). Since (1.4) is a transport equation, it is then natural to ask whether general weak solutions are transported by u in the sense that $\omega(t, X_t^x) = \omega_0(x)$ for a.e. $t \in (0,\infty)$ and a.e. $x \in \Omega$ such that $t_x > t$. This is indeed the case [7, Lemma 3.1], but that does not a priori exclude the possibility of vorticity creation and depletion on $\partial\Omega$ unless $t_x = \infty$ for a.e. $x \in \Omega$ (then $\nabla \cdot u \equiv 0$ shows that $|\Omega \setminus \{X_t^x \,|\, x \in \Omega \text{ and } t_x > t\}| = 0$). If both these properties hold, so that $\omega(t,\cdot)$ is the push-forward of ω_0 via X_t^x for each $t \in (0,\infty)$, we call such ω a *Lagrangian solution*. It is currently an open question whether non-Lagrangian solutions can exist on (sufficiently singular) two-dimensional domains.

Existence of non-Lagrangian solutions would imply non-uniqueness of weak solutions. But even if all weak solutions are Lagrangian on some domain, this does not immediately yield their uniqueness. In fact, while weak solutions are known to be unique on rectangles [2] and on domains that are $C^{1,1}$ except at finitely many corners that are all exact acute angle sectors [4, 11], this remains an open question on more singular domains. The main issue is that the velocity typically is not log-Lipschitz near corners with angles greater than $\frac{\pi}{2}$, which removes a crucial ingredient from the proof of uniqueness. However, one can sometimes obtain a partial result via [12, Proposition 3.2], which shows that a Lagrangian solution remains unique as long as it remains constant near the singular portion of $\partial\Omega$. In particular, if ω_0 is constant near $\partial\Omega$ and each Euler particle trajectory associated with a corresponding solution ω can be shown to never reach $\partial\Omega$ (in which case $\operatorname{dist}(\{X_t^x \,|\, x \in K\}, \partial\Omega) > 0$ for any compact $K \subseteq \Omega$ and any $t > 0$), then ω is the unique Lagrangian solution with initial value ω_0. And if all weak solutions within some class are proved to be Lagrangian, this will yield uniqueness of ω in that class. We note that a similar idea was used in [1] to prove uniqueness of solutions on a sector in $\mathbb{R}^2$ with an obtuse angle, for initial data $\omega_0 \geq 0$ that are supported in some sub-sector (and hence ω_0 need not be constant near the obtuse corner). If that sub-sector does not extend all the way to the left side of the sector, $\operatorname{supp} \omega(t,\cdot)$ will be instantly carried away from the corner and any solution will vanish in its vicinity for each $t > 0$, which [1] was then able to leverage to obtain uniqueness of such solutions.

The above approach was successfully used by Lacave for solutions $\omega \geq 0$ on domains that are $C^{1,1}$ except at finitely many corners that are all exact sectors with angles $> \frac{\pi}{2}$ [10], and later without the sign restriction on ω by Lacave and the second author on domains that are $C^{1,1}$ except at finitely many corners with angles in $(0, \pi)$ [12], as well as by both authors on domains with much lower boundary regularity [7] (including infinitely many corners with angles in $(0, \pi)$). The latter paper in fact contains a sharp criterion for the geometry of $\partial\Omega$ that guarantees that no weak solution has a particle trajectory that reaches $\partial\Omega$ in finite time. This criterion is slightly stronger than exclusion of corners with angles $> \pi$ (in particular, it is satisfied by all convex domains), and it was demonstrated in [9, 12] that particle trajectories for bounded ω on domains that do have such corners can reach $\partial\Omega$ in finite time.

The examples in [9, 12] all involve sign-changing solutions, so in view of [10] it is natural to ask whether signed solutions can exhibit such singular behavior on more irregular domains than those considered in [10]. The main result of the present paper is a negative answer to this question on much more general domains, allowing both infinitely many corners without size or shape restrictions and considerably less boundary smoothness in-between them. In particular, we show that positive (and then obviously also negative) weak solutions on such domains are Lagrangian, with particle trajectories approaching $\partial\Omega$ no faster than double-exponentially, and that these solutions are also unique when ω_0 is constant near $\partial\Omega$.

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open Lipschitz domain with $\partial\Omega$ a Jordan curve, let $L := |\partial\Omega|$ be the arc-length of $\partial\Omega$, and let $\sigma : [0, L] \to \mathbb{C}$ be a counter-clockwise arc-length parametrization of $\partial\Omega$ (so $\sigma(L) = \sigma(0)$). For any $\theta \in [0, L)$, the *unit forward tangent vector* to Ω at $\sigma(\theta)$ is the unit vector

$$\bar{\tau}(\theta) := \lim_{\phi \to \theta+} \frac{\sigma(\phi) - \sigma(\theta)}{|\sigma(\phi) - \sigma(\theta)|}, \tag{1.6}$$

provided the limit exists (we also let $\bar{\tau}(L) := \bar{\tau}(0)$). If it does for each $\theta \in [0, L)$, and $\bar{\tau}$ has one-sided limits everywhere on $[0, L]$, then $\bar{\tau}$ is a *regulated function* (see [3, p.145]) and the domain Ω is also called *regulated* (see [13, p.59]). In that case $\bar{\tau}$ is right-continuous, and if we identify $\mathbb{R}^2$ with $\mathbb{C}$ and let $\arg(z) \in (-\pi, \pi]$ for $z \neq 0$, then

$$\bar{\alpha}(\theta) := \arg\left(\frac{\bar{\tau}(\theta)}{\lim_{\phi \to \theta-} \bar{\tau}(\phi)}\right) \in (-\pi, \pi) \tag{1.7}$$

for $\theta \in (0, L]$ is such that $\pi - \bar{\alpha}(\theta)$ is the *interior angle* of Ω at $\sigma(\theta)$. Note that $\bar{\alpha}(0)$ is not defined, and $\bar{\alpha}(\theta) \in (-\pi, \pi)$ for $\theta \in (0, L]$ because Ω is Lipschitz.

So corners of Ω are precisely the points $\sigma(\theta)$ with $\theta \in (0, L]$ and $\bar\alpha(\theta) \neq 0$, and regulated domains clearly have countably many of them. If also $\sum_{\theta\in(0,L]} |\bar\alpha(\theta)| < \infty$, then

$$\bar\beta_c(\theta) := \bar{\arg}\,(\bar\tau(\theta)) - \sum_{\theta'\le\theta} \bar\alpha(\theta') \tag{1.8}$$

is a continuous function on $[0, L]$ provided we let $\bar{\arg}(\bar\tau(\theta))$ be the argument of $\bar\tau(\theta)$ plus an appropriate θ-dependent integer multiple of 2π (essentially, $\bar\beta_c$ is the argument of the curve that we obtain if we successively eliminate all corners $\sigma(\theta)$ by rotating $\sigma([\theta, L])$ around $\sigma(\theta)$ counterclockwise by angle $\alpha(\theta)$). We will also assume that $\bar\beta_c$ is *Dini continuous* on $[0, L]$, that is, it has a modulus of continuity $m : [0, L] \to [0, \infty)$ with $\int_0^L \frac{m(r)}{r} dr < \infty$ (i.e., $|\bar\beta_c(\theta) - \bar\beta_c(\theta')| \le m(|\theta - \theta'|)$ holds for all $\theta, \theta' \in [0, L]$). We recall that any Hölder modulus of continuity is also a Dini modulus, while m with $m(r) = \frac{1}{|\log r|}$ for $r \in (0, \frac12)$ is not. We can now state our main result.

Theorem 1.1. *Assume that a bounded open Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ with $\partial\Omega$ a Jordan curve is regulated. Let $\bar\tau$ be the forward tangent vector to Ω from (1.6), let $\bar\alpha$ be from (1.7), and assume that $\sum_{\theta\in(0,L]} |\bar\alpha(\theta)| < \infty$ and $\bar\beta_c$ from (1.8) is Dini continuous. Consider any $0 \le \omega_0 \in L^\infty(\Omega)$ and let $\omega \ge 0$ from the Yudovich class be any global weak solution to the Euler equations on Ω with initial value ω_0 (such ω is known to exist by [5]).*

(i) We have $t_x = \infty$ for all $x \in \Omega$ and $\{X_t^x \,|\, x \in \Omega\} = \Omega$ for all $t > 0$, and there is a constant $C_\omega < \infty$ such that for any $\varepsilon > 0$ and all large enough $t > 0$,

$$\sup_{\mathrm{dist}(x,\partial\Omega)\ge\varepsilon} \mathrm{dist}(X_t^x, \partial\Omega) \ge \exp(-e^{C_\omega t}). \tag{1.9}$$

Moreover, $\omega(t, X_t^x) = \omega_0(x)$ for a.e. $(t, x) \in (0, \infty) \times \Omega$ (i.e., ω is Lagrangian), and u is continuous on $[0, \infty) \times \Omega$ and (1.5) holds pointwise.

(ii) If $\mathrm{supp}\,(\omega_0 - a) \cap \partial\Omega = \varnothing$ for some $a \ge 0$, then ω is the unique non-negative weak solution with initial value ω_0.

Remarks. 1. Hence the well-known double-exponential bound on the rate of approach of particle trajectories to the boundaries of smooth domains (going back to [8,15]) still holds on the domains considered here, even though u can be far from log-Lipschitz near $\partial\Omega$ and even unbounded at corners with angles $> \pi$. A partial explanation is that $\omega \ge 0$ forces u to "circulate" around $\partial\Omega$ counter-clockwise, thus keeping any particle trajectory near any corner for only a short time during each passage through its neighborhood. However, our domains can even have everywhere singular boundaries (e.g., a dense set of corners), so all of $\partial\Omega$ could be the set of potential trouble spots rather than just a few individual corners.

2. Part (i) of this result suggests a natural open question: is there any planar domain Ω and a weak solution $\omega \ge 0$ to the Euler equation on it that has a particle trajectory starting

inside Ω and reaching $\partial\Omega$ in finite time? Of course, a second one is whether such solutions, if they exist, can fail to be Lagrangian (this is currently open even for unsigned ω).

