Section 7.4 #1, 5, 6, 8, 12, 13, 44, 53; Section 7.5 #7, 10, 11, 20, 22; Section 7.7 #1, 4, 10, 15, 22, 44

7.4.1 Since $25 - x^2 = (5 - x)(5 + x)$, start with

$$\frac{20}{25 - x^2} = \frac{A}{5 - x} + \frac{B}{5 + x}.$$ 

So,

$$20 = A(5 + x) + B(5 - x).$$

Equating coefficients gives $0 = A - B$, $20 = 5A + 5B$. The solution to this system is $A = 2, B = 2$. Alternatively, we could choose $x = -5$ above, which gives $B = 2$; $x = 5$ gives $A = 2$. Therefore

$$\frac{20}{25 - x^2} = \frac{2}{5 - x} + \frac{2}{5 + x}.$$ 

7.4.5 Since $s^4 - 1 = (s - 1)(s + 1)(s^2 + 1)$, start with

$$\frac{2}{s^4 - 1} = \frac{A}{s - 1} + \frac{B}{s + 1} + \frac{Cs + D}{s^2 + 1}.$$ 

So,

$$2 = A(s + 1)(s^2 + 1) + B(s - 1)(s^2 + 1) + (Cs + D)(s - 1)(s + 1)$$

$$2 = (A + B + C)s^3 + (A - B + D)s^2 + (A + B - C)s + (A - B - D).$$

Equating coefficients gives

$$A + B + C = 0$$
$$A - B + D = 0$$
$$A + B - C = 0$$
$$A - B - D = 2.$$ 

From the first and third equations we find that $A + B = 0$ and $C = 0$. From the second and fourth we find $A - B = 1$ and $D = -1$. So $A = 1/2, B = -1/2$, and

$$\frac{2}{s^4 - 1} = \frac{1/2}{s - 1} - \frac{1/2}{s + 1} - \frac{1}{s^2 + 1}.$$
7.4.6 Since \( y^3 - y^2 + y - 1 = (y - 1)(y^2 + 1) \), start with
\[
\frac{2y}{y^3 - y^2 + y - 1} = \frac{A}{y - 1} + \frac{By + C}{y^2 + 1}.
\]
So,
\[
2y = A(y^2 + 1) + (By + C)(y - 1)
\]
\[
2y = (A + B)y^2 + (-B + C)y + A - C.
\]
Equating coefficients gives
\[
A + B = 0 \quad -B + C = 2 \quad A - C = 0.
\]
The solution to this system is \( A = 1, B = -1, C = 1 \), so
\[
\frac{2y}{y^3 - y^2 + y - 1} = \frac{1}{y - 1} + \frac{-y + 1}{y^2 + 1}.
\]

7.4.8 Using the answer from problem 1,
\[
\int \frac{20}{25 - x^2} \, dx = \int \frac{2}{5 - x} \, dx + \int \frac{2}{5 + x} \, dx = -2 \ln |5 - x| + 2 \ln |5 + x| + C.
\]
Note: this could be rewritten as \( 2 \ln \left( \frac{5 + x}{5 - x} \right) + C \).

7.4.12 Using the answer from problem 5,
\[
\int \frac{2}{s^4 - 1} \, ds = \int \left( \frac{1/2}{s - 1} - \frac{1/2}{s + 1} - \frac{1}{s^2 + 1} \right) \, ds = \frac{1}{2} \ln |s - 1| - \frac{1}{2} \ln |s + 1| - \arctan s + C.
\]

7.4.44 Because \( 5 + y^2 \) appears in the integrand, we should try the substitution \( y = 5 \tan \theta, \, dy = 5 \sec^2 \theta \, d\theta \). So,
\[
\int \frac{y^2}{25 + y^2} \, dy = \int \frac{25 \tan^2 \theta}{25(1 + \tan^2 \theta)} \cdot 5 \sec^2 \theta \, d\theta = 5 \int \tan^2 \theta \, d\theta
\]
\[
= 5 \int (\sec^2 \theta - 1) \, d\theta = 5 \tan \theta - 5\theta + C.
\]
For the last step, we must get back to the original variable \( y \). Recall that \( 5 \tan \theta = y \), and therefore \( \theta = \arctan(y/5) \), so
\[
\int \frac{y^2}{25 + y^2} = y - 5 \arctan(y/5) + C.
\]
7.4.53 The area of the indicated region is
\[ \int_{0}^{\sqrt{2}} \frac{x^3}{\sqrt{4-x^2}} \, dx. \]
We should try the substitution \( x = 2 \sin \theta, \, dx = 2 \cos \theta \, d\theta \) because of the appearance of \( 4 - x^2 \). If we change the limits as well, we won’t have to back-substitute at the end. So, \( x = 0 \) becomes \( \theta = 0 \), and \( x = \sqrt{2} \) becomes \( \theta = \pi/4 \).
\[
\int_{0}^{\sqrt{2}} \frac{x^3}{\sqrt{4-x^2}} \, dx = \int_{0}^{\pi/4} \frac{8 \sin^3 \theta}{\sqrt{4(1-\sin^2 \theta)}} (2 \cos \theta) \, d\theta = 8 \int_{0}^{\pi/4} \sin^3 \theta \, d\theta
\]
\[
= 8 \left[ \sin \theta - \sin \theta \cos^2 \theta \right]_{0}^{\pi/4} = 8 \left( -\cos \theta + \frac{\cos^3 \theta}{3} \right) \bigg|_{0}^{\pi/4}
\]
\[
= 8 \left( \frac{2}{3} - \frac{5\sqrt{2}}{12} \right) \approx .619
\]
(There are many equivalent ways to express the final answer).

