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Undergraduate Analysis Tools

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Numbers
Natural, integer, and rational Numbers

Notation 1.1 Let \( \mathbb{N} = \{1, 2, \ldots \} \) denote the natural numbers, \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \),
\[
\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots \} = \{ \pm n : n \in \mathbb{N}_0 \}
\]
be the integers, and
\[
\mathbb{Q} := \left\{ \frac{m}{n} : m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}
\]
be the rational numbers.

I am going to assume that the reader is familiar with all the standard arithmetic operations (addition, multiplication, inverses, etc.) on \( \mathbb{N}_0, \mathbb{Z}, \) and \( \mathbb{Q} \). However, let us review the important induction axiom of the natural numbers.

Induction Axiom If \( S \subset \mathbb{N} \) is a subset such that \( 1 \in S \) and \( n + 1 \in S \) whenever \( n \in S \), then \( S = \mathbb{N} \).

This axiom leads takes on two other useful forms which we describe in the next Propositions.

Proposition 1.2 (Strong form of Induction). Suppose \( S \subset \mathbb{N} \) is a subset such that \( 1 \in S \) and \( n + 1 \in S \) whenever \( \{1, 2, \ldots, n\} \subset S \), then \( S = \mathbb{N} \).

Proof. Let \( T := \{n \in \mathbb{N} : \{1, 2, \ldots, n\} \subset S\} \). Then \( 1 \in T \) and if \( n \in T \) then \( n + 1 \in T \) by assumption. Therefore by the induction axiom, \( T = \mathbb{N} \) so that \( \{1, 2, \ldots, n\} \subset S \) in for all \( n \in \mathbb{N} \). This suffices to show \( S = \mathbb{N} \).

Proposition 1.3 (Well ordering principle). Suppose \( S \subset \mathbb{N} \) is a non-empty subset, then there exists a smallest element \( m \) of \( S \).

Proof. If \( 1 \in S \), then \( m = 1 \). If \( 1 \notin S \), let \( T = \{ n \in \mathbb{N} : n < s \text{ for all } s \in S \} \). Then \( 1 \in T \) and if \( n \in T \) but \( n + 1 \notin T \) there exists \( s \in S \) such that \( n < s \leq n + 1 \) which would force \( s = n \) which contradicts \( n \notin S \). Thus we must conclude that if \( n \in T \) then \( n + 1 \in T \) and therefore that \( T = \mathbb{N} \). This would then show \( S = \emptyset \).

Remark 1.4. Let us further observe that the well ordering principle implies the induction axiom. Indeed, suppose that \( S \subset \mathbb{N} \) is a subset such that \( 1 \in S \) and \( n + 1 \in S \) whenever \( n \in S \). For sake of contradiction suppose that \( S \neq \mathbb{N} \) so that \( T := \mathbb{N} \setminus S \) is not empty. By the well ordering principle there \( T \) has a unique minimal element \( m \) and in particular \( T \subset \{m, m + 1, \ldots \} \). This implies that \( \{1, 2, \ldots, m - 1\} \subset S \) and then by assumption that \( \{1, 2, \ldots, m\} \subset S \). But this then implies \( T \subset \{m + 1, \ldots \} \) and therefore \( m \notin T \) which violates \( m \) being the minimal element of \( T \). We have arrived at the desired contradiction and therefore conclude that \( S = \mathbb{N} \).

Recall that, for \( q \in \mathbb{Q} \), we define
\[
|q| = \begin{cases} 
q & \text{if } q \geq 0 \\
-q & \text{if } q \leq 0.
\end{cases}
\]

Recall that, for all \( a, b \in \mathbb{Q} \),
\[
|a + b| \leq |a| + |b| \text{ and } |ab| = |a| |b|.
\]
[This will be discussed in more generality in Section 2.2 below.]

Lemma 1.5. If \( a, b \in \mathbb{Q} \), then
\[
||b| - |a|| \leq |b - a|.
\]

Proof. Since both sides of Eq. (1.1) are symmetric in \( a \) and \( b \), we may assume that \( |b| \geq |a| \) so that \( ||b| - |a|| = |b - |a||. \) Since
\[
|b| = |b - a + a| \leq |b - a| + |a|,
\]
it follows that
\[
||b| - |a|| = |b - |a|| \leq |b - a|.
\]

The proof of the previous lemma illustrates one of the key techniques of real analysis, namely adding 0 to an expression. In this case we added 0 in the form of \( -a + a \) to \( b \). The next remark records a couple of other very important “tricks” in this subject. Taking to heart the following remarks will great aid the student in real analysis.

Remark 1.6 (Some basic philosophies of real analysis). Let \( a, b, \varepsilon \) be numbers (i.e. in \( \mathbb{Q} \) or later real numbers).
1. We will often prove that \( a = b \) by proving \( a \leq b \) and \( b \leq a \).
2. Moreover, we will often prove \( a \leq b \) by showing that \( a \leq b + \varepsilon \) for all \( \varepsilon > 0 \).

(See the next theorem.)

**Theorem 1.7.** The rational numbers have the following properties:

1. For any \( p \in \mathbb{Q} \) there exists \( N \in \mathbb{N} \) such that \( p < N \).
2. For any \( \varepsilon \in \mathbb{Q} \) with \( \varepsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that \( 0 < \frac{1}{N} < \varepsilon \).
3. If \( a, b \in \mathbb{Q} \) and \( a \leq b + \varepsilon \) for all \( \varepsilon > 0 \), then \( a \leq b \).

**Proof.** 1. If \( p \leq 0 \) we may take \( N = 1 \). So suppose that \( p = \frac{m}{n} \) with \( m, n \in \mathbb{N} \). In this case let \( N = m \).
2. Write \( \varepsilon = \frac{m}{n} \) with \( m, n \in \mathbb{N} \) and then take \( N = 2n \).
3. If \( a \leq b \) is false happens iff \( a > b \) which is equivalent to \( a - b > 0 \). If we now let \( \varepsilon := \frac{a-b}{2} > 0 \), then

\[
a = b + (b - a) > b + \varepsilon
\]

which would violate the assumption that \( a \leq b + \varepsilon \) for all \( \varepsilon > 0 \).

### 1.1 Limits in \( \mathbb{Q} \)

In this course we will often use the abbreviations, i.o. and a.a. which stand for infinitely often and almost all respectively. For example \( a_n \leq b_n \) a.a. \( n \) means there exists an \( N \in \mathbb{N} \) such that \( a_n \leq b_n \) for all \( n \geq N \) while \( a_n \leq b_n \) i.o. \( n \) means for all \( N \in \mathbb{N} \) there exists a \( n \geq N \) such that \( a_n \leq b_n \). So for example, \( 1/n \leq 1/100 \) for a.a. \( n \) while and \((-1)^n \geq 0 \) i.o. By the way, it should be clear that if something happens for a.a. \( n \) then it also happens i.o. \( n \).

**Definition 1.8.** A sequence \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{Q} \) converges to \( 0 \in \mathbb{Q} \) if for all \( \varepsilon > 0 \) in \( \mathbb{Q} \) there exists \( N \in \mathbb{N} \) such that \( |a_n| \leq \varepsilon \) for all \( n \geq N \). Alternatively put, for all \( \varepsilon > 0 \) we have \( |a_n| \leq \varepsilon \) for a.a. \( n \). This may also be stated as for all \( M \in \mathbb{N} \),

\[
|a_n| \leq \frac{1}{M}
\]

for a.a. \( n \).

**Definition 1.9.** A sequence \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{Q} \) converges to \( a \in \mathbb{Q} \) if \( |a - a_n| \to 0 \) as \( n \to \infty \), i.e. if for all \( N \in \mathbb{N} \), \( |a - a_n| \leq \frac{1}{N} \) for a.a. \( n \). As usual if \( \{a_n\}_{n=1}^{\infty} \) converges to \( a \) we write \( a_n \to a \) as \( n \to \infty \) or \( a = \lim_{n \to \infty} a_n \).