Let us briefly discuss our approach and its relation to [7, 10, 12]. In all four papers, the central ingredient is a non-negative Lyapunov functional on $(0,\infty)\times\overline{\Omega}$ that vanishes only on $(0,\infty)\times\partial\Omega$ and its change on Euler particle trajectories can be controlled sufficiently well to show that it can never become 0 unless it is 0 initially. Lacave first chose this functional to be the stream function $\Psi := -\Delta^{-1}\omega$ of the fluid velocity u [10] because its rate of change in the flow direction u is 0. When ω does not have a sign, then Ψ can vanish inside Ω, and [7, 12] therefore used instead the time-independent function $1-|\mathcal{T}(x)|$, with $\mathcal{T}:\Omega\to\mathbb{D}$ a Riemann mapping. In the present paper we consider again solutions $\omega\ge 0$, and so revisit the idea of using the stream function. However, in Lemmas 2.2–2.5 we obtain sharper and more general estimates on Ψ and $\partial_t\Psi$ than [10], which allows us to include much more general domains, with arbitrary corners as well as considerably less regular boundaries overall.

In the next section we state these estimates and use them to prove Theorem 1.1, leaving the proofs of the estimates and of a formula for $\partial_t\Psi$ for the last two sections.

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2. Proof of Theorem 1.1

We complete the proof in three steps. We always assume that Ω satisfies the hypotheses from Theorem 1.1, and (ω,u) is a weak solution to (1.1)–(1.3) on $(0,\infty)\times\Omega$, as defined next.

2.1. **Weak solutions and space-time differentiability of the stream function.** We consider here weak solutions to (1.1)–(1.3) from the *Yudovich class*

$$\left\{(\omega,u)\in L^\infty\left((0,\infty);L^\infty(\Omega)\times L^2(\Omega)\right)\,\middle|\,\omega=\nabla\times u \text{ and (1.2)–(1.3) all hold weakly}\right\},$$

where the weak form of (1.2)–(1.3) is

$$\int_\Omega u(t,x)\cdot\nabla h(x)\,dx=0 \qquad \forall h\in H^1_{\rm loc}(\Omega)\text{ with }\nabla h\in L^2(\Omega)$$

for a.e. $t\in(0,\infty)$ (see [5, 6]). It is well-known that $\nabla\times u\in L^\infty((0,\infty)\times\Omega)$ implies that u is bounded and log-Lipschitz on any compact $K\subseteq\Omega$ at a.e. time $t\in(0,\infty)$ (and uniformly in these times), after possibly redefining it on a measure zero spatial set for each such t. If we also redefine u at the exceptional measure-zero set of times (and also at $t=0$), then for any compact $K\subseteq\Omega$ we will have

$$\sup_{t\ge 0}\sup_{x,y\in K}\left(|u(t,x)|+\frac{|u(t,x)-u(t,y)|}{|x-y|\max\{1,-\ln|x-y|\}}\right)<\infty \tag{2.1}$$

(this is also shown in the proof of Lemma 2.1 below). Let now $X_t^{s,x}$ for $(s,x) \in [0,\infty) \times \Omega$ be the unique continuous function satisfying

$$\frac{d}{dt} X_t^{s,x} = u(t, X_t^{s,x}) \qquad \text{and} \qquad X_s^{s,x} = x \tag{2.2}$$

a.e. on the maximal interval $I^{s,x} \subseteq [0,\infty)$ (containing s) such that $X_t^{s,x} \in \Omega$ for all $t \in I^{s,x} \setminus \partial I^{s,x}$. That is, $I^{s,x}$ is the (backward and forward) life-span of the particle trajectory $X_t^{s,x}$. Of course, $X_t^{0,x} = X_t^x$ and $I^{0,x} = [0, t_x]$ (or $[0,\infty)$ if $t_x = \infty$) for all $x \in \Omega$.

We say that (ω, u) from the Yudovich class is a *weak solution* to (1.1)–(1.3) on $(0,T) \times \Omega$ (for some $T \in (0,\infty]$) with some initial condition $\omega_0 \in L^\infty(\Omega)$, if

$$\int_0^T \int_\Omega \omega \left(\partial_t \varphi + u \cdot \nabla \varphi\right) \, dx dt = - \int_\Omega \omega_0(x) \varphi(0,x) \, dx \qquad \forall \, \varphi \in C_c^\infty \left([0,T) \times \Omega\right). \tag{2.3}$$

This is in fact the definition of a weak solution ω to the transport equation (1.4) when u is some given vector field, but it is also equivalent to the relevant weak velocity formulation of the Euler equations on Ω (see [6, Remark 1.2]). When $T = \infty$, we call such solutions *global*. Existence of a global weak solution is guaranteed by [5] for any $\omega_0 \in L^\infty(\Omega)$ on very general domains (while uniqueness is still open on most singular domains), and so for the sake of notational simplicity we will always assume that $T = \infty$.

Lemma 3.1 in [7] now shows that for a.e. $t \in (0,\infty)$, a weak solution (ω, u) satisfies $\omega(t, X_t^x) = \omega_0(x)$ for a.e. $x \in \Omega$ such that $t_x < t$. We can therefore redefine ω on a set of measure 0 so that $\omega(t, X_t^x) = \omega_0(x)$ holds for all $x \in \Omega$ and all $t \in (0, t_x)$. Let now $s_1 \in (0,\infty)$ be any Lebesgue point of ω as a function from $(0,\infty)$ to $L^1(\Omega)$. Replacing φ in (2.3) by $\varphi\psi_\varepsilon$, where $\psi_\varepsilon \in C^\infty([0,\infty))$ satisfies $\chi_{[s_1,\infty)} \le \psi_\varepsilon \le \chi_{[s_1-\varepsilon,\infty)}$, and taking $\varepsilon \to 0$ shows that (ω, u) is also a weak solution to (1.1)–(1.3) on $(s_1,\infty) \times \Omega$ with initial condition $\omega(s_1, \cdot)$ (i.e., (2.3) holds with $(0, \omega_0)$ replaced by $(s_1, \omega(s_1, \cdot))$. Doing the same with any $\varphi \in C_c^\infty\left((0,\infty) \times \Omega\right)$ and $\chi_{[0,s_1]} \le \psi_\varepsilon \le \chi_{[0,s_1+\varepsilon]}$ shows that (ω, u) is also a weak solution to (1.1)–(1.3) on $(0, s_1) \times \Omega$ with terminal condition $\omega(s_1, \cdot)$ (which becomes an initial condition if we reverse the direction of time and replace u by $-u$). This and Lemma 3.1 in [7] show that we can redefine ω on a set of measure 0 so that $\omega(t, X_t^{s_1,x}) = \omega(s_1, x)$ holds for all $x \in \Omega$ and all $t \in I^{s_1,x} \setminus \partial I^{s_1,x}$ (clearly the values on the curve $(t, X_t^{s_1,x})$ will not change for any x such that $0 \in I^{s_1,x}$). We can continue this way, with $s_2, s_3, \dots$ consecutively in place of s_1, where $\{s_j\}_{j\ge1}$ is dense in $(0,\infty)$. This allows us to change ω on a measure zero set so that for all $s \in [0,\infty)$ (and with $\omega(0,\cdot) := \omega_0$) we will from now have

$$\omega\left(t, X_t^{s,x}\right) = \omega(s,x) \qquad \forall x \in \Omega \text{ and } t \in I^{s,x} \setminus \partial I^{s,x}. \tag{2.4}$$

It is well known that since Ω is simply connected, ω from any weak solution (ω, u) uniquely defines the velocity u via its *stream function*

$$\Psi(t,\cdot) := -\Delta^{-1}\omega(t,\cdot)$$

for all $t \geq 0$ (the negative sign is chosen so that $\Psi \geq 0$ when $\omega \geq 0$). Namely, after redefinition of u on a measure zero set we have $u = -\nabla^\perp\Psi$, where $(v_1, v_2)^\perp := (-v_2, v_1)$ and $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. We can now use (2.4) to show that Ψ is space-time differentiable (we postpone the proof of this to the last section).

Lemma 2.1. *We have $\Psi \in C^1([0,\infty) \times K)$ for each compact $K \subseteq \Omega$, and $\nabla\Psi = u^\perp$ and*

$$\partial_t\Psi(t,x) = -\frac{1}{2\pi}\int_\Omega \left(\frac{\mathcal{T}(y)-\mathcal{T}(x)}{|\mathcal{T}(y)-\mathcal{T}(x)|^2} - \frac{\mathcal{T}(y)-\mathcal{T}(x)^*}{|\mathcal{T}(y)-\mathcal{T}(x)^*|^2}\right)^T D\mathcal{T}(y)\, u(t,y)\, \omega(t,y)dy \quad (2.5)$$

for each $(t,x) \in [0,\infty) \times \Omega$, where $\mathcal{T} : \Omega \to \mathbb{D}$ is any Riemann mapping.

Note that while (2.5) formally follows from the definition of Ψ and (1.4), we will mainly need to know later that Ψ is C^1. Since Lemma 2.1 now shows that $u = -\nabla^\perp\Psi$ is continuous on $[0,\infty) \times \Omega$, this version of u still satisfies (2.1). Since u is uniquely determined by ω, from now on we will refer to ω as a weak solution to (1.4) (with $u := \nabla^\perp\Delta^{-1}\omega$), instead of to (ω, u) as a weak solution to (1.1)–(1.3).