7.5.7 Let \( f(x) = x^2 \). For each part, divide the interval \( 0 \leq x \leq 6 \) into two intervals of width 3.

(a) LEFT(2) = 3(f(0) + f(3)) = 27

(b) RIGHT(2) = 3(f(3) + f(6)) = 135

(c) TRAP(2) = \frac{27 + 35}{2} = 81

(d) MID(2) = 3(f(1.5) + f(4.5)) = 67.5
7.5.10  Let \( f(x) = \sin x \). For each part, divide the interval \( 0 \leq x \leq \pi \) into two intervals of width \( \pi/2 \).

(a) \[
\text{LEFT}(2) = \frac{\pi}{2} (f(0) + f(\frac{\pi}{2})) = \frac{\pi}{2}
\]

(b) \[
\text{RIGHT}(2) = \frac{\pi}{2} (f(\frac{\pi}{2}) + f(\pi)) = \frac{\pi}{2}
\]

(c) \[
\text{TRAP}(2) = \frac{\pi}{2} + \frac{\pi}{2} = \frac{\pi}{2}
\]

(d) \[
\text{MID}(2) = \frac{\pi}{2} (f(\frac{\pi}{4}) + f(\frac{3\pi}{4})) = \frac{\sqrt{2}\pi}{2}
\]

7.5.11  Let \( f(x) = \frac{1}{1+x^2} \). For part (a), divide the interval \( 0 \leq x \leq 1 \) into eight intervals of width \( \frac{1}{8} \).

(a) (i) \[
\text{LEFT}(8) = \frac{1}{8} \left( f(0) + f(\frac{1}{8}) + f(\frac{2}{8}) + \ldots + f(\frac{7}{8}) \right) \approx 0.8160.
\]

(ii) \[
\text{RIGHT}(8) = \frac{1}{8} \left( f(\frac{1}{8}) + f(\frac{2}{8}) + \ldots + f(\frac{8}{8}) \right) \approx 0.7535.
\]

(iii) \[
\text{TRAP}(8) = \frac{\text{LEFT}(8) + \text{RIGHT}(8)}{2} \approx 0.7847.
\]

(b) \( f(x) \) is a decreasing function for \( x > 0 \). Therefore

\[
\text{RIGHT}(8) \leq \int_0^1 f(x) \, dx \leq \text{LEFT}(8).
\]

In other words, part (i) is an overestimate and part (ii) is an underestimate.
7.5.20

(a) TRAP(4) probably gives the best estimate. We cannot calculate MID(4) because we’d need information that’s not in the table.

\[
\begin{align*}
\text{LEFT}(4) &= 3(f(0) + f(3) + f(6) + f(9)) = 1095 \\
\text{RIGHT}(4) &= 3(f(3) + f(6) + f(9) + f(12)) = 960 \\
\text{TRAP}(4) &= \frac{1095 + 960}{2} = 1027.5
\end{align*}
\]

(b) If we assume there are no points of inflection, the graph is either concave up or concave down. By plotting points we see that it is concave down. Therefore TRAP(4) is an underestimate.

7.5.22

(a) The graph of \( y = \sqrt{2 - x^2} \) is the upper half of a circle of radius \( \sqrt{2} \) centered at the origin. By drawing a picture of the region bounded by this circle, the \( x \)-axis, and the lines \( y = 0, y = 1 \), you can see that

\[
\int_0^1 \sqrt{2 - x^2} \, dx = A + B,
\]

where \( A \) is the area of a certain triangle and \( B \) is the area of a certain sector of a circle. Recall that the area of a triangle is \( \frac{bh}{2} \), and the area of a sector of a circle is \( \frac{r^2\theta}{2} \), where \( r \) is the radius of the circle and \( \theta \) is the angle subtended by the sector (in radians). If you draw the diagram, you’ll see that \( b = h = 1, r = \sqrt{2}, \theta = \pi/4 \). Therefore \( A = 1/2 \) and \( B = \pi/4 \).