**Proposition 1.10.** If \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{Q} \) converges to \( a \in \mathbb{Q} \), then \( \lim_{n \to \infty} |a_n| = |a| \).

\footnote{We will see that the real numbers have these same properties as well.}

**Proof.** From Lemma 1.5 we have,

\[
|a| - |a_n| \leq |a - a_n|.
\]

Thus if \( \varepsilon > 0 \) is given, by definition of \( a_n \to a \) there exists \( N \in \mathbb{N} \) such that \( |a - a_n| < \varepsilon \) for all \( n \geq N \). From the previously displayed equation, it follows that \( |a| - |a_n| < \varepsilon \) for all \( n \geq N \) and hence we may conclude that \( \lim_{n \to \infty} |a_n| \) exists and is equal to \( |a| \).

**Lemma 1.11 (Convergent sequences are bounded).** If \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{Q} \) converges to \( a \in \mathbb{Q} \), then there exists \( M \in \mathbb{Q} \) such that \( |a_n| \leq M \) for all \( n \in \mathbb{N} \).

**Proof.** Taking \( \varepsilon = 1 \) in the definition of \( a = \lim_{n \to \infty} a_n \) implies there exists \( N \in \mathbb{N} \) such that \( |a_n - a| \leq 1 \) for all \( n \geq N \). Therefore,

\[
|a_n| = |a_n - a + a| \leq |a_n - a| + |a| \leq 1 + |a| \text{ for } n \geq N.
\]

We may now take \( M := \max \left( \{ |a_n|_{n=1}^{N} \} \cup \{1 + |a| \} \right) \).

**Theorem 1.12.** If \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{Q} \) converges to \( a \in \mathbb{Q} \setminus \{0\} \), then

\[
\lim_{n \to \infty} \frac{1}{a_n} = \frac{1}{a}.
\]

It is possible that \( a_n = 0 \) for small \( n \) so that \( \frac{1}{a_n} \) is not defined but for large \( n \) this can not happen and therefore it makes sense to talk about the limit which only depends on the tail of the sequences.

**Proof.** Since \( a \neq 0 \) we know that \( |a| > 0 \). Hence, there exists \( M := M|a| \in \mathbb{N} \) such that \( |a_n - a| < \frac{|a|}{2} \) for all \( n \geq M \). Therefore for \( n \geq M \)

\[
|a| = |a - a_n + a_n| \leq |a - a_n| + |a_n| < \frac{|a|}{2} + |a_n|
\]

from which it follows that \( |a_n| > \frac{|a|}{2} \) for all \( n \geq M \). If \( \varepsilon > 0 \) is given arbitrarily, we may choose \( N \geq M \) such that \( |a - a_n| < \varepsilon \) for all \( n \geq M \). Then for \( n \geq N \) we have,

\[
\left| \frac{1}{a_n} - \frac{1}{a} \right| = \left| \frac{a - a_n}{a_n a} \right| = \frac{|a_n - a|}{|a_n| |a|} < \frac{\varepsilon}{\frac{|a|}{2} |a|} = \frac{2 \varepsilon}{|a|^2}.
\]

As \( \varepsilon > 0 \) is arbitrary it follows that \( \frac{2 \varepsilon}{|a|^2} > 0 \) is arbitrarily small as well (replace \( \varepsilon \) by \( \varepsilon |a|^2 / 2 \) if you feel it is necessary), and hence we may conclude that Eq. (1.2) holds.
Definition 1.13. A sequence \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{Q} \) is **Cauchy** if \( |a_n - a_m| \to 0 \) as \( m, n \to \infty \). More precisely we require for each \( \varepsilon > 0 \) in \( \mathbb{Q} \) that \( |a_m - a_n| \leq \varepsilon \) for a.a. pairs \((m, n)\), i.e. there should exists \( N \in \mathbb{N} \) such that \( |a_m - a_n| \leq \varepsilon \) for all \( m, n \geq N \).

Exercise 1.1. Show that all convergent sequences \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{Q} \) are Cauchy.

Exercise 1.2. Show all Cauchy sequences \( \{a_n\}_{n=1}^{\infty} \) are bounded – i.e. there exists \( M \in \mathbb{N} \) such that

\[
|a_n| \leq M \quad \text{for all} \quad n \in \mathbb{N}.
\]

Exercise 1.3. Suppose \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are Cauchy sequences in \( \mathbb{Q} \). Show \( \{a_n + b_n\}_{n=1}^{\infty} \) and \( \{a_n \cdot b_n\}_{n=1}^{\infty} \) are Cauchy.

Exercise 1.4. Assume that \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are convergent sequences in \( \mathbb{Q} \). Show \( \{a_n + b_n\}_{n=1}^{\infty} \) and \( \{a_n \cdot b_n\}_{n=1}^{\infty} \) are convergent in \( \mathbb{Q} \) and

\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n \quad \text{and} \quad \lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n.
\]

Exercise 1.5. Assume that \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are convergent sequences in \( \mathbb{Q} \) such that \( a_n \leq b_n \) for all \( n \). Show \( A := \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n =: B \).

Exercise 1.6 (Sandwich Theorem). Assume that \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are convergent sequences in \( \mathbb{Q} \) such that \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n \). If \( \{x_n\}_{n=1}^{\infty} \) is another sequence in \( \mathbb{Q} \) which satisfies \( a_n \leq x_n \leq b_n \) for all \( n \), then

\[
\lim_{n \to \infty} x_n = a := \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n.
\]

Please note that that main part of the problem is to show that \( \lim_{n \to \infty} x_n \) exists in \( \mathbb{Q} \). Hint: start by showing; if \( a \leq x \leq b \) then \( |x| \leq \max(|a|, |b|) \).

1.2 The Problem with \( \mathbb{Q} \)

The problem with \( \mathbb{Q} \) is that it is full of “holes.” To be more precise, \( \mathbb{Q} \) is not “complete,” i.e. not all Cauchy sequences are convergent. In fact, according to Corollary A.12 below, “most” Cauchy sequences of rational numbers do not converge to a rational number. Let us demonstrate some example pointing out this flaw.

Example 1.14. Let \( S_n := \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q} \) for all \( n \in \mathbb{N} \). For \( n > m \) in \( \mathbb{N} \) we have,

\[
0 \leq S_n - S_m = \sum_{k=m+1}^{n} \frac{1}{k!} = \sum_{j=1}^{n-m} \frac{1}{(m+j)!} = \frac{1}{(m+1)!} + \cdots + \frac{1}{n!} \leq \frac{1}{m!} \left[ \frac{1}{m+1} + \left( \frac{1}{m+1} \right)^2 + \cdots + \left( \frac{1}{m+1} \right)^{n-m} \right] \leq \frac{1}{m!} \left[ \frac{1}{m+1} \right. 
\]

From this inequality it follows that \( \{S_n\}_{n=0}^{\infty} \) is a Cauchy sequence. Suppose that \( e := \lim_{n \to \infty} S_n \) were to exist in \( \mathbb{Q} \). Then letting \( n \to \infty \) in Eq. (1.3) would show,

\[
0 < e - S_m \leq \frac{1}{m \cdot m!}.
\]

Multiplying this inequality by \( m! \) then implies,

\[
0 < m! e - m! S_m \leq \frac{1}{m}.
\]

However for \( m \) sufficiently large \( m! e \in \mathbb{N} \) (as \( e \) is assumed to be rational) and \( m! S_m \) is always in \( \mathbb{N} \) and therefore \( k := m! e - m! S_m \in \mathbb{N} \). But there is no element \( k \in \mathbb{N} \) such that \( 0 < k < \frac{1}{m} \) and hence we must conclude \( \lim_{n \to \infty} S_n \) can not exist in \( \mathbb{Q} \). Moral: the number \( e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n \to \infty} (1 + \frac{1}{n})^{n} \) that you learned about in calculus is not in \( \mathbb{Q} \!.

