Math 148

HW 3: Problem 10

Prove the maximum principle for the heat equation on $\mathbb{R}^N$:

**Theorem.** Let $\Omega = \mathbb{R}^N \times \mathbb{R}^+$ and let $u$ be a $C^{2,1}(\Omega) \cap C(\bar{\Omega})$ solution to the heat equation $u_t = \Delta u$ on $\Omega$. If $u(x,0) \leq M$ for all $x \in \mathbb{R}^N$, then $u(x,t) \leq M$ for all $(x,t) \in \Omega$.

Step 1. Take any $T > 0$, $\epsilon > 0$, and show that the function

$$v_{T,\epsilon}(x,t) = u(x,t) - \frac{\epsilon}{(2T-t)^{N/2}} e^{\frac{|x|^2}{4(2T-t)}}$$

satisfies the heat equation in $\mathbb{R}^N \times (0,2T)$.

Step 2. Take $K \geq M$ such that $|u(x,t)| \leq K$ for all $(x,t) \in \bar{\Omega}$. Show that there is $R$ such that we have $v_{T,\epsilon}(x,t) \leq M$ for all $(x,t) \in (\mathbb{R}^N \setminus B_R(0)) \times [0,T]$.

Step 3. Since $v_{T,\epsilon}(x,0) \leq M$ for all $x \in \mathbb{R}^N$, conclude that $v_{T,\epsilon}(x,t) \leq M$ for all $(x,t) \in B_R(0) \times [0,T]$, using the maximum principle for bounded domains.

Step 4. Take $\epsilon \to 0$ to obtain $u(x,t) \leq M$ for all $(x,t) \in \mathbb{R}^N \times [0,T]$.

Step 5. Take $T \to \infty$ to finish the proof.

**Remark.** If $u$ is only a $C^{2,1}(\Omega) \cap C(\bar{\Omega})$ solution (i.e., not necessarily bounded), but we also have the bound $|u(x,t)| \leq Ke^{L|x|^2}$ for all $(x,t) \in \bar{\Omega}$ (and some $K,L > 0$), the first four steps still work provided $\frac{1}{4T} > L$, i.e., if $T < \frac{1}{4L}$. So this proves the maximum principle only on $\mathbb{R}^N \times [0,\frac{1}{4L}]$. But since now $u(x,\frac{1}{4L}) \leq M$ for all $x \in \mathbb{R}^N$, this version of the theorem proves the maximum principle also on $\mathbb{R}^N \times [\frac{1}{4L},\frac{2}{4L}]$, etc. So the original theorem holds even if we only assume $|u(x,t)| \leq Ke^{L|x|^2}$ for all $(x,t) \in \bar{\Omega}$ (and some $K,L > 0$). It turns out that without this a priori assumption, the theorem is not true for general $C^{2,1}(\Omega) \cap C(\bar{\Omega})$ solutions!