HW2

Exercise. Q6 on p.150

First, we claim that \( \{ f_n \} \) is normal. Since \( f(z) = \lim f_n(z) \) for \( z \in G \). It follows that for each \( z \in G \), there exists \( M_z \) such that \( \sup_n |f_n(z)| < M_z \). Therefore, the closure of \( \{ f_n(z) : n \in \mathbb{N} \} \) is compact in \( G \). Then we need the following lemma to show \( \{ f_n \} \) is equicontinuous on \( G \).

**Lemma 0.1.** For any \( \epsilon > 0 \) and \( z \in G \), there exists \( \eta_z > 0 \) and \( N_z > 0 \) such that \( |f_n(w) - f(w)| < \epsilon \) for \( w \in B_{\eta_z}(z) \subseteq G \) and \( n \geq N_z \).

**Proof.** There exists \( N = N_z > 0 \) such that \( |f_n(z) - f(z)| < \epsilon \) for \( n \geq N \). Since \( f \) and \( f_N \) are continuous at \( z \in G \), there exists \( \eta = \eta_z > 0 \) such that \( |f(z) - f(w)| < \epsilon \) and \( |f_N(z) - f_N(w)| < \epsilon \) for \( w \in B_{\eta}(z) \subseteq G \). It follows that for \( n \geq N \) and \( w \in B_{\eta}(z) \) we have

\[
|f_n(w) - f(w)| \leq |f_N(w) - f(w)| + |f_n(z) - f(z)|
\]

The first inequality is because of the monotonically increasing condition of \( \{ f_n \} \). The lemma follows.

For any \( \epsilon > 0 \) and \( z \in G \), let \( \eta_z \) and \( N_z \) be the same symbol in Lemma 0.1. For any \( \epsilon > 0 \) and \( \eta > 0 \) such that \( |f(z) - f(w)| < \epsilon \) for \( |z - w| < \eta \). Using Lemma 0.1, it shows that for all \( n \geq N_z \)

\[
|f_n(w) - f_n(z)| \leq |f_n(w) - f(w)| + |f(w) - f(z)| + |f_n(z) - f(z)|
\]

\[
\leq 3\epsilon.
\]

Hence, \( \{ f_n \} \) is equicontinuous on \( G \). By Arzela-Ascoli Theorem, \( \{ f_n \} \) is normal, i.e. \( f_n \) converges to \( f \) locally uniformly.

Next, we claim \( f_n \) converge to \( f \) locally uniformly. For any \( \epsilon > 0 \) and any compact set \( S \subseteq G \), there exists \( K > 0 \) such that

\[
|f_{n_k}(w) - f(w)| < \epsilon \text{ for } w \in S \text{ and } k \geq K.
\]

Since \( f_{n_k}(w) \leq f_m(w) \leq f(w) \) for all \( m \geq n_K \), it follows that

\[
|f_m(w) - f(w)| < \epsilon \text{ for } w \in S \text{ and } m \geq n_K.
\]

We have \( f_n \) converges to \( f \) locally uniformly.

**Remark 0.2.** Another method to show \( f_n \to f \): We let \( g_n = f - f_n \). For any \( \epsilon > 0 \), \( z \in K \) which is compact in \( G \). There exists \( N_z \) such that \( |g_n| \leq |g_{N_z}| = g_{N_z} < \epsilon \). We denote \( O_z = \{ w \in G : g_N < \epsilon \} \). It is clear that \( O_z \) is non-empty open set and \( K \subseteq \bigcup_{z \in K} O_z \). Since \( K \) is compact, there is a finite cover \( O_{z_1}, \ldots, O_{z_k} \) to cover \( K \). As a result, take \( N = \min \{ N_z_1, \ldots, N_z_k \} \) and we have \( |f(z) - f_n(z)| = g_n(z) < \epsilon \) for all \( z \in K \) and \( n \geq N \).
Lemma 0.3. Suppose \( \{x_n\} \) is a sequence on a metric space \((X,d)\). Then, \( x_n \) converges to \( x \) if and only if every subsequence of \( x_n \) has a further convergent subsequence which converges to the same limit \( x \).

Proof.

" \Rightarrow " direction is a straightforward argument. Thus, we just show the other direction. Suppose the conclusion is false, i.e. \( x_n \to x \). Then there exist \( \epsilon_0 \) and \( x_{n_k} \) such that \( d(x, x_{n_k}) \geq \epsilon_0 \) for all \( k \in \mathbb{N} \). Therefore, no further convergent subsequence of \( x_{n_k} \) can converge to \( x \). Contradicts with the given condition.