2.2. **Formulation on the unit disc via Riemann mapping.** Let next $\mathcal{T} : \Omega \to \mathbb{D}$ be a Riemann mapping as in Lemma 2.1, extended continuously to $\partial\Omega$, and let $\mathcal{S} := \mathcal{T}^{-1}$. We will now use $\mathcal{T}$ to rewrite u and $\partial_t\Psi$ in terms of integrals over $\mathbb{D}$. We have

$$\Psi(t,x) = -\left[\Delta^{-1}\omega(t,\cdot)\right](x) = -\frac{1}{2\pi}\int_\Omega \ln\frac{|\mathcal{T}(x)-\mathcal{T}(y)|}{|\mathcal{T}(x)-\mathcal{T}(y)^*||\mathcal{T}(y)|}\,\omega(t,y)dy, \quad (2.6)$$

and then

$$u(t,x) = -\nabla^\perp\Psi(t,x) = \frac{1}{2\pi}D\mathcal{T}(x)^T R(t,\mathcal{T}(x)) \quad (2.7)$$

for any $(t,x) \in [0,\infty) \times \Omega$, where

$$R(t,\xi) := \int_\mathbb{D}\left(\frac{\xi - z}{|\xi - z|^2} - \frac{\xi - z^*}{|\xi - z^*|^2}\right)^\perp \det D\mathcal{S}(z)\,\omega(t,\mathcal{S}(z))dz \quad (2.8)$$

for $(t,\xi) \in [0,\infty) \times \mathbb{D}$. We note that the second equality in (2.7) holds because $\mathcal{T} = (\mathcal{T}^1, \mathcal{T}^2)$ is analytic, which means that

$$D\mathcal{T} = \begin{pmatrix} \partial_{x_1}\mathcal{T}^1 & \partial_{x_2}\mathcal{T}^1 \\ \partial_{x_1}\mathcal{T}^2 & \partial_{x_2}\mathcal{T}^2 \end{pmatrix} = \begin{pmatrix} \partial_{x_1}\mathcal{T}^1 & \partial_{x_2}\mathcal{T}^1 \\ -\partial_{x_2}\mathcal{T}^1 & \partial_{x_1}\mathcal{T}^1 \end{pmatrix} \quad (2.9)$$

and so for any $v \in \mathbb{R}^2$ we have

$$\left(\begin{pmatrix} \partial_{x_1}\mathcal{T}^1 & \partial_{x_1}\mathcal{T}^2 \\ \partial_{x_2}\mathcal{T}^1 & \partial_{x_2}\mathcal{T}^2 \end{pmatrix} v \right)^{\perp} = \begin{pmatrix} \partial_{x_2}\mathcal{T}^2 & -\partial_{x_2}\mathcal{T}^1 \\ -\partial_{x_1}\mathcal{T}^2 & \partial_{x_1}\mathcal{T}^1 \end{pmatrix} v^{\perp},$$

Lemma 2.1 and $u \cdot \nabla\Psi \equiv 0$ now yield for any $x \in \Omega$ and $t \in [0, t_x)$,

$$\frac{d}{dt}\Psi(t, X_t^x) = -\frac{1}{2\pi}\int_\Omega \left(\frac{\mathcal{T}(y) - \mathcal{T}(X_t^x)}{|\mathcal{T}(y) - \mathcal{T}(X_t^x)|^2} - \frac{\mathcal{T}(y) - \mathcal{T}(X_t^x)^*}{|\mathcal{T}(y) - \mathcal{T}(X_t^x)^*|^2} \right)^T D\mathcal{T}(y)\, u(t,y)\, \omega(t,y) dy$$

(the parenthesis is replaced by $\frac{\mathcal{T}(y)}{|\mathcal{T}(y)|^2}$ when $\mathcal{T}(X_t^x) = 0$). If we substitute (2.7) here and use

$$D\mathcal{T}(y) D\mathcal{T}(y)^T = \det D\mathcal{T}(y)\, I_2 \tag{2.10}$$

(note that $\det D\mathcal{T} = (\partial_{x_1}\mathcal{T}^1)^2 + (\partial_{x_2}\mathcal{T}^1)^2 > 0$), after a change of variables we obtain

$$\frac{d}{dt}\Psi(t, X_t^x) = -\frac{1}{4\pi^2}\int_{\mathbb{D}} \left(\frac{z - \mathcal{T}(X_t^x)}{|z - \mathcal{T}(X_t^x)|^2} - \frac{z - \mathcal{T}(X_t^x)^*}{|z - \mathcal{T}(X_t^x)^*|^2} \right) \cdot R(t,z)\, \omega(t, \mathcal{S}(z)) dz.$$

Finally, from this and the identity

$$\left| \frac{z}{|z|^2} - \frac{w}{|w|^2} \right| = \frac{|z-w|}{|z|\,|w|} \tag{2.11}$$

for all $z, w \in \mathbb{C} \setminus \{0\}$ we see that (with the fraction below replaced by $\frac{1}{|z|}$ when $\mathcal{T}(X_t^x) = 0$)

$$\left| \frac{d}{dt}\Psi(t, X_t^x) \right| \le \frac{1}{4\pi^2}\int_{\mathbb{D}} \frac{|\mathcal{T}(X_t^x) - \mathcal{T}(X_t^x)^*|}{|z - \mathcal{T}(X_t^x)|\,|z - \mathcal{T}(X_t^x)^*|} |R(t,z)|\, |\omega(t, \mathcal{S}(z))| dz. \tag{2.12}$$

It will also be convenient to re-parametrize the forward tangent vector $\bar\tau$ to Ω to

$$\tau(\theta) := \lim_{\phi \to \theta+} \frac{\mathcal{S}(e^{i\phi}) - \mathcal{S}(e^{i\theta})}{|\mathcal{S}(e^{i\phi}) - \mathcal{S}(e^{i\theta})|},$$

with $\theta \in \mathbb{R}$. Then of course $\tau(\theta) = \bar\tau(\Gamma(e^{i\theta}))$ for all $\theta \in \mathbb{R}$, where $\Gamma := \left(\sigma|_{(0,L]}\right)^{-1} \circ \mathcal{S}$. We now let $\{\bar\theta_j\}_{j\ge1} \subseteq (0, L]$ be the set of all points such that Ω has a corner at $\sigma(\bar\theta_j)$, and define

$$\theta_j := \pi + \arg\left(-\Gamma^{-1}(\bar\theta_j)\right) \in (0, 2\pi] \qquad \text{and} \qquad \alpha_j := \frac{\bar\alpha(\bar\theta_j)}{\pi} \in (-1, 0) \cup (0, 1)$$

for $j \ge 1$. That is, Ω has corners at $\{\mathcal{S}(e^{i\theta_j})\}_{j\ge1}$ with angles $\{\pi - \pi\alpha_j\}_{j\ge1}$. Then we define

$$\beta_c(\theta) := \bar\beta_c(\Gamma(e^{i\theta})) \qquad \text{and} \qquad \beta_d(\theta) := \pi \sum_{\theta_j \le \theta} \alpha_j$$

for $\theta \in (0, 2\pi]$ and extend these two functions to $\mathbb{R}$ so that for all $\theta \in \mathbb{R}$ we have

$$\beta_c(\theta + 2\pi) = \beta_c(\theta) + 2\pi\kappa \qquad \text{and} \qquad \beta_d(\theta + 2\pi) = \beta_d(\theta) + 2\pi(1 - \kappa),$$

where $\kappa := \frac{\bar\beta_c(L) - \bar\beta_c(0)}{2\pi}$ (which means that $\sum_{\theta\in(0,L]} \bar\alpha(\theta) = 2\pi(1-\kappa)$). Then of course β_c is continuous, β_d is piecewise constant, and $\beta := \beta_c + \beta_d$ is the argument of τ in the sense that $e^{i\beta(\theta)} = \tau(\theta)$ for all $\theta \in \mathbb{R}$ (we also have $\beta(\theta + 2\pi) = \beta(\theta) + 2\pi$).

Lemma 1 in [14] shows that Γ and Γ^{-1} are both Hölder continuous, which means that β_c is Dini continuous because $\bar{\beta}_c$ is. Indeed, if $\bar{m}$ is a modulus of continuity for $\bar{\beta}_c$, then β_c has modulus of continuity $m(r) := \bar{m}(Cr^\gamma)$ for some $C, \gamma > 0$, and a simple change of variables shows that $\int_0^1 \frac{\bar{m}(Cr^\gamma)}{r} dr < \infty$ if and only if $\int_0^1 \frac{\bar{m}(r)}{r} dr < \infty$.

We next state the following important formula for $\det D\mathcal{S}$.

Lemma 2.2. *We have*

$$\det D\mathcal{S}(z) = \det D\mathcal{S}(0)\, \Pi_{j\geq 1}|z - e^{i\theta_j}|^{-2\alpha_j} \exp\left(-\frac{2}{\pi}\int_0^{2\pi} \operatorname{Im} \frac{z}{e^{i\theta} - z} \left(\beta_c(\theta) - \kappa\theta\right) d\theta\right)$$

for each $z \in \mathbb{D}$ (this holds even without β_c being Dini continuous), as well as

$$\sup_{z\in\mathbb{D}} \left| \int_0^{2\pi} \operatorname{Im} \frac{z}{e^{i\theta} - z} \left(\beta_c(\theta) - \kappa\theta\right) d\theta \right| < \infty.$$

Proof. Since $\mathcal{S}$ is analytic, $\det D\mathcal{S}(z) = |\mathcal{S}'(z)|^2$, where $\mathcal{S}'$ is the complex derivative when $\mathcal{S}$ is considered as a function on $\mathbb{C}$. Since Ω is regulated, Theorem 3.15 in [13] shows that

$$\mathcal{S}'(z) = |\mathcal{S}'(0)| \exp\left(\frac{i}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\left(\beta(\theta) - \theta - \frac{\pi}{2}\right) d\theta\right)$$

for all $z \in \mathbb{D}$, and from $\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta = 2\pi \in \mathbb{R}$ and $\operatorname{Im} \frac{e^{i\theta}+z}{e^{i\theta}-z} = 2\operatorname{Im} \frac{z}{e^{i\theta}-z}$ we get

$$\det D\mathcal{S}(z) = \det D\mathcal{S}(0) \exp\left(-\frac{2}{\pi}\operatorname{Im}\int_0^{2\pi} \frac{z}{e^{i\theta} - z} \left(\beta(\theta) - \theta\right) d\theta\right) \tag{2.13}$$

(note that $\beta(\theta) - \theta$ is 2π-periodic). We split the integral into two parts, one of which is

$$\int_0^{2\pi} \frac{z}{e^{i\theta} - z} \left(\beta_d(\theta) - (1-\kappa)\theta\right) d\theta = i\int_0^{2\pi} \ln(1 - ze^{-i\theta})\, d\left(\beta_d(\theta) - (1-\kappa)\theta\right),$$

where we used integration by parts. Since $\int_0^{2\pi} \ln(1 - ze^{-i\theta}) d\theta = \ln 1 = 0$, it follows that

$$\begin{aligned}\exp\left(-\frac{2}{\pi}\operatorname{Im}\int_0^{2\pi} \frac{z}{e^{i\theta} - z}\big(\beta_d(\theta) - (1-\kappa)\theta\big)d\theta\right) &= \exp\left(-\frac{2}{\pi}\int_0^{2\pi} \ln|e^{i\theta} - z|\, d\beta_d(\theta)\right)\\ &= \Pi_{j\geq 1}|z - e^{i\theta_j}|^{-2\alpha_j}.\end{aligned}$$

This and (2.13) prove the first claim.