(b) Using the methods of this section, LEFT(5) \( \approx 1.32350 \), RIGHT(5) \( \approx 1.24066 \), TRAP(5) \( \approx 1.28208 \), MID(5) \( \approx 1.28705 \). The exact value is \( \frac{1}{2} + \frac{\pi}{4} \approx 1.28540 \). So, the LEFT error is \( \approx -0.03810 \), the RIGHT error is \( \approx 0.04474 \), the TRAP error is \( \approx 0.00332 \), and the MID error is \( \approx -0.00166 \). So, for this function, MID is the most accurate, followed by TRAP. LEFT and RIGHT are much less accurate (as usual).

7.7.1

(a) This area looks essentially the same as Figure 7.17 (a) in your book. It extends infinitely far along the positive \( x \)-axis. (However, the area is infinite whereas the area in Figure 7.17 (a) is actually finite.)

(b) This area looks essentially the same as figure 7.19 (a) in your book. It extends infinitely far along the positive \( y \)-axis.
7.7.4

(a) $e^{-x^2}$ is a bell-shaped curve; see figure 5.67 in your book. The integral represents the entire area between the $x$-axis and the curve, infinitely far in both directions.

(b) Using a calculator,

\[
\int_{-1}^{1} e^{-x^2} \, dx \approx 1.49365 \\
\int_{-2}^{2} e^{-x^2} \, dx \approx 1.76416 \\
\int_{-3}^{3} e^{-x^2} \, dx \approx 1.77241 \\
\int_{-4}^{4} e^{-x^2} \, dx \approx 1.77245 \\
\int_{-5}^{5} e^{-x^2} \, dx \approx 1.77245
\]

(c) It appears that $\int_{-\infty}^{\infty} e^{-x^2} \, dx \approx 1.77245$.

7.7.10

\[
\int_{1}^{\infty} \frac{x}{4 + x^2} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{x}{4 + x^2} \, dx = \lim_{b \to \infty} \frac{1}{2} \ln |4 + x^2| \bigg|_{1}^{b} = \lim_{b \to \infty} \frac{1}{2} \ln |4 + b^2| - \frac{1}{2} \ln 5.
\]

Since $\lim_{b \to \infty} \ln |4 + b^2| = \infty$, the integral diverges.

7.7.15 This integral is improper because $\frac{1}{v}$ is undefined at $v = 0$. To evaluate it, we must split it into two integrals, $\int_{-1}^{0} \frac{1}{v} \, dv$ and $\int_{0}^{1} \frac{1}{v} \, dv$. But, notice that

\[
\int_{0}^{1} \frac{1}{v} \, dv = \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{v} \, dv = \lim_{a \to 0^+} (- \ln a) = \infty.
\]

Therefore the integral diverges.
7.7.22 We find the antiderivative with the help of the substitution $u = \ln x$, $du = \frac{1}{x} \, dx$. Namely,

$$\int \frac{\ln x}{x} \, dx = \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln x)^2 + C.$$  

This integral is improper because the integrand is undefined at $x = 0$. So, we have to take the limit from the right:

$$\int_0^1 \frac{\ln x}{x} \, dx = \lim_{a \to 0^+} \frac{1}{2} (\ln x)^2 \bigg|_{a}^{1} = \lim_{a \to 0^+} \frac{1}{2} (0 - (\ln a)^2) = -\infty.$$  

Therefore the integral diverges.

7.7.44

(a) The graph starts at the origin, increases to a maximum at $x = 2$, and then decreases back towards the $x$-axis as $x \to \infty$. (Your graph should show these three features clearly, although it’s okay if your sketch is not labeled with specific values.)

(b) People are getting sick fastest when the rate of infection is highest. Therefore we should set $\frac{dr}{dt} = 0$ to find the maximum value:

$$\frac{dr}{dt} = 1000e^{-0.5t} - 1000(0.5)te^{-0.5t} = 500e^{-0.5t}(2 - t),$$

which is zero only at $t = 2$. Therefore $t = 2$ is the moment at which the rate of infection is highest.

(c) The total number of sick people is $\int_0^\infty 1000e^{-0.5t} \, dt$. We’ll use integration by parts to find an antiderivative: let $u = 1000t$, $dv = e^{-0.5t} \, dt$, so $du = 1000 \, dt$, $v = -2e^{-0.5t}$.

$$\int 1000e^{-0.5t} \, dt = uv - \int v \, du = -2000te^{-0.5t} - \int (-2e^{-0.5t})(1000 \, dt)$$

$$= 1000e^{-0.5t}(-2t - 4) + C$$

Now, we can calculate the original improper integral:

$$\int_0^\infty 1000e^{-0.5t} \, dt = \lim_{b \to \infty} 1000e^{-0.5t}(-2t - 4) \bigg|_{0}^{b}$$

$$= 1000 \left[ \lim_{b \to \infty} e^{-0.5b}(-2b - 4) \right] - 1000(1)(-4)$$

However, we know that the limit above is equal to zero because $e^{-b}$ times any polynomial goes to zero as $b \to \infty$. Therefore the total number of infected people is 4000.