Suppose \( f(z) = \lim f_n(z) \) for \( z \in G \). Then, for each \( z \in G \), there exists \( M_z \) such that \( sup_n |f_n(z)| < M_z \). Moreover, \( \{f_n\} \) is equicontinuous on \( G \). Then, by Arzela-Ascoli Theorem, \( \{f_n\} \) is normal in \( G \). Then for any subsequence of \( \{f_n\} \), there is a further convergent subsequence converging to \( f \) because of the assumption that \( f(z) = \lim f_n(z) \) for \( z \in G \). Hence, by using Lemma 0.2, \( f_n \to f \).

Exercise. Q7 on p. 150

We need one lemma before we do this question.

Lemma 1.1

Exercise. Q4 on p. 154

\( \{f_n\}_{n \in \mathbb{N}} \) is locally bounded which means \( \{f_n\}_{n \in \mathbb{N}} \) is normal in \( H(G) \). Then for any subsequence of \( \{f_n\} \), the subsequence has a further convergent subsequence denoted by \( \{\hat{f}_n\} \subseteq \{f_n\} \). Let \( \hat{f}_n \to g \). By the assumption that \( \lim f_n(z) = f(z) \) for \( z \in A \) and \( A \) has a limit point. Therefore, \( A' = \{z \in G : f(z) = g(z)\} \) has a limit point which follows that \( g \equiv f \) on \( G \). Then by using Lemma 0.2, we have \( f_n \to f \).

Exercise. Q8 on p. 154

Suppose there is a sequence of \( M_n \) of positive constant such that \( |a_n| \leq M_n \) and \( \lim sup |M_n|^{1/n} \leq 1 \). Then for any open ball with center \( 0 \) and radius \( \epsilon < 1 \), \( B_\epsilon(0) \), we have for all \( z \in B_\epsilon(0) \),

\[
|f_n(z)| \leq \sum |a_n| |z|^n \leq \sum M_n \epsilon^n < \infty.
\]

The last strict inequality because of \( \lim sup |M_n \epsilon^n|^{1/n} < 1 \). It follows that \( \mathcal{F} \) is locally bounded and hence normal by Montel Theorem.

Suppose \( \mathcal{F} \) is normal in \( H(D) \). By Montel’s Theorem and Lemma 2.8, for any \( 0 < \epsilon < 1 \), there exists \( C_\epsilon > 0 \) such that

\[
|f(z)| \leq C_\epsilon \text{ for all } f \in \mathcal{F} \text{ and } z \in \partial B_\epsilon(0)
\]

where \( B_\epsilon(0) \) be a ball with center \( 0 \) and radius \( \epsilon \). Then, by using Cauchy’s formula, we have

\[
a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z^{n+1}} \, dz \quad (0.1) \tag{equ.1.1}
\]

where \( \gamma \) is an anti-clockwisely oriented closed curve on \( \partial B_\epsilon(0) \). Therefore, using Eq. \((0.1)\), \( |a_n| \) is bounded by the following

\[
|a_n| \leq \frac{C_\epsilon}{\epsilon^n} \text{ for all } 1 > \epsilon > 0.
\]
Then we take \( M_n := \inf_{0<\epsilon<1} \frac{C_\epsilon}{\epsilon} \). And therefore, fixed \( 0 < \epsilon_0 < 1 \), it follows that

\[
\limsup_n |M_n|^{1/n} \leq \limsup_n \left( \inf_{0<\epsilon<1} \frac{|C_\epsilon|^{1/n}}{\epsilon} \right) \\
\leq \limsup_n \frac{|C_{\epsilon_0}|^{1/n}}{\epsilon_0} \leq \frac{1}{\epsilon_0}
\]

because \(|C_\epsilon|^{1/n} \to 1\) as \( n \to \infty \). Therefore, \( \limsup_n |M_n|^{1/n} \leq 1 \) follows immediately if we take \( \epsilon_0 \to 1^- \).

**Exercise.** Q5 on p.163,

Let \( D \) be a unit disk with center zero. By Riemann Mapping Theorem, there exists the unique biholomorphic map \( g \) from \( G \) to \( D \) such that \( g(a) = 0 \) and \( g'(a) > 0 \). Then we let \( F(z) = g \circ f \circ g^{-1}(z) \) is a map from \( D \) to \( D \) such that \( F(0) = 0 \). By Schwarz’s Lemma on p.130, we can conclude that

\[
1 \geq \left| F'(0) \right| = \left| (g \circ f \circ g^{-1})'(z) \right| \\
= \left| g'(a) f'(a) (g^{-1})'(0) \right| = \left| f'(a) \right|.
\]

The second equality is because \( f(a) = a \).