Example 1.15 (Square roots need not exist). The square root, \( \sqrt{2} \), of 2 does not exist in \( \mathbb{Q} \). Indeed, if \( \sqrt{2} = \frac{m}{n} \) where \( m \) and \( n \) have no common factors (in particular are both odd integers), then

\[
\frac{m^2}{n^2} = 2 \implies m^2 = 2n^2.
\]

This shows that \( m^2 \) is even which would then imply that \( m = 2k \) is even (since odd-odd=odd). However this implies \( 4k^2 = 2n^2 \) from which it follows that \( n^2 \) and hence \( n \) is even contradicting the assumption that \( m \) and \( n \) had no common factors (of 2).

Exercise 1.7. Use the following outline to construct another Cauchy sequence \( \{q_n\}_{n=1}^{\infty} \subset \mathbb{Q} \) which is not convergent in \( \mathbb{Q} \).
1. Recall that there is no element \( q \in \mathbb{Q} \) such that \( q^2 = 2 \). To each \( n \in \mathbb{N} \) let \( m_n \in \mathbb{N} \) be chosen so that
\[
\frac{m_n^2}{n^2} < 2 < \frac{(m_n + 1)^2}{n^2}
\] (1.4)
and let \( q_n := \frac{m_n}{n} \).

2. Verify that \( q_n^2 \to 2 \) as \( n \to \infty \) and that \( \{q_n\}_n \) is a Cauchy sequence in \( \mathbb{Q} \).

3. Show \( \{q_n\}_n \) does not have a limit in \( \mathbb{Q} \).

Example 1.16. It is also a fact that \( \pi \notin \mathbb{Q} \) where
\[
\pi = 2 \int_0^\infty \frac{1}{1 + x^2} \, dx = 2 \lim_{N \to \infty} \int_0^N \frac{1}{1 + x^2} \, dx
\]
\[
= 2 \lim_{N \to \infty} \sum_{k=1}^{N^2} \frac{1}{1 + (\frac{k}{N})^2} \cdot \frac{1}{N}
\]
\[
= \lim_{N \to \infty} \sum_{k=1}^{N^2} \frac{2N}{N^2 + k^2}.
\]

The point is that the basic operations from calculus tend to produce "real numbers" which are not rational even though we start with only rational numbers.

1.3 Peano’s arithmetic (Highly Optional)

This section is for those who want to understand \( \mathbb{N} \) at a more fundamental level. Here we start with Peano’s rather minimalistic axioms for \( \mathbb{N} \) and show how they lead to all the standard properties you are used to using for \( \mathbb{N} \). Here are the axioms:

non-empty \( \mathbb{N} \) is a non-empty set which contains a distinguished element, 0.

We let \( \mathbb{N} := \mathbb{N} \setminus \{0\} \) and call these the natural numbers.

Successor Function There is an injective\(^2\) function, \( s : \mathbb{N} \to \mathbb{N} \) and we let \( 1 := s (0) \in \mathbb{N} \).

Induction hypothesis If \( S \subset \mathbb{N} \) is a set such that \( 0 \in S \) and \( s (n) \in S \) whenever \( n \in S \), then \( S = \mathbb{N} \).

\(^2\) This fact also shows that the intermediate value theorem, (see Theorem ?? below.) fails when working with continuous functions defined over \( \mathbb{Q} \).

\(^3\) Injective is the same as one to one. Thus we are assuming that if \( s (n) = s (m) \) then \( n = m \).

Assuming these axioms one may develop all of the properties or \( \mathbb{N} \) that you are accustomed to seeing. I will develop the basic properties of addition, multiplication, and the ordering on \( \mathbb{N} \) in this section. For more on this point and then the further construction of \( \mathbb{Z} \) and \( \mathbb{Q} \) from \( \mathbb{N} \), the reader is referred to the notes: "Numbers" by M. Taylor. You may also consult E. Landau’s book \( [1] \) for a very detailed (but perhaps too long winded) exposition of these topics.

Lemma 1.17. The map \( s : \mathbb{N}_0 \to \mathbb{N} \) is a bijection.

Proof. Let \( S := s (\mathbb{N}_0) \cup \{0\} \subset \mathbb{N}_0 \). Then 0 \( \in \mathbb{N} \) and \( s (0) \in s (\mathbb{N}_0) \subset S \). Moreover if \( x \in \mathbb{N} \cap \mathbb{N} \) then \( s (x) \in s (\mathbb{N}_0) \subset S \) so that \( x \in S \implies s (x) \in S \) and hence \( S = \mathbb{N}_0 \) and therefore \( s (\mathbb{N}_0) = \mathbb{N} \).

Theorem 1.18 (Addition). There exists a function \( p : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \) such that \( p (x, 0) = x \) for all \( x \in \mathbb{N}_0 \) and \( p (x, s (y)) = s (p (x, y)) \) for all \( x, y \in \mathbb{N}_0 \). Moreover, we may construct \( p \) so that \( p (s (x), y) = p (x, s (y)) \) for all \( x, y \in \mathbb{N}_0 \).

This function \( p \) satisfies the following properties;

1. \( p (x, 0) = x = p (0, x) \) for all \( x \in \mathbb{N}_0 \),
2. \( p (x, 1) = p (1, x) = s (x) \) for all \( x \in \mathbb{N}_0 \),
3. \( p (x, y) = p (y, x) \) for all \( x, y \in \mathbb{N}_0 \),
4. \( p (x, p (y, z)) = p (p (x, y), z) \) for all \( x, y, z \in \mathbb{N}_0 \).

Proof. We will construct \( p \) inductively. Let \( S := \{ x \in \mathbb{N} : \exists p_x : \mathbb{N}_0 \to \mathbb{N}_0 \ni p_x (0) = x \text{ and } p_x (s (y)) = s (p_x (y)) \forall y \in \mathbb{N}_0 \} \).

Taking \( p_0 (y) = y \) shows \( 0 \in S \). Moreover if \( x \in S \) we define
\[
p_s (x) (y) := s (p_x (y)) \text{ for all } y \in \mathbb{N}_0.
\]

We then have \( p_s (x) (0) = s (p_x (0)) = s (x) \) and
\[
p_s (x) (s (y)) := s (p_x (s (y))) = s \circ s (p_x (y)) = s (p_s (x) (y))
\]
which shows \( s (x) \in S \). Thus we may conclude \( S = \mathbb{N}_0 \) and we may now define \( p (x, y) := p_x (y) \) for all \( x, y \in \mathbb{N}_0 \). By construction this function satisfies,
\[
p (s (x), y) = s (p (x, y)) = p (x, s (y)).
\]

We now verify the properties in items 1. – 4.

1. By construction \( p (x, 0) = x \) for all \( x \in \mathbb{N}_0 \). Let \( S = \{ x \in \mathbb{N} : p (0, x) = x \} \), then \( 0 \in S \) and if \( x \in S \) we have \( p (0, s (x)) = s (p (0, x)) = s (x) \) so that \( s (x) \in S \). Therefore \( S = \mathbb{N}_0 \) and the first item holds.
Notation 1.19 We now write \( x + y \) for \( p(x, y) \) and refer to the symmetric binary operator, +, as addition.

To summarize we have now shown addition satisfies for all \( x, y, z \in \mathbb{N}_0; \)
1. \( x + 0 = 0 + x = x, \)
2. \( s(x) = x + 1 = 1 + x, \)
3. \( x + y = y + x, \)
4. \( (x + y) + z = x + (y + z). \)
5. The induction hypothesis may now be written as; if \( S \subseteq \mathbb{N}_0 \) is a subset such that \( 0 \in S \) and \( n + 1 \in S \) whenever \( n \in S \), then \( S = \mathbb{N}_0. \)

Proposition 1.20 (Additive Cancellation). If \( x, y, z \in \mathbb{N}_0 \) and \( x + z = y + z \), then \( x = y. \)

Proof. Let \( S \) be those \( z \in \mathbb{N}_0 \) for which the statement \( x + z = y + z \) implies \( x = y \) holds. It is clear that \( 0 \in S. \) Moreover if \( z \in S \) and \( x + (z + 1) = y + (z + 1) \) then \( (x + 1) + z = (y + 1) + z \) and so by the inductive hypothesis \( s(x) = x + 1 = y + 1 = s(y). \) Recall that \( s \) is one to one by assumption and therefore we may conclude \( x = y \) and we have shown \( s(z) \in S. \) Therefore \( S = \mathbb{N}_0 \) and the proposition is proved.

