Section 11.5 #1, 2, 5, 10, 12, 21, 23, 24; Section 11.6 #5, 12, 14, 22, 23, 25; Section 11.7 #4, 7, 10, 12, 13

11.5.1

(a) (I)
(b) (IV)
(c) (II) and (IV)
(d) (II) and (III)

11.5.2  (a) = (I), (b) = (II), (c) = (III). Graph (II) represents an egg originally at 0° C which is moved to the kitchen table (20° C) two minutes after the first egg is moved.

11.5.5

(a) Equilibrium solutions occur where $y' = 0$, so the slope field is horizontal. There are two such lines, $y = 2$ and $y = -1$. $y = 2$ is stable since nearby slopes point towards it, but $y = -1$ is unstable.

(b) Draw solution curves by starting at these points and following the flow of the slope field. Solution curves will flow towards the line $y = 2$ or away from the line $y = -1$. 
11.5.10
(a) The differential equation says that the rate at which a cup of coffee cools is proportional to the difference between the temperature of the coffee and the temperature of the surrounding air. Since $\frac{dT}{dt} < 0$ when $T > 20$, $k$ must be positive.

(b) Using the separation of variables technique,

$$\int \frac{dT}{T - 20} = \int -k \, dt$$

$$\ln |T - 20| = -kT + C$$

$$T = 20 + Ae^{-kt}.$$  

If the coffee is initially boiling, we plug in $t = 0, T = 100$ to find that $A = 80$. To find $k$, we use the fact that $T = 90$ when $t = 2$, so

$$90 = 20 + 80e^{-2k}.$$  

Solving for $k$ gives $k = \frac{1}{2} \ln \frac{8}{7}$.

To find how long the coffee takes to cool to $60^\circ$ C, we set $T = 60$, so

$$60 = 20 + 80e^{-kt}.$$  

Solving for $t$ gives $t = \frac{\ln 2}{k} = \frac{2\ln 2}{\ln \frac{8}{7}} \approx 10$ minutes.

11.5.12
(a) $Q(t)$ is an exponential decay curve which passes through the points $(0, Q_0)$ and $(37, Q_0/2)$.

(b) $\frac{dQ}{dt} = -kQ$. The relationship between $k$ and the half-life is $k = \frac{\ln 2}{t_{1/2}}$ (See problem 23 for a derivation of this rule). Therefore $k = \frac{\ln 2}{37}$.

(c) Since $25% = (\frac{1}{2})^2$ it takes two half-lives = 74 hours for the drug level to reduce to 25%.
11.5.21

(a) \( \frac{dT}{dt} = -k(T - 68) \), where \( T \) is the temperature of the room (in Farenheit), \( t \) is the time in hours since 9 am, and \( k \) is an unknown constant.

(b) Using separation of variables as in problem 10, we’ll get the general solution \( T = 68 + Ae^{-kt} \). Using \( T(0) = 90.3 \) we get \( A = 22.3 \). Using \( T(1) = 89.0 \) we can solve for \( k \):

\[
\begin{align*}
89.0 &= 68 + 22.3e^{-k} \\
21 &= 22.3e^{-k} \\
k &= -\ln\left(\frac{21}{22.3}\right) \approx .06.
\end{align*}
\]

We want to know when \( T \) was equal to 98.6° F, the temperature of a live body. So we solve the following equation:

\[
\begin{align*}
98.6 &= 68 + 22.3e^{-0.06t} \\
\ln\left(\frac{30.6}{22.3}\right) &= -0.06t \\
t &\approx -5.27.
\end{align*}
\]

So, the victim was killed approximately 5\( \frac{1}{2} \) hours prior to 9am, at 3:45 am.

11.5.23

(a) \( C' = -kC \), and so \( C = C_0e^{-kt} \). Since the half-life is 5730 years, \( \frac{1}{2}C_0 = C_0e^{-5730k} \), and we use this equation to solve for \( k \):

\[
\ln\left(\frac{1}{2}\right) = -\ln 2 = -5730k \\
k = \frac{\ln 2}{5730} \approx .000121.
\]

(b) From the given information, \( 0.91 = e^{-kt} \), where \( t \) is the age of the shroud. Solving for \( t \), we’ll get \( t = \frac{-\ln 0.91}{k} \approx 779.4 \) years.
11.5.24

(a) Since speed is the derivative of distance, Galileo’s mistaken conjecture was

$$\frac{dD}{dt} = kD.$$ 

(b) If the conjecture were true, then $D(t) = D_0e^{kt}$, where $D_0$ is the initial distance fallen. But when we drop an object, it starts out not having traveled any distance, so $D_0 = 0$. This implies that $D = 0$ for all $t$.

