There are 7 pages and 4 questions, for a total of 100 points.

No notes, no calculators, no books.

Please turn off all electronic devices.
Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Unless otherwise stated, show all your work for full credit. To maximize credit, cross out incorrect work.

Good luck! ☺

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1. (15 points) The variables $x, y, z$ satisfy the relation $x^2y^2z^2 + z^3 = 12$. Find $\frac{\partial z}{\partial y}$ at the point $(1, 1, 2)$.

\[
x^2y^2z^2 + z^3 = 12
\]

Take $\frac{\partial}{\partial y}$ of both sides.

Get:
\[
x^2 \left[ (2y)z^2 + y^2 \left( 2z \frac{\partial z}{\partial y} \right) \right] + 3z^2 \frac{\partial z}{\partial y} = 0
\]

Substitute $x = 1, y = 1, z = 2$.

\[
1^2 \left[ (2 \cdot 1) \cdot 4 + 1^2 \cdot (2 \cdot 2 \cdot \frac{\partial z}{\partial y}) \right] + 12 \frac{\partial z}{\partial y} = 0
\]

\[
8 + 16 \frac{\partial z}{\partial y} = 0
\]

\[
\frac{\partial z}{\partial y} = -\frac{1}{2}
\]
2. Let \( f(x,y,z) = x^2y^2 - x + yz^2. \) 

(a) (5 points) Find \( \nabla f \) at \((2,1,2)\).

\[
\nabla \Phi = \left< \Phi_x, \Phi_y, \Phi_z \right> = \left< 2xy^2 - 1, 2x^2y + z^2, 2yz^2 \right>
\]

at \((2,1,2)\)

\[
\nabla \Phi(2,1,2) = \left< 4-1, 8+4, 4 \right> = \left< 3, 12, 4 \right>
\]

(b) (5 points) Write the equation for the tangent plane to the surface \( f = 6 \) at the point \((2,1,2)\).

\[
\nabla \Phi(2,1,2) = \left< 3, 12, 4 \right> \text{ normal vector}
\]

plane passes through \((2,1,2)\) so

\[
3x + 12y + 4z = 3 \cdot 2 + 12 \cdot 1 + 4 \cdot 2
\]

\[
3x + 12y + 4z = 26
\]
(c) (10 points) Use a linear approximation to find the approximate value of \( f(1.9, 1.1, 2) \).

\[
\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z
\]

\[
 f(1.9, 1.1, 2) - f(2, 1, 2) \approx f_x (1.9-2) + f_y (1.1-1) + f_z (2-2)
\]

\[
= 3 \cdot (-0.1) + 12 \cdot 0.1 = 0.9
\]

\[
f(2, 1, 2) = 4 \cdot 1 - 2 + 1 \cdot 4 = 6
\]

So \( f(1.9, 1.1, 2) \approx 6 + 0.9 = 6.9 \)

(d) (10 points) Find the directional derivative of \( f \) at \( (2, 1, 2) \) in the direction of \(-i + j\).

\[
\nabla f = -i + j = \langle -1, 1, 0 \rangle \quad \Rightarrow \quad |\nabla f| = \sqrt{2}
\]

\[
\hat{u} = \frac{|\nabla f|}{|\nabla f|} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle
\]

\[
\nabla f \cdot \hat{u} = \nabla f (\hat{u}) \cdot \langle 3, 12, 4 \rangle \cdot \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle
\]

\[
= -\frac{3}{\sqrt{2}} + \frac{12}{\sqrt{2}} + 0 = \frac{9}{\sqrt{2}}
\]
3. (25 points) Use Lagrange multipliers to find the closest point(s) to the origin along the curve $4x^2 + y^2 = 9$. Hint: instead of minimizing the distance to the origin, you might want to consider minimizing the square of the distance.

$$f(x, y) = x^2 + y^2$$
the square of the distance
from $(x, y)$ to the origin.

Want to minimize $f(x, y)$ under constraint

$$4x^2 + y^2 = 9$$

$$g(x, y) = 4x^2 + y^2$$

**Step 1** \(\text{Gradients}\)

$$\nabla f = \langle 2x, 2y \rangle$$

$$\nabla g = \langle 8x, 2y \rangle$$

**Step 2** \(\text{Equations}\)

$$\nabla f = \lambda \nabla g + g = c$$

$$2x = 8 \lambda x$$

$$2y = 2 \lambda y$$

$$4x^2 + y^2 = 9$$

**Step 3** \(\text{Solve}\)

$$2y (1- \lambda) = 0 \Rightarrow y = 0 \text{ or } \lambda = 1$$

If $\lambda = 1$ \(\Rightarrow 2x = 8x \Rightarrow x = 0 \Rightarrow y^2 = 9\)

$$4x^2 + y^2 = 9 \Rightarrow (0, 3), (0, -3)$$

If $y = 0$ \(\Rightarrow 4x^2 = 9 \Rightarrow x = \pm \frac{3}{2} \Rightarrow (\frac{3}{2}, 0), (-\frac{3}{2}, 0)$$

**Step 4** \(\text{Values}\)

$$f(0, 3) = 9 = f(0, -3)$$

$$f(\frac{3}{2}, 0) = \frac{9}{4} = f(-\frac{3}{2}, 0)$$

$$4x^2 + y^2 = 9 \text{ closed} \Rightarrow \text{minimum is one of these points.}$$

Looking at values, see closest points are $$(\frac{3}{2}, 0), (-\frac{3}{2}, 0)$$
4. Consider the function \( f(x, y) = x^3 + xy + y^3 \).

(a) (10 points) In which direction should one go from the point (0, 1) to obtain the most rapid increase in \( f \)? Express your answer as a unit vector.

\[
\nabla f = \langle 3x^2 + y, x + 3y^2 \rangle
\]
\( \nabla f(0, 1) = \langle 1, 3 \rangle \)

Unit vector in this direction \( \frac{\langle 1, 3 \rangle}{\sqrt{1 + 9}} = \frac{\langle 1, 3 \rangle}{\sqrt{10}} \)