Definition 1.21. Given \( x, y \in \mathbb{N}_0, \) we say \( x < y \) if \( f = x + n \) for some \( n \in \mathbb{N} \) and \( x \leq y \) iff \( y = x + n \) for some \( n \in \mathbb{N}_0. \) We further let \( R_x := \{ x + n : n \in \mathbb{N}_0 \} \)
so that \( y \geq x \iff y \in R_x. \)

Proposition 1.22. If \( x, y \in \mathbb{N}_0 \) and \( x \leq y \) and \( y \leq x \) then \( x = y. \) Moreover if \( x \leq y \) then either \( x < y \) or \( x = y. \)

Proof. By assumption there exists \( m, n \in \mathbb{N}_0 \) such that \( x = y + m \) and \( x = y + n \) and therefore \( y = y + (m + n). \) Hence by cancellation it follows that \( m + n = 0. \) If \( n \neq 0 \) then \( n = s(x) \) for some \( x \in \mathbb{N}_0 \) and we have \( m + n = m + s(x) = s(m + n) \in \mathbb{N} \) which would imply \( m + n \neq 0. \) Thus we conclude that \( m = 0 = n \) and therefore \( x = y. \)

If \( x \leq y \) and \( x \neq y \) then \( y = x + n \) for some \( n \in \mathbb{N}_0 \) with \( n \neq 0, \) i.e. \( x < y. \)

Proposition 1.23. If \( x, y \in \mathbb{N}_0 \) then precisely one of the following three choices must hold, 1) \( x < y, \) 2) \( x = y, \) 3) \( y < x. \)

Proof. Suppose that \( x \leq y \) does not hold, i.e. \( y \notin R_x. \) We wish to show that \( y < x, \) i.e. that \( x = y + n \) for some \( n \in \mathbb{N}_0. \) We do this by induction on \( y. \) That is let \( S \) be the the set of \( y \in \mathbb{N}_0 \) such that the statement \( y \notin R_x \) holds. If \( y = 0 \notin R_x \) implies \( n := x \neq 0 \) so that \( x = y + n, \) i.e. \( y < x. \) This shows \( 0 \in S. \) Now suppose that \( y \in S \) and that \( y + 1 \notin R_x = \{ s(x) = 1 + n : n \in \mathbb{N}_0 \} \).
It follows that \( y + 1 = x + m + 1 \) for all \( m \in \mathbb{N}_0 \) and hence that \( y \neq x + m \) for all \( m \in \mathbb{N}_0, \) i.e. \( y \notin R_x. \) So by induction \( y < x \) and therefore \( x = y + k \) for some \( k \in \mathbb{N}_0. \) Since \( k \in \mathbb{N}_0 \) we know there exists \( k' \in \mathbb{N}_0 \) such that \( k = k'. \) and it follows that \( x = y + 1 + k', \) i.e. \( y + 1 \leq x. \) Since \( y + 1 \notin R_x \) we may conclude that in fact \( y + 1 < x \) and therefore \( y + 1 \in S. \) So by induction \( S = \mathbb{N}_0 \) and we have shown if \( x < y \) does not hold iff \( y \leq x. \) Combining this statement with the Proposition \( \ref{prop:1.22} \) completes the proof.

We have now set up a satisfactory addition operations and ordering on \( \mathbb{N}_0. \) Our next goal is to define multiplication on \( \mathbb{N}_0. \)

Theorem 1.24. There exists a function \( M : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0 \) such that \( M(x, y) = 0 \) for all \( x \in \mathbb{N}_0 \) and \( M(x, y + 1) = M(x, y) + x \) for all \( x, y \in \mathbb{N}_0. \) This function \( M \) satisfies the following properties;
1. \( M(x, 0) = 0 = M(0, x) \) for all \( x \in \mathbb{N}_0, \)
2. \( M(x, 1) = M(1, x) = x \) for all \( x \in \mathbb{N}_0, \)
3. \( M(x, y) = M(y, x) \) for all \( x, y \in \mathbb{N}_0, \)
4. \( M(x, y + z) = M(x, y) + M(x, z) \) for all \( x, y, z \in \mathbb{N}_0, \)
5. \( M(x, M(y, z)) = M(M(x, y), z) \) for all \( x, y, z \in \mathbb{N}_0. \)

Proof. Let \( S \) denote those \( x \in \mathbb{N}_0 \) such that there exists a function \( M_x : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \) satisfying \( M_x(0) = 0 \) and \( M_x(y + 1) = M_x(y) + x \) for all \( y \in \mathbb{N}_0. \) Taking \( M_0(y) := 0 \) shows \( 0 \in S. \) Moreover if \( x \in S \) we define \( M_{x+1}(y) := M_x(y) + y. \) Then \( M_{x+1}(0) = 0 \) and
\[
M_{x+1}(y + 1) = M_x(y + 1) + y + 1 = M_x(y) + x + y + 1
\]
This shows that \( x + 1 \in S \) and so by induction \( S = \mathbb{N}_0 \) and we may now define \( M(x, y) := M_x(y) \) for all \( x, y \in \mathbb{N}_0 \). We now prove the properties of \( M \) stated above.

1. By construction \( M(x, 0) = 0 \) for all \( x \). Let \( S := \{ x \in \mathbb{N}_0 : M(0, x) = 0 \} \).
   Then \( 0 \in S \) and if \( x \in S \) we have
   \[
   M(0, x + 1) = M(0, x) + 0 = 0 + 0 = 0
   \]
   which shows \( x + 1 \in S \). Therefore by induction \( S = \mathbb{N}_0 \) and \( M(0, x) = 0 \) for all \( x \in \mathbb{N}_0 \).

2. \( M(x, 1) = M(x, 0 + 1) = M(x, 0) + x = 0 + x = x \) for all \( x \in \mathbb{N}_0 \). Let
   \[
   S := \{ x \in \mathbb{N}_0 : M(1, x) = x \} \text{. Then } 0 \in S \text{ and if } x \in S \text{ we have}
   
   M(1, x + 1) = M(1, x) + 1 = x + 1
   
   which shows \( x + 1 \in S \). Therefore \( S = \mathbb{N}_0 \) and \( M(1, x) = x \) for all \( x \in \mathbb{N}_0 \).

3. Let \( S := \{ x \in \mathbb{N}_0 : M(x, \cdot) = M(\cdot, x) \} \). Then by items 1. and 2. we know that \( 0, 1 \in S \). Now suppose that \( x \in S \), then by construction,
   \[
   M(x + 1, y) = M(x, y) + y
   \]
   while
   \[
   M(y, x + 1) = M(y, x) + y.
   \]
   The last two displayed equations along with the induction hypothesis shows \( x + 1 \in S \) and therefore \( S = \mathbb{N}_0 \) and item 3. is proved.

4. Let \( S \) denotes those \( x \in S \) such that \( M(x, y + z) = M(x, y) + M(x, z) \) for all \( y, z \in \mathbb{N}_0 \). Then \( 0, 1 \in S \) and if \( x \in S \) we have,
   \[
   M(x + 1, y + z) = M(x, y + z) + y + z
   = M(x, y) + M(x, z) + y + z
   = M(x, y) + y + M(x, z) + z
   = M(x + 1, y) + M(x + 1, z)
   \]
   which shows \( x + 1 \in S \). Therefore \( S = \mathbb{N}_0 \) and we have proved item 4.

5. Let
   \[
   S := \{ x \in \mathbb{N}_0 : M(x, M(y, z)) = M(M(x, y), z) \forall y, z \in \mathbb{N}_0 \} \text{.}
   
   Then \( 0 \in S \) and if \( x \in S \) we find,
   \[
   M(x + 1, M(y, z)) = M(x, M(y, z)) + M(y, z)
   \]
   while
   \[
   M(M(x + 1, y), z) = M(M(x, y) + y, z) = M(M(x, y), z) + M(y, z).
   \]
   The last two equations along with the induction hypothesis shows \( x + 1 \in S \) and therefore \( S = \mathbb{N}_0 \) and item 5. is proved.

\[ \text{Notation 1.25} \]

We now write \( x \cdot y \) for \( M(x, y) \) and refer to the symmetric binary operator, \( \cdot \), as multiplication.

