6.2 Taylor’s Theorem Problems

Theorem 6.8 (Taylor’s at order 1 and 2). If \( f(t) \) for \( 0 \leq t \leq 1 \) is twice continuously differentiable, then

\[
  f(1) = f(0) + \int_0^1 f'(t) \, dt \quad \text{and} \quad f'(1) = f'(0) + \frac{1}{1!} \int_0^1 f''(t) (1-t) \, dt.
\]

Proof. The first assertion follows by the fundamental theorem of calculus

\[
  f(1) - f(0) = \int_0^1 f'(t) \, dt.
\]

For the second we integrate by parts as follows;

\[
  \int_0^1 f'(t) \, dt = - \int_0^1 f'(t) \, d(1-t) = -f'(t) (1-t) \bigg|_0^1 + \int_0^1 f''(t) (1-t) \, dt
\]

and therefore

\[
  f(1) - f(0) = \int_0^1 f'(t) \, dt = f'(0) + \frac{1}{1!} \int_0^1 f''(t) (1-t) \, dt.
\]

Exercise 6.1. If \( f : \mathbb{R} \to \mathbb{C} \) is a function which is differentiable to all orders, show for all \( N \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \} \) that

\[
  f(1) = \sum_{k=0}^N \frac{1}{k!} f^{(k)}(0) + \frac{1}{N!} \int_0^1 f^{(N+1)}(t) (1-t)^N \, dt.
\]

Hint: use integration by parts and induction with Theorem 6.8 providing the case \( N = 0 \) and \( N = 1 \).

Exercise 6.2. Recall the if we define \( e^z := e^x (\cos y + i \sin y) \) where \( z = x + iy \), then \( \frac{d}{dz} e^{\ell z} = ze^{\ell z} \). Use Exercise 6.1 with \( f(t) = e^{\ell z} \) to conclude;

\[
  e^z = f(1) = \sum_{k=0}^N \frac{z^k}{k!} + R_N(z) \quad \quad (6.4)
\]

where

\[
  R_N(z) = \frac{1}{N!} \int_0^1 e^{\ell z} (1-t)^N \, dt. \quad \quad (6.5)
\]

Then show that \( \lim_{N \to \infty} |R_N(z)| = 0 \) for all \( z \in \mathbb{C} \) and use this to conclude

\[
  e^z = \sum_{k=0}^\infty \frac{z^k}{k!} \quad \text{for all } z \in \mathbb{C}.
\]