Let $\tilde{\beta}(\theta) = \beta_c(\theta) - \kappa\theta$, which is also 2π-periodic. If β_c has a Dini modulus of continuity m, then $\tilde{\beta}$ has Dini modulus $\tilde{m}(r) := m(r) + |\kappa| r$. So for any $z \in \mathbb{D}$ and $\theta_z := \arg(z)$ we obtain using $\operatorname{Im} \frac{z}{e^{i(-\theta+\theta_z)}-z} = -\operatorname{Im} \frac{z}{e^{i(\theta+\theta_z)}-z}$ and $\left|\frac{z}{e^{i(\theta+\theta_z)}-z}\right| \leq \frac{\pi}{2|\theta|}$ for any $\theta \in \mathbb{R}$ the estimate

$$\left|\int_0^{2\pi} \operatorname{Im}\frac{z}{e^{i\theta} - z}\tilde{\beta}(\theta) d\theta\right| = \left|\int_{-\pi}^{\pi} \operatorname{Im}\frac{z}{e^{i(\theta+\theta_z)} - z}\left(\tilde{\beta}(\theta + \theta_z) - \tilde{\beta}(\theta_z)\right) d\theta\right| \leq \left|\frac{\pi}{2}\int_{-\pi}^{\pi} \frac{\tilde{m}(|\theta|)}{|\theta|} d\theta\right|.$$

Since this is finite, the second claim follows. ■

In view of (2.12), (2.8), and this lemma, of particular concern to us will be corners corresponding to $\alpha_j > 0$ (i.e., those with angles less than π; note that the velocity u on Ω in fact vanishes at these, while it may be infinite at the other corners). We therefore let $\alpha_j^+ := \max\{\alpha_j, 0\}$ and define $\beta_d^+(\theta) := \pi \sum_{\theta_j \le \theta} \alpha_j^+$ for all $\theta \in (0, 2\pi]$. We then extend β_d^+ to $\mathbb{R}$ so that $\beta_d^+(\theta + 2\pi) = \beta_d^+(\theta) + \pi \sum_{j\ge 1} \alpha_j^+$, and choose $\delta \in (0, \frac{1}{8}]$ such that

$$\frac{\beta_d^+(\theta + 3\delta) - \beta_d^+(\theta - 3\delta)}{\pi} \le \alpha_* := \frac{1 + \max_{j\ge 1} \alpha_j^+}{2} \tag{2.14}$$

for each $\theta \in \mathbb{R}$. Note that $\alpha_* \in [\frac{1}{2}, 1)$ because $\max_{j\ge1} \alpha_j^+ < 1$ by $\sum_{j\ge1} |\alpha_j| < \infty$.

2.3. **Estimates on the stream function and conclusion of the proof.** We now state the following three crucial estimates, whose proofs we postpone to the next section. In them, constants C_Ω and C'_Ω only depend on Ω.

Lemma 2.3. *There is $C_\Omega > 0$ such that for each $(t, \xi) \in [0, \infty) \times \mathbb{D}$ we have*

$$|\Psi(t, \mathcal{S}(\xi))| \le C_\Omega \|\omega(t, \cdot)\|_{L^\infty} (1 - |\xi|)^{2 \min\{1-\alpha_*, 1/4\}}.$$

Lemma 2.4. *If $\omega \ge 0$, then for each $(t, \xi) \in [0, \infty) \times \mathbb{D}$ we have*

$$\Psi(t, \mathcal{S}(\xi)) \ge \frac{1 - |\xi|}{100\pi} \int_{\mathbb{D}} \frac{(1 - |z|) \det D\mathcal{S}(z)}{\max\{|z - \xi|, 1 - |\xi|\}^2} \omega(t, \mathcal{S}(z)) dz.$$

Lemma 2.5. *There is $C'_\Omega > 0$ such that for each $(t, \xi) \in [0, \infty) \times (\mathbb{D} \setminus B(0, \frac{3}{4}))$ we have*

$$\int_{\mathbb{D}} \frac{|R(t, z)|}{|z - \xi||z - \xi^*|} dz \le C'_\Omega |\ln(1 - |\xi|)| \left(\int_{\mathbb{D}} \frac{(1 - |z|) \det D\mathcal{S}(z)}{\max\{|z - \xi|, 1 - |\xi|\}^2} |\omega(t, \mathcal{S}(z))| dz + \|\omega(t, \cdot)\|_{L^\infty} \right).$$

Remarks. 1. Lemmas 2.3 and 2.4 are sharper and more general versions of Lemmas 3.1 and 3.2 in [10]. Our use of Lemma 2.5 to estimate $\partial_t \Psi$ is analogous to the use of Proposition 2.4 and Lemma 3.5 in [10], but instead of bounding $|R|$ above by essentially $\|\omega\|_{L^\infty}$ and leaving ω as a function, we bound ω by $\|\omega\|_{L^\infty}$ and leave $|R|$ in (2.12). This is because for the domains Ω considered here, R can blow up at $\partial\mathbb{D}$ (see (4.2) below). In particular, this happens at corners with angles $\le \frac{\pi}{2}$, which is why such corners had to be excluded in [10].

2. Lemma 2.5 easily extends to $\xi \in B(0, \frac{3}{4})$ but we will not need this.

From now assume also that $\omega \ge 0$. Since $(1 - |z|) \det D\mathcal{S}(z)$ is bounded below by a positive constant on $B(0, r)$ for any $r < 1$ due to Lemma 2.2, for any $a > 0$ there is $c_a > 0$ such that

$$\int_{\mathbb{D}} \frac{(1 - |z|) \det D\mathcal{S}(z)}{\max\{|z - \xi|, 1 - |\xi|\}^2} |\omega(t, \mathcal{S}(z))| dz \ge c_a \|\omega(t, \cdot)\|_{L^\infty}$$

whenever $\|\omega(t,\cdot)\|_{L^1} \geq a\|\omega(t,\cdot)\|_{L^\infty}$. From this, the above lemmas, and (2.12) it follows that when $|\mathcal{T}(X_t^x)| \geq \frac{3}{4}$ (in which case also $|\mathcal{T}(X_t^x) - \mathcal{T}(X_t^x)^*| \leq 3(1-|\mathcal{T}(X_t^x)|)$), then we have

$$\begin{aligned} \left|\frac{d}{dt}\Psi(t,X_t^x)\right| &\leq \frac{75C'_\Omega}{\pi}\frac{1+c_a}{c_a}\|\omega(t,\cdot)\|_{L^\infty}|\ln(1-|\mathcal{T}(X_t^x)|)|\Psi(t,X_t^x) \\ &\leq C_{a,\Omega}\|\omega(t,\cdot)\|_{L^\infty}\Psi(t,X_t^x)\left|\ln\frac{\Psi(t,X_t^x)}{C_\Omega\|\omega(t,\cdot)\|_{L^\infty}}\right|, \end{aligned} \tag{2.15}$$

where $C_{a,\Omega} > 0$ is some constant that only depends on (a,Ω).

For each $\varepsilon > 0$ let $\Omega_\varepsilon := \Omega \setminus \bigcup_{x\in\partial\Omega} B(x,\varepsilon)$. For each $\varepsilon > 0$ such that $\Omega_{2\varepsilon} \neq \varnothing$, let

$$T_\varepsilon := \operatorname{dist}(\Omega_{2\varepsilon}, \Omega\setminus\Omega_\varepsilon)\|u\|^{-1}_{L^\infty((0,\infty)\times\Omega_\varepsilon)} > 0.$$

Then $X_t^x \in \Omega_\varepsilon$ for all $(t,x) \in [0,T_\varepsilon]\times\Omega_{2\varepsilon}$, and therefore (2.4) yields $\omega(t,X_t^x) = \omega_0(x)$ for all $(t,x) \in [0,T_\varepsilon]\times\Omega_{2\varepsilon}$. Taking $\varepsilon \to 0$ we obtain

$$\|\omega_0\|_{L^\infty} \leq \liminf_{t\to 0}\|\omega(t,\cdot)\|_{L^\infty} \leq \|\omega\|_{L^\infty},$$

and then from $\nabla\cdot u \equiv 0$ also

$$\|\omega(t,\cdot)\|_{L^1} \geq \|\omega_0\|_{L^1(\Omega_{2\varepsilon})} \geq \|\omega_0\|_{L^1} - |\Omega\setminus\Omega_{2\varepsilon}|\,\|\omega_0\|_{L^\infty} \geq \|\omega_0\|_{L^1} - |\Omega\setminus\Omega_{2\varepsilon}|\,\|\omega\|_{L^\infty} \tag{2.16}$$

for each $\varepsilon > 0$ and all $t \in [0,T_\varepsilon]$.

If now $\omega_0 \not\equiv 0$, let $a := \frac{1}{2}\|\omega_0\|_{L^1}\|\omega\|^{-1}_{L^\infty} > 0$ and let $\varepsilon > 0$ be such that $|\Omega\setminus\Omega_{2\varepsilon}| \leq a$. From (2.16) we obtain

$$\|\omega(t,\cdot)\|_{L^1} \geq a\|\omega\|_{L^\infty} \geq a\|\omega(t,\cdot)\|_{L^\infty}$$

for all $t \in [0,T_\varepsilon]$. Thus (2.15) yields

$$\left|\frac{d}{dt}\Psi(t,X_t^x)\right| \leq C_{a,\Omega}\|\omega\|_{L^\infty}\Psi(t,X_t^x)\left|\ln\frac{\Psi(t,X_t^x)}{C_\Omega\|\omega\|_{L^\infty}}\right| \tag{2.17}$$

for all $(t,x) \in [0,T_\varepsilon]\times\Omega$ such that $|\mathcal{T}(X_t^x)| \geq \frac{3}{4}$. This and Gronwall's inequality show that $X_t^x \in \Omega$ for all $(t,x) \in [0,T_\varepsilon]\times\Omega$. Therefore $\omega(t,X_t^x) = \omega_0(x)$ for all $(t,x) \in [0,T_\varepsilon]\times\Omega$, and in particular, $\|\omega(T_\varepsilon,\cdot)\|_{L^1} = \|\omega_0\|_{L^1}$. We can therefore repeat this argument with the same a and ε on the time interval $[T_\varepsilon, 2T_\varepsilon]$, then on $[2T_\varepsilon, 3T_\varepsilon]$, etc.