To summarize Theorem [1.24] we have shown multiplication satisfies for all \( x, y, z \in \mathbb{N}_0 \):

1. \( x \cdot 0 = 0 = 0 \cdot x \),
2. \( x \cdot 1 = x = 1 \cdot x \),
3. \( x \cdot y = y \cdot x \),
4. \( x \cdot (y + z) = x \cdot y + x \cdot z \),
5. \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \).

\[ \text{Proposition 1.26 (Multiplicative Cancellation).} \]

If \( x, y \in \mathbb{N}_0 \) and \( z \in \mathbb{N} \) such that \( x \cdot z = y \cdot z \), then \( x = y \).

\[ \text{Proof.} \]

If \( x \neq y \), say \( x < y \), then \( y = x + n \) for some \( n \in \mathbb{N} \) and therefore
   \[
   y \cdot z = (x + n) \cdot z = x \cdot z + n \cdot z.
   \]
Hence if \( x \cdot z = y \cdot z \), then by additive cancellation we must have \( n \cdot z = 0 \). As \( n, x \in \mathbb{N} \) we may write \( n = n' + 1 \) and \( z = z' + 1 \) with \( n', z' \in \mathbb{N}_0 \) and therefore,
   \[
   0 = n \cdot z = (n' + 1) \cdot (z' + 1) = n' \cdot z' + n' + z' + 1 \neq 0
   \]
which is a contradiction. \[ \text{\blacksquare} \]

As mentioned above one can formalize \( \mathbb{Z} \) and \( \mathbb{Q} \) using \( \mathbb{N}_0 \) constructed above.
I will omit the details here and refer the reader to the references already mentioned.
Fields

Before going to the real numbers, let us step back and formalize the properties that we would like any number system to possess.

**Definition 2.1 (Fields, i.e. “numbers”).** A field is a set \( \mathbb{F} \) equipped with two operations called addition and multiplication, and denoted by + and \( \cdot \), respectively, such that the following axioms hold; (subtraction and division are defined implicitly in terms of the inverse operations of addition and multiplication:)

1. **Closure** of \( \mathbb{F} \) under addition and multiplication. For all \( a, b \in \mathbb{F} \), both \( a + b \) and \( a \cdot b \) are in \( \mathbb{F} \) (or more formally, + and \( \cdot \) are binary operations on \( \mathbb{F} \)).
2. **Associativity of addition and multiplication.** For all \( a, b, \) and \( c \) in \( \mathbb{F} \), the following equalities hold: \( a + (b + c) = (a + b) + c \) and \( a \cdot (b \cdot c) = (a \cdot b) \cdot c \).
3. **Commutativity of addition and multiplication.** For all \( a \) and \( b \) in \( \mathbb{F} \), the following equalities hold: \( a + b = b + a \) and \( a \cdot b = b \cdot a \).
4. **Additive and multiplicative inverses.** There exists an element of \( \mathbb{F} \), called the additive identity element and denoted by \( 0 \) or \( 0_\mathbb{F} \), such that for all \( a \) in \( \mathbb{F} \), \( a + 0 = a \). Likewise, there is an element, called the multiplicative identity element and denoted by \( 1 \) or \( 1_\mathbb{F} \), such that for all \( a \) in \( \mathbb{F} \), \( a \cdot 1 = a \). For technical reasons, the additive identity and the multiplicative identity are required to be distinct.
5. **Additive and multiplicative inverses.** For every \( a \) in \( \mathbb{F} \), there exists an element \( -a \) in \( \mathbb{F} \), such that \( a + (-a) = 0 \). Similarly, for any \( a \in \mathbb{F} \) other than \( 0 \), there exists an element \( a^{-1} \) in \( \mathbb{F} \), such that \( a \cdot a^{-1} = 1 \). (The elements \( a + (-b) \) and \( a \cdot b^{-1} \) are also denoted \( a - b \) and \( a/b \), respectively.) In other words, subtraction and division operations exist.
6. **Distributivity of multiplication over addition.** For all \( a, b \) and \( c \) in \( \mathbb{F} \), the following equalities hold: \( a \cdot (b + c) = (a \cdot b) + (a \cdot c) \).

(Note that all but the last axiom are exactly the axioms for a commutative group, while the last axiom is a compatibility condition between the two operations.)

2.1 Basic Properties of Fields

Here are some sample properties about fields. For more information about Fields see 5-8 of Rudin.

**Lemma 2.2.** Let \( \mathbb{F} \) be a field, then;

1. There is only one additive and multiplicative inverses.
2. If \( x, y, z \in \mathbb{F} \) with \( x \neq 0 \) and \( xy = xz \) then \( y = z \).
3. \( 0 \cdot x = 0 \) for all \( x \in \mathbb{F} \).
4. If \( x, y \in \mathbb{F} \) such that \( xy = 0 \) then \( x = 0 \) or \( y = 0 \).
5. \( (-x) y = -(xy) \).
6. \( -(x) = x \) for all \( x \in \mathbb{F} \).
7. \( -(x)(-y) = xy \) or all \( x, y \in \mathbb{F} \).

**Proof.** We take each item in turn.

1. Suppose that \( x + y = 0 = x + y' \), then adding \(-x\) to both sides of this equation shows \( y = y' \). Taking \( y = -x \) then shows \( y = -x = y' \), i.e. additive inverses are unique. Similarly if \( x \neq 0 \) and \( xy = 1 \) then multiplying this equation by \( x^{-1} \) shows \( y = x^{-1} \) and so there is only one multiplicative inverse.
2. If \( xy = xz \) then multiplying this equation by \( x^{-1} \) shows \( y = z \).
3. \[ 0 \cdot x + x = 0 \cdot x + 1 \cdot x = (0 + 1) \cdot x = 1 \cdot x = x. \]

Adding \(-x\) to both side of this equation using associativity and commutativity of addition then implies \( 0 \cdot x = 0 \).
4. If \( x \in \mathbb{F} \setminus \{0\} \) and \( y \in \mathbb{F} \) such that \( xy = 0 \), then
\[ 0 = x^{-1} \cdot 0 = x^{-1} (xy) = (x^{-1}x) y = 1y = y. \]
5. \( (-x) y + xy = (-x + x) y = 0 \cdot y = 0 \implies (-x) y = -(xy) \).
6. Since \( (-x) + x = 0 \) we have \( -(-x) = x \).
7. \( (-x)(-y) = -(x \cdot (-y)) = -(-xy) = xy \) by 6.

**Example 2.3.** Here are a few examples of Fields:

1. \( \mathbb{F}_2 = \{0, 1\} \) with \( 0 + 0 = 0 = 1 + 1 \), and \( 0 + 1 = 1 + 0 = 0 \) and \( 0 \cdot 1 = 1 \cdot 0 = 0 \). In this case \(-1 = 1, 1^{-1} = 1 \) and \(-0 = 0 \).
2. \( \mathbb{Q} \) – the rational numbers with the usual addition and multiplication of fractions. \( \left( \frac{m}{n} \right)^{-1} = \frac{n}{m} \) if \( m \neq 0 \) and \(-\frac{m}{n} = \frac{-m}{n} \).
3. \( F = \mathbb{Q}(t) \) where
\[
\mathbb{Q}(t) = \left\{ \frac{p(t)}{q(t)} : p(t) \text{ and } q(t) \text{ are polynomials over } \mathbb{Q} \ni q(t) \neq 0 \right\}.
\]

Again the multiplication and addition are as usual.

**Example 2.4.** \( \mathbb{Z} \) is not a field. 2 has no multiplicative inverse. \( 2^{-1} \) should be \( \frac{1}{2} \) but this is not in \( \mathbb{Z} \).

