Math 10A Final Exam
March 20, 2009

Notes allowed (one page), graphing calculators allowed, show work
100 points total, each problem is worth 10 points

1. Find the derivatives of the following functions. Do not simplify derivatives.
   a. (3 points) \( p(x) = \frac{2x \ln x}{e^x - 1} \). Use Quotient and Product Rules.
      \[
p'(x) = \frac{2 \left( \ln x + x \cdot \frac{1}{x} \right) (e^x - 1) - 2x \ln x \left( e^x \right)}{(e^x - 1)^2}
\]
   b. (3 points) \( g(x) = 2^x \sin^2 \left( \sqrt{x} \right) \). Use Product and Chain Rules.
      \[
g'(x) = 2x \ln 2 \left( 2^x \right) \left( \sin^2 \left( \sqrt{x} \right) \right) + 2^x \left( \frac{1}{2 \sqrt{x}} \right) \left( \cos \sqrt{x} \right) \left( 2 \sin \sqrt{x} \right)
\]
   c. (4 points) \( j(t) = (\tan(2t) - \cos(3t))^3 \). Use Chain Rule twice.
      \[
j'(t) = \left( 2 \left( \frac{1}{\cos^2(2t)} \right) \right) - 3(-\sin(3t)) \left( 3 \tan(2t) - \cos(3t))^2 \right)
\]

2. Given the position function \( s(t) = 3t^2 - 2t \),
   a. (5 points) Find the average velocity between \( t = 1 \) and \( t = 2 \).
      \[
      v_{avg} = \frac{s(2) - s(1)}{2 - 1} = \frac{8 - 1}{1} = 7
      \]
   b. (5 points) Use the limit definition \( s'(a) = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h} \) to compute the
      instantaneous velocity at \( t = 2 \).
      \[
s'(2) = \lim_{h \to 0} \frac{s(2 + h) - s(2)}{h} = \lim_{h \to 0} \frac{(3(2 + h)^2 - 2(2 + h)) - 8}{h}
      \]
      \[
      \lim_{h \to 0} \frac{3(4 + 4h + h^2) - 2(2 + h) - 8}{h} = \lim_{h \to 0} \frac{12 + 12h + 3h^2 - 4 - 2h - 8}{h} = \lim_{h \to 0} \frac{10h + 3h^2}{h} = \lim_{h \to 0} 10 + 3h = 10
      \]

3. If possible, choose constants \( a \) and \( b \) so that the following function is continuous and
differentiable at \( x = 0 \):
   \[
g(x) = \begin{cases} 
   \sin(ax) & x < 0 \\
   e^x + b & x \geq 0 
\end{cases}
\]
Continuity: \( \sin(a \cdot 0) = e^0 + b \) or \( 0 = 1 + b \), so \( b = -1 \).

Differentiability: \( (\sin(ax))'(0) = \left(e^x - 1\right)'(0) \) or \( (\cos(ax))(0) = \left(e^x\right)(0) \) or \( a\cos(a \cdot 0) = e^0 \) or \( a = 1 \)

4. For the implicit function \( xy - y^2 = 2 \):
   a. (7 points) Find the equation of the tangent line to its graph at the point \((3, 2)\).
      Find \( dy/dx \). \( y + x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0 \), so \( \frac{dy}{dx} = \frac{y}{2y - x} \).
      Find \( m \). \( m = \frac{2}{2(2) - 3} = 2 \)
      Find tangent line. \( y = 2(x - 3) + 2 = 2x - 4 \)
   b. (3 points) Use the line in part (a) to estimate the value of \( y \) when \( x = 3.1 \).
      \( y \approx 2(3.1) - 4 = 2.2 \)

5. Find constants \( a \) and \( b \) in the function \( f(x) = ax^2 e^{-bx} \) such that \( f \) has a local maximum at the point \( \left( 2, \frac{1}{e^2} \right) \).
   Find \( f'(x) \). \( f'(x) = a \left( 2xe^{-bx} - bx^2 e^{-bx} \right) = axe^{-bx} (2 - bx) \)
   Find critical points. \( axe^{-bx} (2 - bx) = 0 \) so \( x = 0 \) or \( x = \frac{2}{b} \). Since \( \left( 2, \frac{1}{e^2} \right) \) is a local maximum, it is a critical point and \( 2 = \frac{2}{b} \). Therefore, \( b = 1 \) and \( f(x) = ax^2 e^{-x} \). Since \( \left( 2, \frac{1}{e^2} \right) \) is a point on the graph of \( f \), then \( \frac{1}{e^2} = a \left( \frac{2^2}{e^2} \right) \left( e^{-2} \right) \). Therefore, \( a = \frac{1}{4} \).

