Practice Problem 5a

1. \( F = (2yz, -x + 3y + z, x^2 + z), \ S: x^2 + y^2 = 4, \ 0 \leq z \leq 1. \)

\( C_1(t) = (2\cos t, 2\sin t, 0), \ 0 \leq t \leq 2\pi, \ C_2(t) = (2\sin t, 2\cos t, 1) \)

\[ \int_S DxF \cdot dS = \int_{C_1} \int_{C_2} \int_{S} \left( \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \right) dS \]

\[ = \int_0^{2\pi} \int_0^1 \int_0^2 (2z + 12\sin t + 8\cos^2 t) \cdot (2\cos t, 2\sin t, 0) \ dt \]

\[ + \int_0^{2\pi} \int_1^2 \int_0^1 (4\cos t - 2\sin t + 6\cos^2 t, 4\cos^2 t, 0) \ dt \]

\[ = \int_0^{2\pi} -4\cos^2 t + 12\cos t + 8\cos^2 t + 4\sin^2 t - 12\sin t \cos t \ dt = \int_0^{2\pi} 4 \ dt = 8\pi \]

2. \( \Delta F = \left| \begin{array}{cc} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} & \frac{\partial F}{\partial z} \end{array} \right| = \left| \begin{array}{cc} 2y & 2z \\ x^2 & y^2 \end{array} \right| = (-1, 2y - 2x, -1 - 2t) \quad S: \ \text{Annulus} = (2\cos t, 2\sin t, 2) \)

\[ T_0 \times T_2 = \left| \begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{array} \right| = (2\cos t, 2\sin t, 0) \]

\[ SS_0 DxF \cdot dS = \int_{C_1} \int_{C_2} \int_{S} \left( \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \right) dS \]

\[ = \int_0^{2\pi} \int_0^1 \int_0^2 (-4\sin t - 8\cos t \sin t) \cdot (2\cos t, 2\sin t, 0) \ dt \]

\[ = \int_0^{2\pi} -2\cos t + \sin t + \sin t \cos t \ dt = \int_0^{2\pi} -2\cos t + 4 \ dt = 8\pi \]

3. \( y \leq x \quad C_1(t) = (t, t^2), \ 0 \leq t \leq 1 \)

\( y \geq x^2 \quad C_2(t) = (1 - t, 1 - t), \ 0 \leq t \leq 1 \)

By Green's Thm: \( A(\mathbb{D}) = \frac{1}{2} \int_D xdy - ydx = \frac{1}{2} \int_{\partial D} (-y, x) \cdot n \ ds \)

\[ = \frac{1}{2} \int_0^1 (-y^2, 1 - t^2) - t^2, 1 - t) \ dt + \frac{1}{2} \int_0^1 (1 - t, 1 - t) \cdot (1, 1) \ dt \]

\[ = \frac{1}{2} \int_0^1 -y^2 + 2t^2 + (1-t) - (1-t) \ dt = \frac{1}{2} \int_0^1 \frac{t^3}{3} \ dt = \frac{1}{6} \int_0^1 \frac{1}{3} \ dt = \frac{1}{6} \]

\( \text{Test: } \int_0^1 x - x^2 dx = \left[ \frac{1}{2} x^2 - \frac{1}{3} x^3 \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \]

4. \( F = (2xy, -y^2) \)

\[ S F \cdot n \ ds = \int_D \int_S \int_F \cdot n \ ds = \int_D \int_D \int_{\partial D} \cdot n \ ds \]

\[ = \frac{1}{2} \int_0^1 \int_0^2 (2xy, -y^2) \ ds = \frac{1}{2} \int_0^1 \int_0^2 \ ds = \int_0^1 0 = 0 \]
\[ S \int (x^2+xy-y^2) \, dx + (2x-xy+y^3) \, dy \]
\[ = \iiint_S \frac{\partial}{\partial y} (2x-xy+y^3) - \frac{\partial}{\partial x} (x^2+xy-y^3) \, dV \]
\[ = \iiint_S (2-2x) - (x-3y^2) \, dV = S_0 \left( x^2-\frac{1}{2}y^2-xy+y^3 \right) \, dx \]
\[ = S_0 \left[ \frac{5}{2}x - \frac{1}{2}x^2 \right]_0 = 2. \]

6) \( F = (x,y) \):
   a) \( 3 \) e
   b) \( 4 \) e
   c) \( 5 \) e

7) \( \dot{C} \) starts at \((0,0,0)\), ends at \((1,2,3)\)
\[ F = ((y+xyz) \, e^{x^2}, x \, e^{x^2}, y \, e^{x^2}) = D \dot{F} \]
\[ \int_C F \cdot ds = \int_C \dot{F} \cdot \dot{s} \]
\[ = \int_0^3 \dot{F} \cdot \dot{s} \, dt = \int_0^3 D \cdot \dot{s} \, dt = \int_0^3 1 \cdot 2 \cdot e^{3} = 2 \cdot e^{3} \]

8) \( S = \) unit sphere = \( D \omega \) where \( \omega \) unit ball.
\[ F = (x-y, y-z, z-x) \]
\[ \iiint_S \nabla \cdot F \cdot dV = \iiint_S 3 \, dV = \frac{4}{3} \pi \cdot 3 = 4 \pi \]

9) \( S \), \( W \)
\[ F = (3xy^2, 3x^2y, z^3) \]
\[ \iiint_S \nabla \cdot F \cdot dV = \iiint_S 3y^2 + 3x^2 + 3z^2 \, dV \]
\[ = 3 \int_0^{2\pi} \int_0^{2\pi} \int_0^r p^2 (p^2 \cos \theta \sin \phi) \, dp \, d\theta \, d\phi = 6 \pi \left[ \frac{5}{5} p^5 \right]_0^1 \left[ -\cos \phi \right]_0^{\pi} \]
\[ = 12 \pi \]

\[ \frac{5}{5} \]