Problem 1. Let $V_1$ and $V_2$ be subspaces of $\mathbb{R}^n$. Show that $V = \{v_1 + v_2 | v_1 \in V_1, v_2 \in V_2\}$ is a subspace of $\mathbb{R}^n$.

Problem 2. Show that $V = \{[x \ y] | x + y + axy = b\}$ is a subspace of $\mathbb{R}^2$ if and only if $a = b = 0$.

Problem 3. Let $A$ and $B$ be symmetric $n \times n$ matrices. Show that $AB$ is symmetric if and only if $AB = BA$.

Problem 4. The matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ has an inverse of the form $A = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$. Find $x, y, z$.

Problem 5. Let $A$ and $B$ be $n \times n$ matrices. Show that if $A$ and $B$ are invertible, then $AB$ is invertible. Show that if $A^k$ is invertible, for some $k \geq 1$, then $A$ is invertible.

Problem 6. Let $A$ be an $m \times n$ matrix, $\vec{v} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^m$. Prove that $A\vec{v} \cdot \vec{w} = \vec{v} \cdot A^T \vec{w}$.

Problem 7. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$. Show that $|\vec{x} - \vec{y}| \leq |\vec{x} - \vec{z}| + |\vec{y} - \vec{z}|$.

Problem 8. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ such that $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{z} = \vec{z} \cdot \vec{x} = 0$. Show that at least one of the vectors $\vec{x}, \vec{y}, \vec{z}$ is equal to zero.

Problem 9. Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ be two given vectors. What is the largest value that the volume of the parallelepiped spanned by $\vec{x}, \vec{y}, \vec{z}$ can take, when $\vec{z} \in \mathbb{R}^3$ is a unit vector?

Problem 10. For what values of $a$ and $b$ does the following system of linear equations have infinitely many solutions?

\begin{align*}
x + ay &= 0 \\
y + bz &= 1 \\
x + z &= b
\end{align*}