It follows that ω is a Lagrangian solution to (1.4) on $(0,\infty)\times\Omega$ and $\|\omega(t,\cdot)\|_{L^p} = \|\omega_0\|_{L^p}$ for all $(t,p) \in [0,\infty)\times[1,\infty]$. Integrating (2.17) shows that there is a constant C'_ω (depending on $\|\omega_0\|_{L^\infty}, \|\omega_0\|_{L^1}, \Omega$) such that for each $\varepsilon > 0$ and all large enough $t > 0$ we have $\Psi(t,X_t^x) \geq \exp(-e^{C'_\omega t})$ whenever $\Psi(0,x) \geq \varepsilon$. Since Lemma 2.3 yields $C''_\omega > 0$ such that $\Psi(t,\mathcal{S}(\xi)) \leq C''_\omega(1-|\xi|)^{2\min\{1-\alpha_*,1/4\}}$ for all $(t,\xi) \in [0,\infty)\times\mathbb{D}$, and $\mathcal{T}$ is Hölder continuous on $\overline{\Omega}$ (see [14, Lemma 1]), this shows (1.9). Using also that (1.5) can clearly be solved backwards in time with the same estimate on the boundary approach rate, we find that $\{X_t^x \,|\, x \in \Omega\} = \Omega$, thus finishing the proof of Theorem 1.1(i) for $\omega_0 \not\equiv 0$.

If $\omega_0 \equiv 0$, then $\omega \equiv 0$ is clearly a Lagrangian solution to (1.4) on $(0,\infty) \times \Omega$ with $X_t^x = x$, which satisfies Theorem 1.1(i) except for (1.9). If $\omega \ge 0$ is a different global weak solution, then the above arguments with time 0 replaced by any $T' > 0$ such that $\omega(T', \cdot) \not\equiv 0$ show that for all $t \in (T', \infty)$ we have $\|\omega(t,\cdot)\|_{L^1} = \|\omega(T',\cdot)\|_{L^1}$. But then $\|\omega(t,\cdot)\|_{L^1}$ must be constant on the time interval (T'', ∞), where $T'' \in [0,\infty)$ is the infimum of times t with $\omega(t,\cdot) \not\equiv 0$ (and that constant is then positive). This contradicts continuity of ω as an $L^1(\Omega)$-valued function of time because $\omega(0,\cdot) = \omega_0 \equiv 0$ (note that as in [12], boundedness of ω shows that u is uniformly in time bounded on any compact subset of Ω, which together with (2.4) yields $\omega \in C([0,\infty); L^p(\Omega))$ for any $p \in [1,\infty)$).

Theorem 1.1(ii) follows immediately from Theorem 1.1(i) and Proposition 3.2 in [12]. We note that the latter result shows that Lagrangian solutions are unique as long as they remain constant near $\partial\Omega$ (more specifically, near the non-$C^{2,\gamma}$ portion of $\partial\Omega$ for some $\gamma > 0$).

3. Proofs of Lemmas 2.3–2.5

Let us first state an auxiliary technical result.

Lemma 3.1. *Let β be a (positive) measure on $\mathbb{R}$ and let $I := (\theta^* - 2\delta, \theta^* + 2\delta)$ for some $\theta^* \in \mathbb{R}$ and $\delta \in (0, \frac{\pi}{2}]$. Let $H \subseteq \mathbb{D}$ be an open region such that if $re^{i(\theta^*+\phi)} \in H$ for some $r \in (0,1)$ and $|\phi| \le \pi$, then $re^{i(\theta^*+\phi')} \in H$ whenever $|\phi'| \le |\phi|$ (i.e., H is symmetric and angularly convex with respect to the line connecting 0 and $e^{i\theta^*}$). If $F : [0,\infty) \to [0,\infty)$ is non-decreasing and convex, then*

$$\int_H f(z) F\left(g(z) + \frac{1}{\beta(I)} \int_I h(|z - e^{i\theta}|) d\beta(\theta) \right) dz \le \int_H f(z) F\left(g(z) + h(|z - e^{i\theta^*}|) \right) dz$$

holds for any non-increasing $h : (0,\infty) \to [0,\infty)$ and non-negative $f, g \in L^1(H)$ such that $f(re^{i(\theta^*+\phi')}) \ge f(re^{i(\theta^*+\phi)})$ and $g(re^{i(\theta^*+\phi')}) \ge g(re^{i(\theta^*+\phi)})$ whenever $r \in (0,1)$ and $|\phi'| \le |\phi|$.

The proof of this result is identical to that of Lemma 4.1 in [7], which was stated with $F(s) = s^\alpha$ for some $\alpha \ge 1$, because the only properties of F used in it were that it is non-decreasing and convex. We will be using it here with $F(s) := e^s$, $g \equiv 0$, and $h(s) := 2\beta(I) \ln_+ \frac{2}{s}$, so that for any β, I, H, f as above we have

$$\int_H f(z) \exp\left(-2 \int_I \ln |z - e^{i\theta}| d\beta(\theta) \right) dz \le \int_H f(z) |z - e^{i\theta^*}|^{-2\beta(I)} dz. \tag{3.1}$$

Since Lemmas 2.3–2.5 are all stated at a single time t, we will drop t from our notation in the proofs below. Hence we will have $\omega(x)$, $\Psi(x)$, and $R(z)$. For $z \in \mathbb{D}$ we will also denote

$$\Lambda(z) := \det D\mathcal{S}(z) \, |\omega(\mathcal{S}(z))| \ge 0.$$

We note that

$$\int_{\mathbb{D}} \Lambda(z)dz \le \|\omega\|_{L^\infty} \int_{\mathbb{D}} \det D\mathcal{S}(z)dz = |\Omega|\, \|\omega\|_{L^\infty}, \tag{3.2}$$

and that constants $C_1, C_2, \dots$ below will always be allowed to depend (only) on Ω.

3.1. **Proof of Lemma 2.4.** We have

$$\begin{aligned}\frac{|\xi - z|^2}{|\xi - z^*|^2|z|^2} &= 1 - \frac{|\xi z - z^* z|^2 - |\xi - z|^2}{|\xi - z^*|^2|z|^2} \\ &= 1 - \frac{(|\xi|^2|z|^2 - 2\mathrm{Re}\,(\xi\bar{z}) + 1) - (|\xi|^2 - 2\mathrm{Re}\,(\xi\bar{z}) + |z|^2)}{|\xi - z^*|^2|z|^2} \\ &= 1 - \frac{(1-|\xi|^2)(1-|z|^2)}{|\xi - z^*|^2|z|^2} \end{aligned} \tag{3.3}$$

for $\xi, z \in \mathbb{D}$ with $z \neq 0$, which also means that $\frac{|\xi - z|^2}{|\xi - z^*|^2|z|^2} \in (0,1)$ when $z \neq 0, \xi$. Hence

$$-\ln \frac{|\xi - z|}{|\xi - z^*||z|} = -\frac{1}{2}\ln\left(1 - \frac{(1-|\xi|^2)(1-|z|^2)}{|\xi - z^*|^2|z|^2}\right) \ge \frac{1}{2}\frac{(1-|\xi|^2)(1-|z|^2)}{|\xi - z^*|^2|z|^2},$$

and so for each $\xi \in \mathbb{D}$ we have

$$\Psi(\mathcal{S}(\xi)) \ge \frac{1}{4\pi}\int_{\mathbb{D}} \frac{(1-|\xi|^2)(1-|z|^2)}{|\xi - z^*|^2|z|^2}\Lambda(z)dz \ge \frac{1-|\xi|}{4\pi}\int_{\mathbb{D}} \frac{(1-|z|)}{|\xi - z^*|^2|z|^2}\Lambda(z)dz.$$

Given any $z, \xi \in \mathbb{D}$, let $M := \max\{|z-\xi|, 1-|\xi|\}$. Then $1 - |z| \le 1 - |\xi| + |z - \xi| \le 2M$, so

$$|\xi - z^*|\,|z| \le |\xi - z| + |z - z^*|\,|z| = |z - \xi| + 1 - |z|^2 \le |z - \xi| + 2(1-|z|) \le 5M$$

when $z \neq 0$, and the result follows.

3.2. **Proof of Lemma 2.3.** Identity (3.3) and $-\ln(1-r) \le (\frac{r}{1-r})^{\frac{1}{2}}$ for $r \in [0,1)$ (equality holds for $r = 0$ and the right-hand side has a larger derivative on $(0,1)$) show that

$$-\ln \frac{|\xi - z|}{|\xi - z^*||z|} \le \frac{1}{2}\left(\frac{\frac{(1-|\xi|^2)(1-|z|^2)}{|\xi - z^*|^2|z|^2}}{\frac{|\xi - z|^2}{|\xi - z^*|^2|z|^2}}\right)^{\frac{1}{2}} \le \frac{(1-|\xi|)^{\frac{1}{2}}(1-|z|)^{\frac{1}{2}}}{|\xi - z|}.$$

Hence it suffices to show that there is $C_1 > 0$ such that

$$\int_{\mathbb{D}} \frac{(1-|z|)^{\frac{1}{2}}}{|z - \xi|}\Lambda(z)dz \le C_1\|\omega\|_{L^\infty}(1-|\xi|)^{2\hat{\alpha}-\frac{1}{2}}, \tag{3.4}$$

where $\hat{\alpha} := \min\{1 - \alpha_*, \frac{1}{4}\}$ (note that $2\hat{\alpha} - \frac{1}{2} \le 0$). From (3.2) we see that it in fact suffices to replace $\mathbb{D}$ by $A_1 := B(\xi, \delta) \cap \mathbb{D}$ in (3.4).