11.6.5

(a) Since the rate of change is proportional to the amount present, $\frac{dy}{dt} = ky$ for some constant $k$.

(b) The general solution is $y = Ae^{kt}$, where $A$ is the initial amount. Since 100 grams becomes 54.9 grams in one hour, $54.9 = 100e^k$, so $k = \ln(54.9/100) \approx -0.5997$.

Therefore, after 10 hours, $y(10) = 100e^{-0.5997\times10} \approx 0.2486$ grams.

11.6.12 The rate at which the volume $V$ is decreasing is proportional to the surface area $A$, so

$$\frac{dV}{dt} = -kA.$$ 

We want to rewrite this differential equation in terms of the radius $r$. Since $V = \frac{4}{3}\pi r^3$, using the chain rule we have $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Also, $A = 4\pi r^2$, so the above equation becomes

$$4\pi r^2 \frac{dr}{dt} = -k4\pi r^2$$

$$\frac{dr}{dt} = -k.$$ 

That is, the radius decreases at a constant rate, so $r(t) = -kt + b$, which is the equation of a line. $b$ is the initial radius of 1 cm, and $k$ is the slope of 0.5 cm/month. Therefore

$$r(t) = -t/2 + 1.$$ 

To find $t$ when $r = 0.2$ is just basic algebra; solving the above equation gives $t = 1.6$ months.
(a) If $P =$ pressure (in inches of mercury) and $h =$ height (in feet), the problem says that $\frac{dP}{dh} = -kP$, where $k = 3.7 \times 10^{-5}$. Therefore

$$P = P_0 e^{-kh},$$

where $P_0 =$ pressure at sea level $= 29.92$.

At the top of Mt. Whitney, the pressure is $P = 29.92 e^{-3.7 \times 10^{-5} (14500)} \approx 17.60$. At the top of Mt. Everest, the pressure is $\approx 10.23$.

(b) We want to find $h$ when $P = 15$, so we have the equation

$$15 = 29.92 e^{-kh}.$$

Solving for $h$ gives $h = \frac{-1}{k} \ln \frac{15}{29.92} \approx 18662$ feet.
11.6.22

(a) This problem is a little easier to understand in terms of the variable \( Q(t) = \) total volume of carbon monoxide in the room. Of course, the concentration is just \( C(t) = Q(t)/60 \), since 60 m\(^3\) is the total volume of the room.

The flow of carbon monoxide into the room is \( 0.05 \times 0.002 \) m\(^3\)/min. The flow out is proportional to the current concentration, so it’s given by \( 0.002 \frac{Q(t)}{60} \). Altogether we have

\[
\frac{dQ}{dt} = 0.05 \times 0.002 - 0.002 \frac{Q}{60} = 0.002(0.05 - \frac{Q}{60})
\]

Now we must rewrite this in terms of the concentration \( C = \frac{Q}{60} \), so

\[
\frac{d(60C)}{dt} = 0.002(0.05 - C) \\
\frac{dC}{dt} = \frac{0.002}{60}(0.05 - C).
\]

(b) Now we can solve this using separation of variables:

\[
\int \frac{dC}{0.05 - C} = \frac{0.002}{60} \int \frac{0.002}{60} dt \\
- \ln |0.05 - C| = \frac{0.002}{60} t + K \\
0.05 - C = e^{-\frac{0.002}{60} t - K} \\
C = 0.05 - Ae^{-\frac{0.002}{60} t}
\]

(Remember that \( K \) in the exponent comes down as a multiplicative constant \( A \).) Since \( C(0) = 0 \) we find that \( A = 0.05 \), and so

\[
C(t) = 0.05(1 - e^{-3.33 \times 10^{-5} t}).
\]

(c) Since \( e^{-kt} \to 0 \) as \( t \to \infty \), we see that in the long run, the carbon monoxide concentration approaches 0.05, which is the concentration of the incoming air.