**Definition 2.5.** We say a map \( \varphi : \mathbb{Z} \to \mathbb{F} \) is a (ring) homomorphism iff \( \varphi(1) = 1_\mathbb{F} \), \( \varphi(0) = 0_\mathbb{F} \), and for all \( x, y \in \mathbb{Z} \);
\[
\varphi(x + y) = \varphi(x) + \varphi(y) \text{ and } \varphi(xy) = \varphi(x) \varphi(y).
\]

**Lemma 2.6.** For every field \( \mathbb{F} \) there a unique (ring) homomorphism, \( \varphi : \mathbb{Z} \to \mathbb{F} \). In \( \varphi(n) = n1_\mathbb{F} \) for all \( n \in \mathbb{Z} \) where \( 0 \cdot 1_\mathbb{F} = 0_\mathbb{F} \),
\[
\begin{align*}
&\text{if } n \in \mathbb{N} \text{ and } n1_\mathbb{F} := \underbrace{1_\mathbb{F} + \cdots + 1_\mathbb{F}}_n, \\
&\text{if } n \in -\mathbb{N} \text{ and } (-n)1_\mathbb{F} := -(n1_\mathbb{F}).
\end{align*}
\]

[The map \( \varphi \) need not be injective as is seen by taking \( \mathbb{F} = \mathbb{F}_2 \).]

**Proof.** Let us first work on \( \mathbb{N}_0 \subset \mathbb{Z} \). We must define \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \) and then \( \varphi \) inductively by \( \varphi(n + 1) = \varphi(n) + \varphi(1) = \varphi(n) + 1_\mathbb{F} \) so that
\[
\varphi(n) = \underbrace{1_\mathbb{F} + \cdots + 1_\mathbb{F}}_n.
\]

We now write \( n1_\mathbb{F} \) for \( \varphi(n) \) with the convention that \( 01_\mathbb{F} = 0_\mathbb{F} \). For \( n \in -\mathbb{N} \) we must set \( \varphi(-n) = -\varphi(n) = -(n1_\mathbb{F}) \). Thus we have \( \varphi(n) = n1_\mathbb{F} \) for all \( n \in \mathbb{Z} \).

We now must show \( \varphi \) is a homomorphism.

**Additive homomorphism:** First suppose that \( m, n \in \mathbb{N}_0 \) and let
\[
S := \{ m \in \mathbb{N}_0 : \varphi(m + n) = \varphi(m) + \varphi(n) \text{ for all } n \in \mathbb{N}_0 \}.
\]

One easily sees that \( 0 \in S \) and that \( 1 \in S \) by construction. Moreover if \( m \in S \), then
\[
\varphi((m + 1) + n) = \varphi(m + n + 1) = \varphi(m) + \varphi(n + 1)
\]
\[
= \varphi(m) + \varphi(n) + 1_\mathbb{F}
\]
\[
= \varphi(m) + 1_\mathbb{F} + \varphi(n) = \varphi(m + 1) + \varphi(n)
\]
which shows \( m + 1 \in S \). Therefore by induction, \( S = \mathbb{N}_0 \) and \( \varphi(m + n) = \varphi(m) + \varphi(n) \) for all \( m, n \in \mathbb{N}_0 \).

If \( m \in \mathbb{N}_0 \) we have \( \varphi(-m) = -\varphi(m) \) by construction. If \( n > m \in \mathbb{N}_0 \), then
\[
\varphi(n + (-m)) + \varphi(m) = \varphi(n - m) + \varphi(m) = \varphi(n)
\]
so that
\[
\varphi(n + (-m)) = \varphi(n) + (-\varphi(m)) = \varphi(n) + \varphi(-m).
\]

If \( n < m \in \mathbb{N}_0 \), then
\[
\varphi(n + (-m)) = -\varphi(m - n) = -[\varphi(m) - \varphi(n)] = \varphi(n) + \varphi(-m)
\]
and if \( m, n \in \mathbb{N}_0 \), then
\[
\varphi(-n + (-m)) = \varphi(-n + m) = -\varphi(n + m)
\]
\[
= -[\varphi(n) + \varphi(m)] = -\varphi(n) - \varphi(m)
\]
\[
= \varphi(-n) + \varphi(-m).
\]

Putting all of this together shows \( \varphi \) is an additive homomorphism.

**Multiplicative homomorphism:** First suppose that \( m, n \in \mathbb{N}_0 \) and let
\[
S := \{ m \in \mathbb{N}_0 : \varphi(mn) = \varphi(m)\varphi(n) \text{ for all } n \in \mathbb{N}_0 \}.
\]

It is easily seen that \( 0, 1 \in S \). Moreover if \( m \in S \) and \( n \in \mathbb{N}_0 \), then
\[
\varphi((m + 1)n) = \varphi(mn + n) = \varphi(mn) + \varphi(n)
\]
\[
= \varphi(m)\varphi(n) + \varphi(n) = (\varphi(m) + 1_\mathbb{F})\varphi(n)
\]
\[
= \varphi(m + 1)\varphi(n),
\]
which shows \( m + 1 \in S \). Therefore by induction, \( S = \mathbb{N}_0 \) and \( \varphi(mn) = \varphi(m)\varphi(n) \) for all \( m, n \in \mathbb{N}_0 \).

If \( m, n \in \mathbb{N}_0 \), then
\[
\varphi((-m)n) = \varphi(-mn) = -\varphi(mn) = -[\varphi(m)\varphi(n)] = [-\varphi(m)]\varphi(n) = \varphi(-m)\varphi(n)
\]
and
\[
\varphi((-m)(-n)) = \varphi(mn) = \varphi(mn) = (\varphi(m))(-\varphi(n)) = -\varphi(m)\varphi(-n)
\]
which completes the verification that \( \varphi \) is a multiplicative homomorphism. \( \blacksquare \)
2.2 Ordered Fields

**Definition 2.7.** We say \( \mathbb{F} \) is an ordered field if there exists, \( P \subset \mathbb{F} \), called the positive elements, such that

1. \( \mathbb{F} \) is the disjoint union of \( \{0\} \), and \(-P\), i.e. if \( x \in \mathbb{F} \) then precisely one of following happens; \( x \in P \), \( x = 0 \), or \(-x \in P \).
2. \( P + P \subset P \) and \( P \cdot P \subset P \).

**Lemma 2.8.** Let \((\mathbb{F}, P)\) be an ordered field, then;

1. For all \( x \in \mathbb{F} \setminus \{0\} \), \( x^2 \in P \). In particular \( 1^2 = 1 \in P \).
2. If \( x \in P \) and \( y \in -P \) then \( xy \in -P \).
3. If \( x \in P \) then \( x^{-1} \in P \).

**Proof.** If \( x \in P \) then \( x^2 \in P \) while if \( x \in -P \) then \( -x \in P \) and \( x^2 = (-x)^2 \in P \). For item 3. we have \( x \cdot x^{-1} = 1 \). \( \square \)

**Example 2.9.** The field \( \mathbb{F} = \{0,1\} \) is not ordered. The only possible choice for \( P = \{1\} \) which does not work since \( 1 + 1 = 0 \notin P \).

**Example 2.10.** Take \( \mathbb{F} = \mathbb{Q} \) and \( P = \left\{ \frac{m}{n} : m, n > 0 \right\} \). This is in fact the unique choice we can make for \( P \) in this case. Indeed suppose that \( P \) is any order on \( \mathbb{Q} \). By Lemma 2.8, we know \( 1 \in P \) and then by induction it follows that \( \mathbb{N} \subset P \). Then again by Lemma 2.8, we must have \( m \cdot n^{-1} \in P \) for all \( m, n \in \mathbb{Q} \).

**Example 2.11.** Take \( \mathbb{F} = \mathbb{Q}(t) \) and 

\[
P = \left\{ \frac{p(t)}{q(t)} \in \mathbb{F} : \frac{p(t)}{q(t)} > 0 \text{ for } t > 0 \text{ large} \right\},
\]

i.e. \( \frac{p(t)}{q(t)} \in P \) iff the highest order coefficients of \( p(t) \) and \( q(t) \) have the same sign. For example \( \frac{t^2 - 25t + 7}{t^7} \in P \) while \( \frac{-t^2 + 25t - 7}{t^7} \in -P \).

