1. Let \( f(x, y) = x \cos(x - 2y) + xy \).
   
   (a) (5 points) Find \( \nabla f \) at \((2, 1)\).
   
   \[
   \begin{align*}
   f_x &= \cos(x - 2y) + x(-\sin(x - 2y)) + y \\
   f_y &= (-2)x(-\sin(x - 2y)) + x
   \end{align*}
   \]
   
   at \((2, 1)\) get \( \nabla f(2, 1) = (\cos 0 + 2 \cdot \sin 0 + 1, -2 \cdot (-\sin 0) + 2) \)
   \[
   \nabla f = (1, 2)
   \]
   
   (b) (5 points) Write the equation for the tangent plane to the surface \( z = f(x, y) \) at the point \((2, 1, 4)\).
   
   \[
   z - 4 = 1 \cdot (x - 2) + 2 \cdot (y - 1)
   \]
   
   \[ x + 2y - z = 0 \]

   (c) (10 points) Use a linear approximation to find the approximate value of \( f(1.9, 1.1) \).
   
   \[
   \Delta f \approx f_x \Delta x + f_y \Delta y
   \]
   
   \[
   f(1.9, 1.1) - f(2, 1) \approx 1 \cdot (1.9 - 2) + 2 \cdot (1.1 - 1)
   \]
   
   \[
   = -0.1 + 0.2 = 0.1
   \]

   \[
   f(2, 1) = 2 \cdot \cos 0 + 2 \cdot 1 = 2 + 2 = 4
   \]

   \[
   f(1.9, 1.1) \approx 4.1
   \]
2. (30 points) Find the area of the ellipse $(2x + 5y - 3)^2 + (3x - 7y + 8)^2 < 1$. Hint: change variables. You can use without proof the fact that the area of the unit disk is $\pi$. It might not be the best idea to try to sketch the ellipse since it can be very time consuming.

**Change of variables**

\[
u = 2x + 5y - 3 \\
v = 3x - 7y + 8\]

**Region becomes** $u^2 + v^2 < 1$: unit disk

**Area ellipse**

\[
\iint (2x + 5y - 3)^2 + (3x - 7y + 8)^2 < 1 \, \, dx \, dy
\]

**Jacobian**

\[
J = \frac{\Theta(u,v)}{\Theta(x,y)} = \begin{vmatrix}
  u_x & u_y \\
  v_x & v_y \\
\end{vmatrix} = \begin{vmatrix}
  2 & 5 \\
  3 & -7 \\
\end{vmatrix} = -14 + 15 = -29
\]

so $du \, dv = |J| \, dx \, dy = 29 \, dx \, dy \Rightarrow dx \, dy = \frac{du \, dv}{29}$

**Area ellipse**

\[
\iint \frac{1}{29} \cdot \frac{1}{\sqrt{u^2 + v^2}} \, du \, dv = \frac{1}{29} \iint \frac{du \, dv}{\sqrt{u^2 + v^2} < 1} \Rightarrow \frac{\pi}{29} \frac{\text{Area unit disk}}{29} = \frac{\pi}{29} \frac{\pi}{29} = \frac{\pi^2}{29^2}
\]
3. Let $\vec{F} = (ax^2y^2 + y^3 + 2)i + (4x^3y + bxy^2 + 5)j$ be a vector field where $a$ and $b$ are constants.

(a) (10 points) Find the values of $a$ and $b$ for which $\vec{F}$ is a gradient field.

Want $a,b$ such that $M_y = N_x$

$M_y = 2ax^2y + 3y^2$
$N_x = 12x^2y + by^2$

$\Rightarrow 2a = 12 \rightarrow a = 6$
$b = 3$

(b) (20 points) For the values of $a$ and $b$ that you found in part (a), find $f(x,y)$ such that $\vec{F} = \nabla f$.

$\vec{F} = \nabla f = (f_x, f_y)$

Integrate $f_x$ w.r.t. $x$ and get

$f = 2x^3y^2 + xy^3 + 2x + g(y)$

So $f_y = 4x^2y + 3xy^2 + g'(y)$

But $f_y = 4x^2y + 3xy^2 + 5$

$g'(y) = 5 \rightarrow g(y) = 5y + c$.

Plug into $(*)$ and get

$f = 2x^3y^2 + xy^3 + 2x + 5y + c$
4. (a) (5 points) Sketch the region $R$ bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere of radius $\sqrt{2}$ centered at the origin.

(b) (15 points) Set up an iterated triple integral in rectangular coordinates giving the volume of the solid $R$. Give the integrand and the bounds, but DO NOT EVALUATE.

\[
\begin{align*}
\text{top: sphere} & \quad x^2 + y^2 + z^2 = 2 \implies z = \sqrt{2 - x^2 - y^2} \\
\text{bottom: paraboloid} & \quad z = x^2 + y^2 \\
\text{shadow on xy-plane} & \quad z = x^2 + y^2 \\
\text{disk of radius} 1 & \quad x^2 + y^2 \leq 1 \\
\text{volume} & \quad \iiint_R d\mathbf{v} = \int_{-1}^{\sqrt{2-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{2-x^2-y^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz \, dy \, dx
\end{align*}
\]