Let us decompose A_1 into $A_2 := B(\xi, \varepsilon) \cap A_1$ with $\varepsilon := \frac{1-|\xi|}{2}$, $A_3 := B(\tilde{\xi}, \varepsilon) \cap A_1$ with $\tilde{\xi} := \frac{\xi}{|\xi|}$, and $A_4 := A_1 \backslash (A_2 \cup A_3)$. Now Lemma 2.2 and (3.1) with $H := A_1$, $I :=$

$(\arg(\xi)-2\delta, \arg(\xi)+2\delta)$, $f(z) := \frac{(1-|z|)^{1/2}}{|z-\xi|}$, and $\beta := \sum_{\theta_j\in I}\alpha_j^+\delta_{\theta_j}$, where δ_{θ_j} is the Dirac mass at θ_j, yield

$$\begin{aligned}\int_{A_1}\frac{(1-|z|)^{\frac12}}{|z-\xi|}\Lambda(z)dz \le C_2\|\omega\|_{L^\infty}\int_{A_1}\frac{(1-|z|)^{\frac12}}{|z-\xi|}\Pi_{\theta_j\in I}|z-e^{i\theta_j}|^{-2\alpha_j^+}dz\\ \le C_2\|\omega\|_{L^\infty}\int_{A_1}\frac{(1-|z|)^{\frac12}}{|z-\xi|}|z-\tilde\xi|^{-2\alpha_*}dz\\ \le C_2\|\omega\|_{L^\infty}\left(\int_{A_2}\frac{\varepsilon^{\frac12-2\alpha_*}}{|z-\xi|}dz+\int_{A_3}\frac{|z-\tilde\xi|^{\frac12-2\alpha_*}}{\varepsilon}dz+\int_{A_4}3^3|z-\xi|^{-\frac12-2\alpha_*}dz\right)\\ \le C_3\|\omega\|_{L^\infty}\varepsilon^{\frac32-2\alpha_*}\le 2C_3\|\omega\|_{L^\infty}(1-|\xi|)^{2\hat\alpha-\frac12}\end{aligned}$$

because (2.14) shows that $\sum_{\theta_j\in I}\alpha_j^+\le\alpha_*<1$. This therefore finishes the proof of (3.4).

3.3. **Proof of Lemma 2.5.** First integrate over $A_0 := \mathbb{D}\backslash B(\xi,\delta)$. Then (2.11), (3.2), and

$$|z-\tilde z^*|\ge|\tilde z^*|-1\ge\frac{|\tilde z-\tilde z^*|}{2}\ge 1-|\tilde z| \tag{3.5}$$

for any $z,\tilde z\in\mathbb{D}$ yield

$$\begin{aligned}\int_{A_0}\frac{|R(z)|}{|z-\xi||z-\xi^*|}dz &\le \frac{1}{\delta^2}\int_{A_0}\int_{\mathbb{D}}\frac{|\tilde z-\tilde z^*|}{|z-\tilde z||z-\tilde z^*|}\Lambda(\tilde z)d\tilde z dz\\ &\le\frac{2}{\delta^2}\int_{\mathbb{D}}\int_{A_0}\frac{dz}{|z-\tilde z|}\Lambda(\tilde z)d\tilde z\\ &\le\frac{4\pi}{\delta^2}\int_{\mathbb{D}}\Lambda(\tilde z)d\tilde z\\ &=\frac{4\pi|\Omega|}{\delta^2}\|\omega\|_{L^\infty}.\end{aligned}$$

So it remains to integrate over $A_1 := B(\xi,\delta)\cap\mathbb{D}$. From (3.5), $|\xi|-\delta\ge\frac58$, and (3.2) we have

$$\int_{A_1}\frac{1}{|z-\xi||z-\xi^*|}\int_{B(0,1/2)}\frac{|\tilde z-\tilde z^*|}{|z-\tilde z||z-\tilde z^*|}\Lambda(\tilde z)d\tilde z dz\le C_1|\ln(1-|\xi|)|\,\|\omega\|_{L^\infty}, \tag{3.6}$$

where we also used that with $B_\xi := B(\xi,\frac{|\xi-\xi'|}{2})\cap\mathbb{D}$ and $B_{\xi'} := B(\xi',\frac{|\xi-\xi'|}{2})\cap\mathbb{D}$ we have

$$\int_{\mathbb{D}}\frac{dz}{|z-\xi||z-\xi'|}\le 3\int_{\mathbb{D}\backslash(B_\xi\cup B_{\xi'})}\frac{dz}{|z-\xi|^2}+\frac{4}{|\xi-\xi'|}\int_{B_\xi}\frac{dz}{|z-\xi|}\le 6\pi\ln_+\frac{1}{|\xi-\xi'|}+50 \tag{3.7}$$

for any $\xi,\xi'\in\mathbb{C}$.

We now let $\varepsilon := 1-|\xi|$ and split A_1 into $A_2 := B(\xi,\frac{\varepsilon}{4})$ and $A_3 := A_1\setminus A_2$. We start with A_2, and let $E_1 := B(\xi,\frac{\varepsilon}{2})$ and $E_2 := \mathbb{D}\backslash(B(0,\frac12)\cup B(\xi,\frac{\varepsilon}{2}))$. We also denote

$M(\xi,z) := \max\{|z-\xi|, 1-|\xi|\}$. When $(z,\tilde z)\in A_2\times E_1$, then (3.5), $|z-\xi^*|\ge\varepsilon$, and (3.7) show that

$$\int_{A_2}\frac{1}{|z-\xi||z-\xi^*|}\int_{E_1}\frac{|\tilde z-\tilde z^*|}{|z-\tilde z||z-\tilde z^*|}\Lambda(\tilde z)d\tilde z dz \le \frac{2}{\varepsilon}\int_{E_1}\Lambda(\tilde z)\int_{A_2}\frac{dz}{|z-\xi||z-\tilde z|}d\tilde z$$
$$\le \frac{C_2}{\varepsilon}\int_{E_1}\Lambda(\tilde z)\,|\ln|\tilde z-\xi||\,d\tilde z.$$

From Lemma 2.2 and (2.14) we see that $\det D\mathcal{S}(\tilde z)\le C_3(1-|\tilde z|)^{-2}$ for some C_3 and all $\tilde z\in\mathbb{D}$, hence

$$\int_{B(\xi,\varepsilon^2)}\Lambda(\tilde z)\,|\ln|\tilde z-\xi||\,d\tilde z \le 4C_3\varepsilon^{-2}||\omega||_{L^\infty}\int_{B(\xi,\varepsilon^2)}|\ln|\tilde z-\xi||\,d\tilde z \le C_4||\omega||_{L^\infty}\varepsilon^2|\ln\varepsilon|.$$

From the last two estimates and $M(\xi,\tilde z)=\varepsilon\le 2(1-|\tilde z|)$ for $\tilde z\in E_1$ it now follows that

$$\int_{A_2}\frac{1}{|z-\xi||z-\xi^*|}\int_{E_1}\frac{|\tilde z-\tilde z^*|}{|z-\tilde z||z-\tilde z^*|}\Lambda(\tilde z)d\tilde z dz \le C_2C_4||\omega||_{L^\infty}\varepsilon|\ln\varepsilon| + \frac{C_2|\ln\varepsilon|}{\varepsilon}\int_{E_1}\Lambda(\tilde z)d\tilde z$$
$$\le C_5|\ln\varepsilon|\left(\int_{E_1}\frac{1-|\tilde z|}{M(\xi,\tilde z)^2}\Lambda(\tilde z)d\tilde z + ||\omega||_{L^\infty}\right).$$

Moreover, for all $(z,\tilde z)\in A_2\times E_2$ we have $|z-\tilde z^*|\ge|z-\tilde z|\ge\frac{|\tilde z-\xi|}{2}\ge\frac{1-|\xi|}{4}$ and $|\tilde z-\tilde z^*|\le 3(1-|\tilde z|)$, therefore

$$\int_{A_2}\frac{1}{|z-\xi||z-\xi^*|}\int_{E_2}\frac{|\tilde z-\tilde z^*|}{|z-\tilde z||z-\tilde z^*|}\Lambda(\tilde z)d\tilde z dz \le 48\int_{E_2}\frac{1-|\tilde z|}{M(\xi,\tilde z)^2}\Lambda(\tilde z)d\tilde z\int_{A_2}\frac{dz}{|z-\xi||z-\xi^*|}$$
$$\le C_6|\ln(1-|\xi|)|\int_{E_2}\frac{1-|\tilde z|}{M(\xi,\tilde z)^2}\Lambda(\tilde z)d\tilde z,$$

where we also used (3.7). The last two estimates and (3.6) show that

$$\int_{A_2}\frac{|R(z)|}{|z-\xi||z-\xi^*|}dz \le (C_1+C_5+C_6)|\ln(1-|\xi|)|\left(\int_{\mathbb{D}}\frac{1-|z|}{M(\xi,z)^2}\Lambda(z)dz + ||\omega||_{L^\infty}\right),$$

so it remains to integrate over A_3.

Let $F_1 := B(\xi,\frac{\varepsilon}{8})$, $F_2 := \mathbb{D}\setminus(B(0,\frac12)\cup B(\xi,2\delta))$, and $F_3 := (B(\xi,2\delta)\cap\mathbb{D})\setminus B(\xi,\frac{\varepsilon}{8})$. Then for all $(z,\tilde z)\in A_3\times F_1$ we have $|z-\tilde z^*|\ge|z-\tilde z|\ge\frac{1-|\xi|}{8}\ge|\tilde z-\xi|$ and $|\tilde z-\tilde z^*|\le 3(1-|\tilde z|)$, which together with (3.7) yields

$$\int_{A_3}\frac{1}{|z-\xi||z-\xi^*|}\int_{F_1}\frac{|\tilde z-\tilde z^*|}{|z-\tilde z||z-\tilde z^*|}\Lambda(\tilde z)d\tilde z dz \le 192\int_{F_1}\frac{1-|\tilde z|}{M(\xi,\tilde z)^2}\Lambda(\tilde z)d\tilde z\int_{A_3}\frac{dz}{|z-\xi^*||z-\xi|}$$
$$\le C_7|\ln(1-|\xi|)|\int_{F_1}\frac{1-|\tilde z|}{M(\xi,\tilde z)^2}\Lambda(\tilde z)d\tilde z.$$