Notice that \( t > n \) for all \( n \in \mathbb{N} \) and \( \frac{1}{t} < \frac{1}{n} \) for all \( n \in \mathbb{N} \). This is kind of strange and explains why you have to prove the “obvious” in this course!

**Moral:** obvious statements are often false.

**Definition 2.12.** Suppose that \( \mathbb{F} \) and \( \mathbb{G} \) are fields. A map, \( \varphi : \mathbb{F} \to \mathbb{G} \) is a (field) homomorphism iff \( \varphi(1_{\mathbb{F}}) = 1_{\mathbb{G}} \), \( \varphi(0_{\mathbb{F}}) = 0_{\mathbb{G}} \), and for all \( x, y \in \mathbb{F} \):

\[
\varphi(x+y) = \varphi(x) + \varphi(y) \quad \text{and} \quad \varphi(xy) = \varphi(x) \varphi(y).
\]

**Lemma 2.13 (\( \mathbb{Q} \) embeds into an ordered field).** For every ordered field \((\mathbb{F}, P)\), there a unique field homomorphism, \( \varphi : \mathbb{Q} \to \mathbb{F} \). In fact,

\[
\varphi \left( \frac{m}{n} \right) = \frac{m}{n} \cdot 1_{\mathbb{F}} := m_{1_{\mathbb{F}}} \cdot (n_{1_{\mathbb{F}}})^{-1}
\]

where \( n_{1_{\mathbb{F}}} := 1_{\mathbb{F}} + \cdots + 1_{\mathbb{F}} \) and \((-n)_{1_{\mathbb{F}}} := -(n_{1_{\mathbb{F}}}) \) for \( n \in \mathbb{N} \) and \( 0 \cdot 1_{\mathbb{F}} = 0_{\mathbb{F}} \). Moreover;

1. \( \varphi(x) \in P \) whenever \( x > 0 \),
2. and \( \varphi \) is injective. Thus we may identify \( \mathbb{Q} \) with \( \varphi(\mathbb{Q}) \) and consider \( \mathbb{Q} \) as a sub-field of \( \mathbb{F} \).

[In particular, ordered fields must be fields with an infinite number of elements in it.]

**Proof.** From Lemma 2.6 we know there is a unique ring homomorphism, \( \varphi : \mathbb{Z} \to \mathbb{F} \), given by \( \varphi(m) = m \cdot 1_{\mathbb{F}} \). So for \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \) we must have

\[
\varphi \left( \frac{m}{n} \right) \cdot n_{1_{\mathbb{F}}} = \varphi \left( \frac{m}{n} \right) \cdot (m_{1_{\mathbb{F}}})^{-1} = \varphi(m) = m_{1_{\mathbb{F}}}
\]

which forces us to define \( \varphi \) as in Eq. (2.1). Notice that is easy to verify by induction that \( n_{1_{\mathbb{F}}} = \varphi(n) \in P \) for all \( n \in \mathbb{N} \) and in particular \( n_{1_{\mathbb{F}}} \neq 0 \) for \( n \in \mathbb{N} \). In particular if \( x = m/n > 0 \) then \( \varphi \left( \frac{m}{n} \right) = m_{1_{\mathbb{F}}} \cdot (n_{1_{\mathbb{F}}})^{-1} \in P \) by Lemma 2.8. We must still check that \( \varphi \) is well defined homomorphism.

**Well defined.** Suppose that \( k \in \mathbb{N} \), we must show

\[
(km)_{1_{\mathbb{F}}} \cdot ((kn)_{1_{\mathbb{F}}})^{-1} = m_{1_{\mathbb{F}}} \cdot (n_{1_{\mathbb{F}}})^{-1}.
\]

By cross multiplying, this will happen iff

\[
(km)_{1_{\mathbb{F}}} \cdot (n_{1_{\mathbb{F}}})^{-1} = ((kn)_{1_{\mathbb{F}}}) \cdot m_{1_{\mathbb{F}}}
\]

which is the case as \( \varphi : \mathbb{Z} \to \mathbb{F} \) is a ring homomorphism.

**Homomorphism property.** We have

\[
\varphi \left( \frac{m}{n} + \frac{p}{q} \right) = \varphi \left( \frac{mp + pq}{nq} \right) = \varphi(m) + \varphi(p) \cdot \varphi(n)^{-1}
\]

\[
= \varphi(m) + \varphi(p) \cdot \varphi(n)^{-1} = \varphi(m) \cdot \varphi(n)^{-1} + \varphi(p) \cdot \varphi(n)^{-1}
\]

and

\[
\varphi \left( \frac{m}{n} \right) \varphi \left( \frac{q}{p} \right) = \varphi(m) \cdot \varphi(n) \cdot \varphi(q) \cdot \varphi(p)^{-1}
\]

\[
= \varphi(m) \cdot \varphi(q) \cdot \varphi(n) \cdot \varphi(p)^{-1} = \varphi(mq) \cdot \varphi(np)^{-1} = \varphi \left( \frac{mq}{np} \right).
\]
Injectivity. If $0 = \varphi \left( \frac{m}{n} \right)$ then

$$0 = \varphi (m) \cdot \varphi (n)^{-1}$$

which implies $\varphi (m) = 0$ which happens iff $m = 0$, i.e. $m/n = 0$.

**Notation 2.14.** If $(\mathbb{F}, P)$ is an ordered field we write $x > y$ iff $x - y \in P$. We also write $x \geq y$ iff $x > y$ or $x = y$.

Notice that if $x, y \in \mathbb{F}$ then either $x - y = 0$ (i.e. $x = y$), or $x - y \in P$ (i.e. $x > y$), or $x - y \in -P$ (i.e. $y - x \in P$ and $y > x$). Also in this notation we have $P = \{ x \in \mathbb{F} : x > 0 \}$,

$\neg P = \{ x \in \mathbb{F} : x < 0 \text{ (i.e. } 0 > x) \}.$

**Lemma 2.15.** Suppose that $x < y$ and $y < z$ and $a > 0$. Then $x < z$ and $ax < ay$.

**Proof.** By assumption $y - x \in P$ and $z - y \in P$, therefore $z - x = (y - x) + (z - y) \in P$, i.e. $z > x$. Moreover, $a \in P$ and $(y - x) \in P$ implies $P \ni a \cdot (y - x) = ay - ax$.

That is $ay > ax$.

**Definition 2.16.** Given $x \in \mathbb{F}$, we say that $y \in \mathbb{F}$ is a square root of $x$ if $y^2 = x$. [From Lemma 2.8, it follows that if $x \in \mathbb{F}$ has a square root then $x \geq 0$.]

**Lemma 2.17.** Suppose $x, y \in \mathbb{F}$ with $x^2 = y^2$, then either $x = y$ or $x = -y$. In particular, there are at most 2 square roots of any number $x \geq 0$ in $\mathbb{F}$.

**Proof.** Observe that

$$(x - y)(x + y) = (x - y)x + (x - y)y = x^2 - xy + xy - y^2 = x^2 - y^2 = 0.$$ 

Thus it follows that either $x - y = 0$ or $x + y = 0$, i.e. $x = y$ or $x = -y$.

**Definition 2.18.** If $x > 0$ admits a square root we let $\sqrt{x}$ be the unique positive root. We also define $\sqrt{0} = 0$.

**Lemma 2.19.** Suppose that $0 < x < y$, i.e. $x, y - x \in P$, then $x^2 < y^2$.

**Proof.** By Lemma 2.15 we know $x \cdot x < x \cdot y$ and $x \cdot y < y \cdot y$ and therefore $x^2 < y^2$.

**Corollary 2.20.** If $0 \leq x < y$ and $\sqrt{x}$ and $\sqrt{y}$ exists, then $0 \leq \sqrt{x} < \sqrt{y}$.

**Proof.** If $\sqrt{x} = \sqrt{y}$ then $x = (\sqrt{x})^2 = (\sqrt{y})^2 = y$ which is impossible. Similarly if $\sqrt{x} > \sqrt{y}$ then

$$x = (\sqrt{x})^2 > (\sqrt{y})^2 = y$$

which is again false.