11.6.23 We want to solve for \( t \) when \( C = 0.001 \), so

\[
0.001 = 0.05(1 - e^{-3.33 \times 10^{-5} t}),
\]

which yields \( t \approx 673 \) min = 11 hours 13 minutes.
(a) Newton’s Law of Motion says that Force = (mass) × (acceleration), and acceleration is \( \frac{dv}{dt} \). Using the formula for the gravitational force, we have

\[
-\frac{mgR^2}{(R+h)^2} = m\frac{dv}{dt},
\]

so

\[
\frac{dv}{dt} = -\frac{gR^2}{(R+h)^2}.
\]

(b) Since \( v = \frac{dh}{dt} \), the chain rule gives

\[
\frac{dv}{dt} = \frac{dv}{dh} \frac{dh}{dt} = \frac{dv}{dh} v.
\]

Substituting into the equation from (a) gives

\[
\frac{v}{dh} \frac{dv}{dh} = -\frac{gR^2}{(R+h)^2}.
\]

(c) Separating variables gives

\[
\int v \, dv = - \int \frac{gR^2}{(R+h)^2} \, dh
\]

\[
\frac{v^2}{2} = \frac{gR^2}{(R+h)} + C.
\]

Since \( v = v_0 \) when \( h = 0 \),

\[
\frac{v_0^2}{2} = \frac{gR^2}{R} + C, \quad C = \frac{v_0^2}{2} - gR.
\]

Substituting this value for \( C \) gives the solution

\[
\frac{v^2}{2} = \frac{v_0^2}{2} + \frac{gR^2}{(R+h)} - gR.
\]

(Those of you who have studied physics might recognize this equation in terms of kinetic and potential energy, if you multiply through by \( m \)).

(d) Escape velocity means that the velocity never slows to zero; so \( v^2 > 0 \) for all \( h \geq 0 \). To ensure that \( v^2 \) remains positive as \( h \to \infty \), we must have

\[
\frac{v_0^2}{2} \geq gR
\]

since the middle term on the right approaches zero. Therefore \( v_0 = \sqrt{2gR} \) is the (minimum) escape velocity.
11.7.4  The US population in 1860 was 31.4 million. If between 1860 and 1870 the population had increased at the same rate as previous decades, 34.7%, the population in 1870 would have been 42.3 million. In actuality it was 38.6 million. This is a shortfall of 3.7 million people.

History records that about 618,000 soldiers died (total, both sides) in the Civil War. This accounts for only 1/6 of the shortfall (roughly). The rest of the shortfall was undoubtedly caused by many factors, including civilian deaths, declined birthrate due to absent males, and unwillingness to have children in harsh wartime conditions.

11.7.7

<table>
<thead>
<tr>
<th>Year</th>
<th>( P )</th>
<th>( \frac{dP}{dt} \approx \frac{1}{20}(P(t+10) - P(t-10)) )</th>
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<tbody>
<tr>
<td>1790</td>
<td>3.9</td>
<td></td>
</tr>
<tr>
<td>1800</td>
<td>5.3</td>
<td>.165</td>
</tr>
<tr>
<td>1810</td>
<td>7.2</td>
<td>.215</td>
</tr>
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<td>1820</td>
<td>9.6</td>
<td>.285</td>
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<td>12.9</td>
<td>.375</td>
</tr>
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</tr>
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<td>1850</td>
<td>23.2</td>
<td>.715</td>
</tr>
<tr>
<td>1860</td>
<td>31.4</td>
<td>.770</td>
</tr>
<tr>
<td>1870</td>
<td>38.6</td>
<td>.940</td>
</tr>
<tr>
<td>1880</td>
<td>50.2</td>
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</tr>
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<td>1890</td>
<td>62.9</td>
<td>1.290</td>
</tr>
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<td>1900</td>
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<tr>
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<td>1.395</td>
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<tr>
<td>1950</td>
<td>150.7</td>
<td></td>
</tr>
</tbody>
</table>

According to these calculations, the largest growth rate \( \frac{dP}{dt} \) occurs in 1920. The population in 1920 was 105.7 million. If we assume that the limiting value \( L \) is twice the population when it is changing the most quickly (as in the logistic equation), then \( L = 2 \times 105.7 = 211.4 \) million. This is greater than the estimate of 187 million computed in the text.

