How do we measure speed? (2.1)

Velocity graphs

PLOT Table 2.1. This data show the height of a small object thrown straight upward in feet above the ground as a function of time, measured in seconds.

GRAPH table and label data points.

1. WHAT does this graph represent?
2. DOES it represent the trajectory of the object? DRAW trajectory and label data points.
3. WHAT was the initial height of the object?
4. WHAT was the maximum height of the object? HOW do we know that the height was greater than 162 feet?
5. WHEN did the object reach its maximum height?
6. HOW fast was the object thrown initially? WHY is the initial velocity more than 84 ft/sec?
7. WHEN did the object hit the ground? HOW do we know it wasn't more than 7 seconds?
8. The average velocity during the first second was 84 ft/sec. WAS the velocity constant during the first second?

Average velocity

Average velocity is defined to be equal to \( \frac{\text{Change in position}}{\text{Change in time}} = \frac{\Delta s}{\Delta t} = \frac{s(b) - s(a)}{b - a} \).

Show graphically that the slope of the line segment joining two points on the graph represents the average velocity for that interval of time. This line segment is called a secant. The secant represents how the object would have traveled if its velocity were constant over that time interval; the slope of the secant represents this constant velocity.

The difference quotient, \( \frac{s(b) - s(a)}{b - a} \), is simply the slope of the secant and is equal to the average velocity on the time interval \((a, b)\). A difference quotient was defined in section 1.1 as the quotient of two differences, here the difference in height divided by the difference in time.

Instantaneous velocity

The average velocity during the first second, \((0, 1)\), is 84 ft/s and during the second second, \((1, 2)\), 52 ft/s. WHAT is the instantaneous velocity when \(t\) is exactly equal to 1? See Fig 2.2. We can estimate the answer more accurately if we take smaller and smaller time intervals around \(t = 1\).

The average velocity on \((0.9, 1)\) is \(\frac{90 - 83.04}{1 - 0.9} = \frac{6.96}{0.1} = 69.6\) ft/sec,

the average velocity on \((0.99, 1)\) is \(\frac{90 - 89.318}{1 - 0.99} = \frac{0.682}{0.01} = 68.2\) ft/sec,

and the average velocity on \((0.999, 1)\) is \(\frac{90 - 89.932}{1 - 0.999} = \frac{0.068}{0.001} = 68.0\) ft/sec.

WHAT do we notice as our interval shrinks? So we estimate the instantaneous velocity at \(t = 1\) to be approximately 68 ft/sec.
Estimating instantaneous velocities numerically

**ESTIMATE** the instantaneous velocity at \( t = 2 \) given that \( s(t) = t^2 + 1 \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>1.999</th>
<th>2</th>
<th>2.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s(t) )</td>
<td>4.996001</td>
<td>5</td>
<td>5.004001</td>
</tr>
</tbody>
</table>

The average velocity on the interval \((1.999, 2)\) is equal to \( \frac{s(2) - s(1.999)}{2 - 1.999} = \frac{5 - 4.996001}{0.001} = 3.999 \). The average velocity on the interval \((2, 2.001)\) is equal to \( \frac{s(2.001) - s(2)}{2.001 - 2} = \frac{5.004001 - 5}{0.001} = 4.001 \). Therefore, we can estimate the instantaneous velocity at \( t = 2 \) to be approximately 4. HOW does our solution method for this problem differ from our solution method in the previous problem?

Estimating instantaneous velocities graphically

Since the slope of the secant joining any two points on a position graph represents the average velocity for that interval, we can guess that the slope of the tangent to the graph at a point represents the instantaneous velocity for that input value. GRAPH \( s(t) = t^2 + 1 \) using a graphing calculator and ZOOM IN around \( t = 2 \). Notice that the graph appears to look more and more like a (straight) line as we zoom in. This effect is called **local linearity**. WHAT is the slope of this tangent line at \( t = 2 \)? CALCULATE the slope between two points on the graph near \( t = 2 \) to show the slope is approximately equal to 4.

CALCULATE and DRAW the tangent line to the graph at \( t = 2 \). ZOOM IN on the point \((2, 5)\). Notice that the graph of \( s \) and the tangent line appear to coincide as we zoom in.

Calculating instantaneous velocities algebraically

If \( s(t) \) represents the position of an object at time \( t \), then the instantaneous velocity at \( t = a \) is defined as the limit of the difference quotient of the average velocity as the time interval shrinks around \( a \), \( \lim_{h \to 0} \frac{s(a + h) - s(a)}{h} \). Note that \( h \) can be either positive or negative, i.e., we can approach \( a \) from either the left or the right.

For \( s(t) = t^2 + 1 \), the instantaneous velocity of \( s \) at \( t = 2 \) is equal to
\[
\lim_{h \to 0} \frac{s(2 + h) - s(2)}{h} = \lim_{h \to 0} \frac{(2 + h)^2 + 1 - 5}{h} = \lim_{h \to 0} \frac{4 + 4h + h^2}{h} = 4.
\]

Therefore, the instantaneous velocity of \( s \) at \( t = 2 \) is equal to 4.

**ESTIMATE** the instantaneous velocity for \( s(t) = \frac{1}{t} \) at \( t = 1 \). CALCULATE the instantaneous velocity for \( s(t) = \frac{1}{t} \) at \( t = 1 \).