Follow the instructions below and possibly others made during the exam (1 point):

• Write down your name, PID number, TA’s name, section number and “Version A” on the front of your blue book.

• Draw the same grading table as on the blackboard on the front of your blue book.

• Write your solutions clearly in your bluebook:
  – Indicate the number and letter of each question and question part.
  – Present your answers in the same order they appear in the exam.
  – Start a new question on a new side of a page.

• You may use one piece of notes. No textbook or calculator is allowed.

• Unless otherwise stated, full justification is required and no credit will be given to unsupported answers.

All functions mentioned below have continuous derivatives of arbitrary order.

1. (6 points) $f(x, y) = (1 + x^2)^{(\sin y)}$, i.e. $(1 + x^2)$ is the base and $(\sin y)$ is the exponent
   (a) (3 points) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$
   (b) (3 points) Compute $g'(t)$ for $g(t) = f(t, t^2)$. (Your answer should be a function of $t$ with no other variables.)

2. (6 points) $z = f(x, y)$ is a function of $x, y$ such that $F(x, y, z) = F(x, y, f(x, y)) = 0$ for all $(x, y)$. We also know that $f(1, 2) = 3$, $\nabla F(1, 2, 3) = \langle -2, 2, 1 \rangle$.
   (a) (3 points) Compute $f_x(1, 2)$ and $f_y(1, 2)$ (You may use any formula you know without justification.)
   (b) (3 points) Estimate $f(1.1, 1.9)$ by linear approximation.

3. (8 points) Let $f(x, y) = x^3 - 4x^2 + 2xy - y^2$
   (a) (4 points) Find all critical points of $f$.
   (b) (4 points) Classify each critical point found in the last part as a local maximum, local minimum or saddle point.

4. (4 points) The global minimum of $f(x, y)$ on the circle $x^2 + y^2 = 5$ is achieved at $(2, 1)$, and we also know that $\nabla f(2, 1) = \langle 4, b \rangle$. Find $b$. 


1.(a) 
\[ \frac{\partial f}{\partial x} = (\sin y) \cdot (1 + x^2)^{\sin y - 1} \cdot (2x) \]
(by the power rule and chain rule);
\[ \frac{\partial f}{\partial y} = (1 + x^2)^{\sin y} \cdot \ln(1 + x^2) \cdot \cos y \]
(by the exponential rule and the chain rule).

1.(b) By multi-variable chain rule,
\[ g'(t) = f_x(t, t^2) \frac{dt}{dt} + f_y(t, t^2) \frac{dt^2}{dt} \]
\[ = (\sin t^2) \cdot (1 + t^2)^{\sin t^2 - 1} \cdot (2t) + (1 + t^2)^{\sin t^2} \cdot \ln(1 + t^2) \cdot (\cos t^2) \cdot (2t) \]

2.(a) By definition of gradient,
\[ \nabla F(1, 2, 3) = (F_x(1, 2, 3), F_y(1, 2, 3), F_z(1, 2, 3)) = (-2, 2, 1) \]
And therefore by implicit differentiation,
\[ f_x(1, 2) = -\frac{F_x(1, 2, 3)}{F_z(1, 2, 3)} = 2, f_y(1, 2) = -\frac{F_y(1, 2, 3)}{F_z(1, 2, 3)} = -2 \]

2.(b) 
\[ f(1.1, 1.9) \approx f(1, 2) + f_x(1, 2)(1.1 - 1) + f_y(1, 2)(1.9 - 2) \]
\[ = 3 + 2 \cdot 0.1 + (-2) \cdot (-0.1) = 3.4 \]
3.(a) $f_x(x, y) = 3x^2 - 8x + 2y$ and $f_y(x, y) = 2x - 2y$, thus the critical points are solutions of the system of equations,

$$3x^2 - 8x + 2y = 0$$
$$2x - 2y = 0$$

From the second equation, we have $x = y$, and this turns the first one to be $3x^2 - 8x + 2x = 3x^2 - 6x = 3x(x - 2) = 0 \Leftrightarrow x = 0, \text{ or } x = 2$. Thus the solutions are $(0, 0)$ and $(2, 2)$.

3.(b) $f_{xx}(x, y) = 6x - 8, f_{xy} = f_{yx} = 2, f_{yy} = -2$. And the discriminant is

$$\Delta(x, y) = -12x + 16 - 4 = 12(1 - x)$$

When $x = y = 0$, $\Delta(0, 0) = 12 > 0$ and $f_{xx}(0, 0) = -8 < 0$. Thus $f(x, y)$ has a local maximum at $(0, 0)$.

When $x = y = 2$, $\Delta(2, 2) = -12 < 0$, and thus $f(x, y)$ has a saddle point at $(2, 2)$.

4. Let $g(x, y) = x^2 + y^2 - 5$, then $\nabla g(x, y) = \langle 2x, 2y \rangle$.

As $f(x, y)$ has a global minimum at $(2, 1)$ on the circle, there is Lagrange multiplier $\lambda$ such that $\nabla f(2, 1) = \lambda \cdot g(2, 1) = \lambda \langle 4, 2 \rangle$. So we only need to solve the following equations,

$$4 = 4\lambda$$
$$b = 2\lambda$$

It’s clear that the first equation tells us $\lambda = 1$, and then the second one gives $b = 2$. 