However, this estimate of carrying capacity must be viewed with caution, since the current US population as of 2007 is estimated at over 300 million and still growing. (Also, this logistic model is generally considered too simplistic to use for human populations, as humans are rather unique in their ability to change their own carrying capacity depending on social structure).
11.7.10

(a) Let $P(t)$ represent the population at time $t$. In this problem it is important to know the definition of relative growth rate $G$, which means absolute growth rate divided by the current population: $G = \frac{dP}{dt} / P$.

We assume that the relative growth rate is a linear function of population, so

$$G(P) = mP + b.$$ 

The relative growth rate equals the relative birth rate minus the relative death rate. Therefore $G(600) = .35 - .15 = .20$, and $G(800) = .30 - .20 = .10$. Using standard algebra methods to compute the line through these two points, we find that $G(P) = \frac{1}{2} - \frac{P}{2000}$. Therefore our differential equation is

$$\frac{dP}{dt} / P = \frac{1}{2} - \frac{P}{2000}.$$ 

We can rewrite this in standard form for a logistic equation:

$$\frac{dP}{dt} = \frac{1}{2000}P(1000 - P).$$ 

(b) Equilibrium population occurs when $\frac{dP}{dt} = 0$. From above, we see that $P = 0$ (extinction) and $P = 1000$ (carrying capacity) are the equilibrium points. We expect a population of 900 to increase to the equilibrium value of 1000.

(c) If additional elk are added, the population 1350 elk is above carrying capacity, so we expect it to decrease to 1000.

(d) Prior to the addition of the elk, we’ll have an S-shaped (sigmoid) curve that increases to 900. Your graph should have a discontinuous jump to 1350 with the addition of elk, then decrease back towards 1000.

Importing more elk is ecologically unsound because, according to this model, it would cause them to become overpopulated and start dying off.
(a) By the chain rule,
\[
\frac{dP}{dt} = \frac{dP}{du} \frac{du}{dt}.
\]
(This is an alternative, equivalent form of the chain rule that is very useful for this kind of problem). Since \( P = \frac{1}{u} \), we have \( \frac{dP}{du} = -\frac{1}{u^2} \), and so
\[
\frac{dP}{dt} = -\frac{1}{u^2} \frac{du}{dt}.
\]

(b) Substituting into the equation
\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{L}\right)
\]
gives
\[
-\frac{1}{u^2} \frac{du}{dt} = \frac{k}{u} \left(1 - \frac{1}{Lu}\right)
\]
\[
\frac{du}{dt} = -k \left(u - \frac{1}{L}\right).
\]
Now we separate variables, integrate, and simplify:
\[
\int \frac{du}{u - 1/L} = -\int k \, dt
\]
\[
\ln |u - 1/L| = -kt + C
\]
\[
u - 1/L = Ae^{-kt}
\]
\[
u = 1/L + Ae^{-kt}.
\]

(c) Finally, substitute \( u = 1/P \) to get back to the original variable:
\[
1/P = 1/L + Ae^{-kt}
\]
\[
P = \frac{1}{1/L + Ae^{-kt}} = \frac{L}{1 + LAt e^{-kt}}.
\]
11.7.13

(a) This question involves drawing a slope field. Importantly, everywhere along the lines $y = 0$ and $y = 4$ the slope should be zero. In between, the slope should be negative, and elsewhere the slope should be positive.

(b) The solution curves follow the slope field in (a). If the initial value $y_0$ is between 0 and 4, the slope is negative and so the solution curves down towards 0. If $y_0 = 4$, the solution is at equilibrium and remains at 0. If $y_0 > 4$, the solution will quickly increase away from 0.

(c) As above, the two equilibrium values are $P = 0$ and $P = 4$. The first, representing extinction, is stable. The second is unstable. Also, we could obtain the equilibrium values by setting the

$$\frac{dP}{dt} = .02P^2 - .08P = .02P(P - 4) = 0.$$