6. The total cost \( C(q) \) of producing \( q \) goods is given by \( C(q) = 2q^2 - 7q + 20 \). Each item is sold for $25. Assume all \( q \) items are sold.
   a. (2 points) What is the revenue function, \( R(q) \)?
      \( R(q) = 25q \)
   b. (2 points) What is the profit function, \( \pi(q) \)?
      \( \pi(q) = R(q) - C(q) = 25q - \left( 2q^2 - 7q + 20 \right) = -2q^2 + 32q - 20 \)
   c. (5 points) At what quantity is profit maximized?
      Find critical points. \( \pi'(q) = -4q + 32 = 0 \), so \( q = 8 \).
      Classify critical points. By the FDT, this is a local maximum.
   d. (1 point) What is the maximum profit?
      \( \pi(8) = -2\left( 8^2 \right) + 32(8) - 20 = $108 \)

7. Do the following for the function \( f(x) = x^3(2 - x)^2 \) on \([0, 3] \):
   a. (3 points) Find the critical points of \( f \).
\[ f'(x) = 3x^2 (2 - x)^2 + x^3 (-1)(2 - x) = x^2 (2 - x)(6 - 5x) = 0. \]
So the critical points are \( x = 0, 2, \frac{6}{5} \).

b. (3 points) Determine if these critical points are local maxima or local minima or neither.

By the FDT, \( x = 0 \) is neither, \( x = 2 \) is a local minimum, and \( x = \frac{6}{5} \) is a local maximum.

c. (4 points) Find the global maximum and global minimum values of \( f \), if they exist.

Find values for critical points and endpoints.

\[ f(0) = 0, \quad f(2) = 0, \quad f\left(\frac{6}{5}\right) = \frac{3456}{3125}, \quad f(3) = 27. \]
Therefore, the global maximum value is 27 and the global minimum value is 0.

8. The graph of the derivative function \( f'(x) \) is shown. THIS ISN'T GRAPH OF \( f(x) \).

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a. (3 points) Determine the \( x \)-value(s) where \( f(x) \) has a local minimum.

\( x = 3 \)

b. (3 points) On which interval(s) is \( f(x) \) decreasing.

\( (1, 3) \)

c. (2 points) Estimate the interval(s) where \( f(x) \) is concave down.

Approximately \( (-\infty, 1.8) \) and \( (3.8, 5) \)

d. (2 points) Estimate the \( x \)-values of any inflection point(s) of \( f(x) \).
9. A right triangle has one vertex at the origin, another vertex on the curve $y = xe^{-x}$, and the remaining vertex on the positive $x$-axis. What is the maximum possible area of this triangle?

$$
\text{Area} = A(x) = \frac{1}{2} bh = \frac{1}{2} x(xe^{-x}) = \frac{1}{2} x^2 e^{-x}
$$

Find critical points. $A'(x) = \frac{1}{2} \left( 2xe^{-x} - x^2 e^{-x} \right) = \frac{1}{2} x e^{-x}(2-x) = 0$. Therefore, the critical point is $x = 2$ (since $x > 0$).

Classify critical points. By the FDT, this is a local maximum.

Determine maximum area. $A(2) = \frac{1}{2} \left( x^2 \right) \left( e^{-2} \right) = \frac{2}{e^2}$

10. The volume of a spherical balloon is increasing at a constant rate of $16 \pi \text{ cm}^3/\text{sec}$. The volume of a sphere is given by the formula, \text{Volume} = \frac{4}{3} \pi r^3$. How fast is the radius changing when the volume is $36\pi \text{ cm}^3$? Indicate units.

Given $\frac{dV}{dt} = 16\pi$, $V = 36\pi$, and $\frac{dr}{dt}$ unknown.

Find relationship between $V$ and $r$. $V(t) = \frac{4}{3} \pi (r(t))^3$.

Differentiate. $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Substitute known values. $16\pi = 4\pi \left( 3^2 \right) \frac{dr}{dt}$, since the radius is 3 when the volume is $36\pi$ (from $V(t) = \frac{4}{3} \pi (r(t))^3$).

Solve for unknown rate. $\frac{dr}{dt} = \frac{4 \text{ cm}}{9 \text{ sec}}$