And from (3.5), (3.2), and (3.7) we obtain

$$\int_{A_3} \frac{1}{|z-\xi||z-\xi^*|} \int_{F_2} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) d\tilde{z} dz \leq \frac{2}{\delta} \int_{F_2} \Lambda(\tilde{z}) d\tilde{z} \int_{A_3} \frac{dz}{|z-\xi||z-\xi^*|}$$
$$\leq C_8 |\ln(1-|\xi|)| \, ||\omega||_{L^\infty}.$$

For the integral involving $(z, \tilde{z}) \in A_3 \times F_3$, let $F_4 := F_3 \cap B(0, 1-\varepsilon^{\frac{1}{1-\alpha_*}})$ and for $\tilde{z} \in F_3$ let $A_{\tilde{z}} := B(\tilde{z}, \frac{|\tilde{z}-\xi|}{2}) \cap A_3$. From $|\xi|, |\tilde{z}| \geq \frac{1}{2}$ and (2.11) we get

$$|\tilde{z}-\xi^*| \leq |\tilde{z}-\tilde{z}^*| + 4|\tilde{z}-\xi| \leq |\tilde{z}-\tilde{z}^*| + 8|\tilde{z}-z| \leq 10|z-\tilde{z}^*|$$

when also $z \notin A_{\tilde{z}}$. This, (3.7), $|\tilde{z}-\tilde{z}^*| \leq 3(1-|\tilde{z}|)$, and $|\tilde{z}-\xi^*| \geq |\tilde{z}-\xi| \geq \frac{1-|\xi|}{8}$ for $\tilde{z} \in F_3$, and $|\tilde{z}-\xi^*| \geq \frac{|\tilde{z}-\xi|}{2}$ show that

$$\begin{aligned}
&\int_{A_3} \frac{1}{|z-\xi||z-\xi^*|} \int_{F_4} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) d\tilde{z} dz \\
&\qquad \leq 4 \int_{F_4} \int_{A_{\tilde{z}}} \frac{1}{|\tilde{z}-\xi||\tilde{z}-\xi^*|} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) dz d\tilde{z} \\
&\qquad\quad + 20 \int_{F_4} \int_{A_3 \setminus A_{\tilde{z}}} \frac{1}{|z-\xi||z-\xi^*|} \frac{|\tilde{z}-\tilde{z}^*|}{|\tilde{z}-\xi||\tilde{z}-\xi^*|} \Lambda(\tilde{z}) dz d\tilde{z} \\
&\qquad \leq C_9 \int_{F_4} \frac{|\tilde{z}-\tilde{z}^*|}{|\tilde{z}-\xi||\tilde{z}-\xi^*|} \Lambda(\tilde{z}) \left(|\ln(1-|\tilde{z}|)| + |\ln(1-|\xi|)| \right) d\tilde{z} \\
&\qquad \leq C_{10} |\ln(1-|\xi|)| \int_{F_4} \frac{1-|\tilde{z}|}{M(\xi, \tilde{z})^2} \Lambda(\tilde{z}) d\tilde{z}.
\end{aligned}$$

Finally, let $F_5 := F_3 \setminus F_4$. From (3.5), (3.7), Lemma 2.2, and (3.1) with $H := F_5$, $I := (\arg(\xi) - 3\delta, \arg(\xi) + 3\delta)$, $f \equiv 1$, and $\beta := \sum_{\theta_j \in I} \alpha_j^+ \delta_{\theta_j}$ we obtain

$$\begin{aligned}
\int_{A_3} \frac{1}{|z-\xi||z-\xi^*|} \int_{F_5} \frac{|\tilde{z}-\tilde{z}^*|}{|z-\tilde{z}||z-\tilde{z}^*|} \Lambda(\tilde{z}) d\tilde{z} dz \\
&\leq 2 \int_{F_5} \Lambda(\tilde{z}) \int_{A_3} \frac{dz}{|z-\xi||z-\xi^*||z-\tilde{z}|} d\tilde{z} \\
&\leq 8 \int_{F_5} \Lambda(\tilde{z}) \left(\int_{A_{\tilde{z}}} \frac{dz}{|\tilde{z}-\xi|^2|\tilde{z}-z|} + \int_{A_3 \setminus A_{\tilde{z}}} \frac{dz}{|z-\xi||z-\xi^*||\tilde{z}-\xi|} \right) d\tilde{z} \\
&\leq C_{11} \int_{F_5} \Lambda(\tilde{z}) \left(\frac{1}{|\tilde{z}-\xi|} + \frac{|\ln(1-|\xi|)|}{|\tilde{z}-\xi|} \right) d\tilde{z} \\
&\leq C_{12} \frac{|\ln(1-|\xi|)|}{\varepsilon} ||\omega||_{L^\infty} \int_{F_5} \Pi_{\theta_j \in I} |\tilde{z} - e^{i\theta_j}|^{-2\alpha_j^+} d\tilde{z} \\
&\leq C_{13} \frac{|\ln(1-|\xi|)|}{\varepsilon} ||\omega||_{L^\infty} \int_{F_5} \left| \tilde{z} - \frac{\xi}{|\xi|} \right|^{-2\alpha_*} d\tilde{z} \\
&\leq C_{14} |\ln(1-|\xi|)| \, ||\omega||_{L^\infty},
\end{aligned}$$

where in the last inequality we used that $|F_5| \leq \varepsilon^{\frac{1}{1-\alpha_*}}$, which is less than the area of a disc with radius $\varepsilon^{\frac{1}{2-2\alpha_*}}$. Combining the above estimates and (3.6) yields

$$\int_{A_3} \frac{|R(z)|}{|z-\xi||z-\xi^*|} dz \leq (C_1 + C_7 + C_8 + C_{10} + C_{14}) |\ln(1-|\xi|)| \left(\int_{\mathbb{D}} \frac{1-|z|}{M(\xi,z)^2} \Lambda(z) dz + ||\omega||_{L^\infty} \right),$$

and the result follows.

4. Proof of Lemma 2.1

We see from (2.7), a change of variables in the integral from (2.5), and (2.10) that we need to show boundedness and continuity of R and

$$Q(t,\xi) := \int_{\mathbb{D}} \left(\frac{z-\xi}{|z-\xi|^2} - \frac{z-\xi^*}{|z-\xi^*|^2} \right) \cdot R(t,z) \, \omega(t, \mathcal{S}(z)) dz$$

on $[0,\infty) \times K$ for any compact $K \subseteq \mathbb{D}$, as well as that $\partial_t \Psi(t,x) = -\frac{1}{2\pi} Q(t, \mathcal{T}(x))$ holds for each $(t,x) \in [0,\infty) \times \Omega$.

So fix any such K and let $d := \text{dist}(K, \partial\mathbb{D}) > 0$, then fix any $(t,\xi) \in [0,\infty) \times K$ and let $B := B(\xi, \frac{d}{2})$ and $B' := \overline{B(\xi, \frac{d}{4})}$. With $C_d := \sup_{|z| \leq 1-d/2} \det D\mathcal{S}(z)$, and using (2.11),

$|w-z^*| \ge |w-z|$ for all $z,w\in\mathbb{D}$, (3.7), and (3.2), we obtain for any $(t',\xi')\in[0,\infty)\times B'$,

$$\begin{aligned}|R(t,\xi)-R(t,\xi')| \le& ||\omega||_{L^\infty}\left(\int_B+\int_{\mathbb{D}\setminus B}\right)\left(\frac{|\xi-\xi'|}{|\xi-z||\xi'-z|}+\frac{|\xi-\xi'|}{|\xi-z^*||\xi'-z^*|}\right)\det D\mathcal{S}(z)dz\\ \le& 2||\omega||_{L^\infty}|\xi-\xi'|\left(6\pi C_d\ln_+\frac{1}{|\xi-\xi'|}+50C_d+\frac{8|\Omega|}{d^2}\right)\end{aligned}$$

and (using also $|z-z^*|\le 2|\xi'-z^*|$ and Hölder's inequality)

$$\begin{aligned}|R(t,\xi')-R(t',\xi')| \le& \int_{\mathbb{D}}\frac{|z-z^*|}{|\xi'-z||\xi'-z^*|}\det D\mathcal{S}(z)|\omega(t,\mathcal{S}(z))-\omega(t',\mathcal{S}(z))|dz\\ \le& 2\left(\int_{\mathbb{D}}|\xi'-z|^{-\frac{3}{2}}\det D\mathcal{S}(z)dz\right)^{\frac{2}{3}}||\omega(t,\cdot)-\omega(t',\cdot)||_{L^3(\Omega)}. \end{aligned} \tag{4.1}$$

(Note also that the first of these estimates and (4.2) below prove (2.1).) Since the last integral is bounded in $\xi'\in B'$ by Lemma 2.2 and (3.2), and $\omega\in C([0,\infty);L^p(\Omega))$ for any $p\in[1,\infty)$ (see the end of Section 2), these two estimates show that R is continuous at (t,ξ).

Boundedness of R on $[0,\infty)\times K$ follows from the estimate

$$|R(t,\xi)|\le C_\Omega||\omega||_{L^\infty}(1-|\xi|)^{1-2\alpha_*} \tag{4.2}$$

for all $(t,\xi)\in[0,\infty)\times\mathbb{D}$, with α_* from (2.14) and some Ω-dependent constant C_Ω. To obtain it, first note that $|z-z^*|\le 2|\xi-z^*|$ and (3.2) yield (with δ from (2.14))

$$\int_{\Omega\setminus B(\xi,\delta)}\frac{|z-z^*|}{|\xi-z||\xi-z^*|}\det D\mathcal{S}(z)dz\le\frac{2}{\delta}\int_{\Omega\setminus B(\xi,\delta)}\det D\mathcal{S}(z)dz\le\frac{2|\Omega|}{\delta}.$$

Then use Lemma 2.2, and (3.1) with $H:=B(\xi,\delta)$, $I:=(\arg(\xi)-2\delta,\arg(\xi)+2\delta)$, $f(z):=\frac{1}{|\xi-z|}$, and $\beta:=\sum_{\theta_j\in I}\alpha_j^+\delta_{\theta_j}$ to get (with $\varepsilon:=\frac{1-|\xi|}{2}$ and $\tilde\xi=\frac{\xi}{|\xi|}$)

$$\begin{aligned}\int_{B(\xi,\delta)}\frac{|z-z^*|}{|\xi-z||\xi-z^*|}\det D\mathcal{S}(z)dz\le& C'\int_{B(\xi,\delta)}\frac{|\tilde\xi-z|^{-2\alpha_*}}{|\xi-z|}dz\\ \le& C'\left(\int_{B(\xi,\varepsilon)}\frac{\varepsilon^{-2\alpha_*}}{|\xi-z|}dz+\int_{B(\tilde\xi,\varepsilon)}\frac{|\tilde\xi-z|^{-2\alpha_*}}{\epsilon}dz+9\int_{B(\xi,\delta)\setminus(B(\xi,\varepsilon)\cup B(\tilde\xi,\varepsilon))}|\xi-z|^{-1-2\alpha_*}dz\right)\\ \le& C''(1-|\xi|)^{1-2\alpha_*}\end{aligned}$$

with some Ω-dependent constant C',C'' because $\sum_{\theta_j\in I}\alpha_j^+\le\alpha_*<1$ by (2.14). The last two estimates now imply (4.2).

