Shortcuts to differentiation

We have learned four ways to estimate or calculate a derivative at a point, \( f'(a) \): numerically, using estimation and calculating the average rate of change on the interval \([a, a+h]\) for a small \( h \) graphically, by zooming in on the point \((a, f(a))\) on the graph and estimating the slope of the tangent line there algebraically, \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \) calculating \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \), then evaluating \( f'(a) \)

Differentiation rules

In this chapter, we will learn more rules that will allow us to find derivative functions for a far greater number of functions. Previously, we have proven the following rules for derivative functions.

- **Function** \( f(x) = k \)
- **Derivative Function or Rule** \( f'(x) = 0 \)
- **Function** \( f(x) = mx + b \)
- **Derivative Function or Rule** \( f'(x) = m \)
- **Function** \( f(x) = ax^2 + bx + c \)
- **Derivative Function or Rule** \( f'(x) = 2ax + b \)

Derivatives of power functions

**FIND** the derivative of \( f(x) = x^3 \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} = \lim_{h \to 0} \frac{3hx^2 + 3h^2x + h^3}{h} = 3x^2 + 3hx + h^2 = 3x^2.
\]

What about higher powers of \( x \)? We could use our limit definition to calculate derivative functions for other power functions.

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( x^2 )</td>
<td>( 2x )</td>
</tr>
<tr>
<td>( x^3 )</td>
<td>( 3x^2 )</td>
</tr>
<tr>
<td>( x^4 )</td>
<td>( 4x^3 )</td>
</tr>
<tr>
<td>( x^5 )</td>
<td>( 5x^4 )</td>
</tr>
</tbody>
</table>

**WHAT** do you guess the rule (derivative function) would be for the power function, \( f(x) = x^n \)? **PROVE** \( \frac{d}{dx}(x^n) = nx^{n-1} \) for any positive integer \( n \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{(x^n + nhx^{n-1} + \ldots + nxh^{n-1} + h^n) - x^n}{h} = \lim_{h \to 0} \frac{nhx^{n-1} + \ldots + nxh^{n-1} + h^n}{h} = \lim_{h \to 0} nx^{n-1} + \left(\frac{n}{2}\right)hx^{n-2} + \ldots + nxh^{n-2} + h^{n-1} = nx^{n-1} \). The text extends this to any real number \( n \) on p.112. This derivative of a power function is called the *Power Rule*. **FIND** the derivative function for \( f(x) = \frac{1}{x^2} \).
Derivative function of a constant multiple

LET \( f(x) = x^2 \) and \( g(x) = 2x^2 \). We want to show numerically that \( g'(1) = 2f'(1) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.999</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.998001</td>
<td>1</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>1.996002</td>
<td>2</td>
</tr>
</tbody>
</table>

We estimate \( f'(1) = \frac{f(1) - f(0.999)}{1 - 0.999} = \frac{1 - 0.998001}{0.001} = \frac{0.001999}{0.001} = 1.999 \), and we estimate \( g'(1) = \frac{g(1) - g(0.999)}{1 - 0.999} = \frac{2 - 1.996002}{0.001} = \frac{0.003998}{0.001} = 3.998 = 2f'(1) \).

This principle is stated in Theorem 3.1: If \( f \) is differentiable and \( c \) is a constant,
then \( \frac{d}{dx}[cf(x)] = cf'(x) \). FIND the derivative function for \( f(x) = -\frac{2}{3}x^{3/4} \).

Derivatives of sums and differences

Another useful theorem is Theorem 3.2 that states that the derivative of a sum or difference of two functions is equal to the sum or difference of the separate derivatives: If \( f \) and \( g \) are differentiable, then
\[
\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x).
\]
FIND the derivative function for \( g(x) = x^2 - x^3 \).

Derivatives of polynomials

We can now use the Constant Multiple, Sum and Difference, and Power rules to find the derivatives of all polynomials.

FIND the derivative of \( f(x) = 4x^3 - 3x^2 + x - 10 \).

FIND the second derivative of \( g(x) = ax^2 + bx + c \) and interpret its sign. If \( a > 0 \), then \( g \) is concave up; if \( a < 0 \), then \( g \) is concave down.

Derivatives of other functions

SOLVE problem 3.1.29.

SOLVE problem 3.1.53.