Section 11.4. Separation of variables

Example 1

Figure 1 shows the slope field of the differential equation

\[ \frac{dy}{dx} = y \]

and Figure 2 shows the graphs of eight solutions. (a) Use the differential equation to explain the pattern of the slope lines. (b) Find an equation for all solutions.

Answer: (a) One description and explanation: The lines in the slope field of \( \frac{dy}{dx} = y \) in Figure 1 have the same slope along each horizontal line because the formula on the right does not involve \( x \). The lines are horizontal along the \( x \)-axis where \( y = 0 \), have positive slopes above the \( x \)-axis where \( y > 0 \), and have negative slopes below the \( x \)-axis where \( y < 0 \), and they become steeper as \( y \) increases through positive values or decreases through negative values.

(b) The solutions are \( y = Ce^x \) with arbitrary constants \( C \).

Example 2

Find the solution of the initial-value problem \( \frac{dy}{dx} = 2y \cos x, \ y(0) = 4 \).

Answer: \( y = 4e^{2 \sin x} \).

Example 3

Check the result of Example 2.

Answer: Set \( y = 4e^{2 \sin x} \). The initial condition is satisfied. \( dy \)

\[ \frac{dy}{dx} = \frac{d}{dx}(4e^{2 \sin x}) = 8(\cos x)e^{2 \sin x} = 2y \cos x \]

The differential equation is satisfied.

Example 4

Find the solutions of the differential equation

\[ \frac{dy}{dx} = -2xy^2 \]

with the initial conditions (a) \( y(0) = 1 \) \( \bullet \) \( y(0) = -\frac{1}{4} \).

Answer: (a) \( y = \frac{1}{x^2 + 1} \) \( (b) y = \frac{1}{x^2 - 4} \). (Figure A4a shows the slope field for the differential equation (6), and Figure A4b gives the graphs of the solutions.)
Example 5
Solve the initial-value problem \( K'(x) = \sqrt{xK(x)} \), \( K(1) = 1 \)

Answer: \( K = \left( \frac{1}{3} x^{3/2} + \frac{2}{3} \right)^2 \)

Example 6
Find all nonzero solutions of \( \frac{dQ}{dx} = -3Q^{1/4} \).

Answer: \( Q = \left( C - \frac{9}{4} x \right)^{4/3} \)

Example 6
(a) A two-gram object is moving on an s-axis with distances measured in centimeters. Its velocity in the positive direction is 1 centimeter per second at time \( t = 0 \) (seconds) and the force on it at time \( t > 0 \) is \( 4tv^2 \) dynes in the positive s-direction if its velocity is \( v \) centimeters per second at that time. Give an initial-value problem satisfied by \( v = v(t) \). (b) Give a formula for \( v \) for \( t \geq 0 \). (c) What happens to the velocity as \( t \to \infty \)? (The slope field and graph of the solution are in Figure 3.)

Answer: (a) Initial-value problem: \( \frac{dv}{dt} = 2tv^2 \), \( v(0) = 1 \). (b) \( v = \frac{1}{1 - t^2} \). (c) \( v \to \infty \) as \( t \to 1^- \)

Interactive Examples
Work the following Interactive Examples on Shenk’s web page, http://www.math.ucsd.edu/~ashenk/:

Section 9.1: Examples 1–3, 5, 6, 8

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†The chapter and section numbers on Shenk’s web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.