Let us now turn to Q. Fix any K as above, then fix any $(t,\xi)\in[0,\infty)\times K$ and let d,B,B' be as above (without loss assume that $d\le\frac{1}{4}$). Then for any $(t',\xi')\in[0,\infty)\times B'$ we have from (2.11),

$$|Q(t,\xi)-Q(t,\xi')| \le ||\omega||_{L^\infty} \int_{\mathbb{D}} \left(\frac{|\xi-\xi'|}{|\xi-z||\xi'-z|} + \frac{|\xi^*-\xi'^*|}{|\xi^*-z||\xi'^*-z|} \right) |R(t,z)|dz,$$

where the second fraction is just $\frac{1}{|\xi^*-z|}$ when $\xi'=0$ and $\frac{1}{|\xi'^*-z|}$ when $\xi=0$. Using (2.11), splitting the integration to $z\in B$ and $z\in\mathbb{D}\backslash B$, and applying (4.2) and (3.7) yields

$$|Q(t,\xi)-Q(t,\xi')| \le C'||\omega||_{L^\infty}|\xi'-\xi| \left(d^{1-2\alpha_*} \left(1+\ln_+ \frac{1}{|\xi-\xi'|}\right) + d^{-2} \right)$$

for some Ω-dependent constant C'. Next, we have

$$\begin{aligned} |Q(t,\xi')-Q(t',\xi')| \le & ||\omega||_{L^\infty} \int_{\mathbb{D}} \frac{|\xi'-\xi'^*|}{|\xi'-z|\,|\xi'^*-z|} |R(t,z)-R(t',z)|dz \\ & + \int_{\mathbb{D}} \frac{|\xi'-\xi'^*|}{|\xi'-z|\,|\xi'^*-z|} |R(t',z)|\,|\omega(t,\mathcal{S}(z))-\omega(t',\mathcal{S}(z))|dz. \end{aligned}$$

Splitting the first integration into $z\in B'$ and $z\in\mathbb{D}\backslash B'$, and then using $|\xi'-\xi'^*|\le 2|\xi'^*-z|$, (4.1), and (4.2) shows that the first integral is bounded above by

$$C_d||\omega(t,\cdot)-\omega(t',\cdot)||_{L^3(\Omega)} + \frac{4}{d}\int_{\mathbb{D}} |R(t,z)-R(t',z)|\,dz$$

for some (Ω,d)-dependent constant C_d. This converges to 0 as $t'\to t$ by $\omega\in C([0,\infty);L^3(\Omega))$, together with (4.1) and integrability of the right-hand side of (4.2).

Using $|\xi'-\xi'^*|\le 2|\xi'^*-z|$, (4.2), and Lemma 2.2, the second integral is bounded by

$$\begin{aligned} C' \left[\int_{\mathbb{D}} \left(\frac{(1-|z|)^{1-2\alpha_*}}{|\xi'-z| \det D\mathcal{S}(z)^{\frac{1}{p}}} \right)^q dz \right]^{\frac{1}{q}} \left(\int_{\mathbb{D}} \det D\mathcal{S}(z) |\omega(t,\mathcal{S}(z))-\omega(t',\mathcal{S}(z))|^p dz \right)^{\frac{1}{p}} \\ \le C_d||\omega(t,\cdot)-\omega(t',\cdot)||_{L^p(\Omega)} \end{aligned}$$

for some Ω-dependent C' and (d,Ω)-dependent C_d, provided $p\in(2,\infty)$ is large enough so that with $q:=\frac{p}{p-1}$ we have $(1-2\alpha_*-\frac{1}{p}\sum_j \alpha_j^+)q>-1$. The above estimates thus together show that Q is continuous at (t,ξ).

We can also use (2.11), $|\xi-\xi^*|\le 2|\xi^*-z|$, and (4.2) to get

$$|Q(t,\xi)| \le 2C_\Omega ||\omega||^2_{L^\infty} \int_{\mathbb{D}} \frac{(1-|z|)^{1-2\alpha_*}}{|\xi-z|}dz \tag{4.3}$$

for all $(t,\xi)\in[0,\infty)\times\mathbb{D}$, showing boundedness of Q on $[0,\infty)\times K$ for each compact $K\subseteq\mathbb{D}$.

Hence it remains to show $\partial_t\Psi(t,x) = -\frac{1}{2\pi}Q(t,\mathcal{T}(x))$ pointwise, which will follow from

$$-\frac{1}{2\pi}\int_{t_0}^{t_1} Q(t,\mathcal{T}(x_0))dt = \Psi(t_1,x_0)-\Psi(t_0,x_0) \tag{4.4}$$

for all $0\le t_0<t_1$ and $x_0\in\Omega$ because Q is continuous. So fix any such (t_0,t_1,x_0).

Let

$$\phi(x) := -\frac{1}{2\pi}\ln\frac{|\mathcal{T}(x_0)-\mathcal{T}(x)|}{|\mathcal{T}(x_0)-\mathcal{T}(x)^*||\mathcal{T}(x)|} = -\frac{1}{2\pi}\ln\frac{|\mathcal{T}(x)-\mathcal{T}(x_0)|}{|\mathcal{T}(x)-\mathcal{T}(x_0)^*||\mathcal{T}(x_0)|}$$

(so $\Psi(t_j,x_0)=\int_\Omega \phi(x)\omega(t_j,x)dx$ for $j=0,1$) and

$$\psi(x) := \nabla\phi(x) = -\frac{1}{2\pi}D\mathcal{T}(x)^T\left(\frac{\mathcal{T}(x)-\mathcal{T}(x_0)}{|\mathcal{T}(x)-\mathcal{T}(x_0)|^2}-\frac{\mathcal{T}(x)-\mathcal{T}(x_0)^*}{|\mathcal{T}(x)-\mathcal{T}(x_0)^*|^2}\right)$$

for each $x\in\Omega$ (recall (2.9)). Also, for each $r\in(0,\frac{t_1-t_0}{2})$ let $g_r\in C_c^\infty([0,\infty))$ be such that

$$\chi_{[t_0+r,t_1-r]}\le g_r\le \chi_{(t_0,t_1)}$$

and g_r is non-increasing on $[0,t_1]$ and non-decreasing on $[t_1,\infty)$; and for each $h\in(0,1]$ let $f_h\in C^\infty([0,\infty))$ be such that

(1) $f_h(x)=0$ for $x\in[0,\frac{h}{3}]$,
(2) $f_h(x)=x$ for $x\in[h,\frac{1}{h}]$,
(3) $f_h(x)=\frac{1}{h}+h$ for $x\in[\frac{1}{h}+h,\infty)$,
(4) $0\le f_h'(x)\le 2$ for $x\in[0,\infty)$.

Now for any $h,r\in(0,\min\{1,\frac{t_1-t_0}{2}\})$ and $(t,x)\in[0,\infty)\times\Omega$ let

$$\varphi_{r,h}(t,x) := g_r(t)f_h(\phi(x)).$$

Then clearly $\varphi_{r,h}\in C_c^\infty([0,\infty)\times\Omega)$ and $\varphi_{r,h}(0,\cdot)\equiv 0$, so plugging it into (2.3) yields

$$\int_0^\infty\int_\Omega \omega(t,x)g_r(t)f_h'(\phi(x))\,u(t,x)\cdot\psi(x)dxdt+\int_0^\infty\int_\Omega\omega(t,x)g_r'(t)f_h(\phi(x))dxdt=0.$$

Since $\omega(t,x)g_r(t)f_h'(\phi(x))\psi(x)$ is a bounded function and $u\in L^\infty((0,\infty);L^2(\Omega))$, we can use the dominated convergence theorem to pass to the limit $r\to 0$ and obtain

$$\int_{t_0}^{t_1}\int_\Omega\omega(t,x)f_h'(\phi(x))\,u(t,x)\cdot\psi(x)dxdt+\int_\Omega\omega(t_0,x)f_h(\phi(x))dx-\int_\Omega\omega(t_1,x)f_h(\phi(x))dx=0,$$

where in the second integral above we used that $\omega\in C([0,\infty);L^1(\Omega))$. If we can show that $u\cdot\psi\in L^\infty((0,\infty);L^1(\Omega))$, then taking $h\to 0$ will yield

$$\int_{t_0}^{t_1}\int_\Omega\psi(x)^T u(t,x)\,\omega(t,x)dxdt=\int_\Omega\phi(x)\omega(t_1,x)dx-\int_\Omega\phi(x)\omega(t_0,x)dx$$

via the dominated convergence theorem. But this is precisely (4.4) due to (2.7) and (2.10).

If $B:=B(x_0,\frac{1}{2}\mathrm{dist}(x_0,\partial\Omega))$, then $u\cdot\psi\in L^\infty((0,\infty);L^1(B))$ because u is bounded on $[0,\infty)\times B$ by (4.2). From (2.9) we see that there is C_{x_0} such that

$$|\psi(x)|\le C_{x_0}\|D\mathcal{T}(x)\|\le 2C_{x_0}|\det D\mathcal{T}(x)|^{\frac{1}{2}}$$

for all $x\in\Omega\setminus B$, so $\psi\in L^2(\Omega)$ by $\int_\Omega\det D\mathcal{T}(x)dx=|\mathbb{D}|$. So $u\cdot\psi\in L^\infty((0,\infty);L^1(\Omega\setminus B))$, which indeed yields $u\cdot\psi\in L^\infty((0,\infty);L^1(\Omega))$ and thus finishes the proof.

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