**Definition 2.21.** The absolute value, $|x|$, of $x$ in ordered field $\mathbb{F}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$ 

Alternatively we may define $|x| = \sqrt{x^2}$.

**Proposition 2.22.** For all $x, y \in \mathbb{F}$, then

1. $|x| \geq 0$
2. $|xy| = |x| |y|$
3. $|x + y| \leq |x| + |y|$.

**Proof.** 1. holds by definition since $-x > 0$ if $x < 0$.
2. As $|x| |y| \geq 0$ and $((|x| |y|)^2 = |x|^2 |y|^2 = x^2 y^2 = (xy)^2$, we have $|x| |y| = \sqrt{(xy)^2} = |xy|$.
3. It suffices to show $|x + y|^2 \leq (|x| + |y|)^2$. However,

$$|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy \leq x^2 + y^2 + 2|x|y (xy \leq |xy|)$$

$$= |x|^2 + |y|^2 + 2|x| |y|$$

$$= (|x| + |y|)^2.$$ 

**Definition 2.23.** Let $(\mathbb{F}, P)$ be an ordered field and $S$ be a subset of $\mathbb{F}$.

1. We say that $S \subseteq \mathbb{F}$ is bounded from above if there exists $x \in \mathbb{F}$ such that $x \geq s$ for all $s \in S$. Any such $x$ is called an upper bound of $S$.
2. If $S$ is bounded from above, we say that $y \in \mathbb{F}$ is a least upper bound for $S$ if $y$ is an upper bound for $S$ and $y \leq x$ for any other upper bound, $x$, of $S$. 


Notice that least upper bounds are unique if they exists. We will write \( y = \text{l.u.b.}(S) = \text{sup}(S) \) if \( y \) is the least upper bound for \( S \).

**Example 2.24.** Let \( F = \mathbb{Q} \), then:

1. \( S = \mathbb{N} \) is not bounded from above.
2. \( S = \{1 - \frac{1}{n} : n \in \mathbb{N}\} \) is bounded from above and \( 1 = \text{sup}(S) \).
3. Let
   \[
   S = \{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, \ldots \}
   \]
   where I am getting these digits from the decimal expansion of \( \sqrt{2} \):
   \[
   \sqrt{2} \cong 1.41421356237309504880168872420969807856967187537694807317667973799...
   \]
   In this case \( S \) is bounded above by 2, or 1.42, or 1.415, etc. Nevertheless \( \sqrt{2} = \text{sup}(S) \) does not exists in \( \mathbb{Q} \).

**Example 2.25.** Now let \( F = \mathbb{Q}(t) \) be the field of rational functions described in Example 2.11, then; \( S = \mathbb{N} \) is bounded from above. For example \( t \) is an upper bound but there is not least upper bound. For example \( \frac{1}{n}t \) is also an upper bound for \( S \).
Real Numbers

As we saw in Section [1.2], \( \mathbb{Q} \) is full of holes and calculus tends to produce answers which live in these holes. So it is imperative that we fill the holes. Doing so will lead to the real numbers provided we fill in the holes without adding too much extra filler along the way.

**Definition 3.1.** An order preserving field isomorphism between two ordered fields, \((F_1, P_1)\) and \((F_2, P_2)\), is a bijection, \( f : F_1 \to F_2 \) such that 
\[ f(0) = 0, \quad f(1) = 1, \quad f(P_1) = P_2, \] and 
\[ f(x + y) = f(x) + f(y) \quad \text{and} \quad f(xy) = f(x)f(y) \] for all \( x, y \in F_1 \).

**Definition 3.2.** An ordered field \((F, P)\) is complete if every non-empty subset, \( S \subset F \), which is bounded from above possesses a least upper bound in \( F \).

**Theorem 3.3 (The real numbers).** Up to order preserving field isomorphism (see Definition 3.1), there is exactly one complete ordered field. It is this field that we refer to as the real numbers and denote by \( \mathbb{R} \).

The existence part of the proof of this theorem (based on Cauchy sequences) will be relegated to the appendix below. For an alternative existence proof using “Dedekind cuts” is covered in Rudin [2, pages 17-21.]. We will prove the uniqueness assertion at the end of this section. One may also construct the Real numbers using decimal expansions, see T. Gower’s notes on real numbers as decimals.

Observe that \( \mathbb{Q}, \mathbb{Q}(t), \mathbb{R}(t) \) are not complete and hence are not the real numbers, \( \mathbb{R} \). For example \( \mathbb{N} \subset \mathbb{Q}(t) \) (or \( \mathbb{N} \subset \mathbb{R}(t) \)) is bounded by \( t \) say but has no least upper bound. However, we do know that \( \mathbb{Q} \subset \mathbb{R} \) by Lemma 2.13.

We will soon see that \( \mathbb{Q} \) is “dense” in \( \mathbb{R} \). We now pause to discuss some of the basic properties of \( \mathbb{Q} \).

**Theorem 3.4.** Suppose that \( \mathbb{R} \) is a complete ordered field which we assume we have already embedded \( \mathbb{Q} \) into \( \mathbb{R} \). Then:

1. For all \( x \geq 0 \) there exists \( n \in \mathbb{N} \) such that \( n \geq x \).
2. For all \( \varepsilon > 0 \) in \( \mathbb{R} \) there exists \( n \in \mathbb{N} \) such that \( 0 < \frac{1}{n} \leq \varepsilon \).
3. If \( \varepsilon \geq 0 \) satisfies \( \varepsilon \leq 1/n \) for all \( n \in \mathbb{N} \) then \( \varepsilon = 0 \).
4. If \( a, b \in \mathbb{R} \) and \( a \leq b + \frac{1}{n} \) for all \( n \in \mathbb{N} \) or \( a \leq b + \varepsilon \) for all \( \varepsilon > 0 \), then \( a \leq b \).
5. For all \( m \in \mathbb{R} \), we have,
\[ m = \sup \{ y \in \mathbb{Q} : y < m \} \].
6. If \( a, b \in \mathbb{R} \) with \( a < b \), then there exists \( q \in \mathbb{Q} \) such that \( a < q < b \).

**Proof.** We take each item in turn.

1. If \( n < x \) for all \( n \in \mathbb{N} \), then \( \mathbb{N} \) is bounded from above so \( \sup(\mathbb{N}) \) exists in \( \mathbb{R} \) by the completeness axiom. As \( a \) the least upper bound for \( \mathbb{N} \) there must be an \( n \in \mathbb{N} \) such that \( n > a - 1 \). However this implies \( n + 1 > a \) which violates the assumption that \( a \) is the least upper bound for \( \mathbb{N} \).
2. If \( \varepsilon > 0 \) in \( \mathbb{R} \) and \( \frac{1}{n} > \varepsilon \) for all \( n \in \mathbb{N} \), then \( n < \frac{1}{\varepsilon} \) for all \( n \in \mathbb{N} \) which is impossible by item 1.
3. If there exists \( \varepsilon > 0 \) such that \( \varepsilon \leq \frac{1}{n} \) for all \( n \) then \( n \leq 1/\varepsilon \) for all \( n \) which is again impossible by item 1.
4. It suffices to prove the first assertion. We may assume \( a \geq b \) for otherwise we are done. If \( a \leq b + \frac{1}{n} \) for all \( n \), then \( 0 \leq a - b \leq \frac{1}{n} \) for all \( n \in \mathbb{N} \) and hence \( a = b \) and in particular \( a \leq b \).
5. Let \( \mathbb{Q}_m := \{ y \in \mathbb{Q} : y < m \} \) and \( M := \sup \mathbb{Q}_m \). Then \( M \leq m \). If \( M \neq m \) then \( M < m \). To see this last case is not possible \( \varepsilon := m - M > 0 \) and choose \( n \in \mathbb{N} \) such that \( 0 < \frac{1}{n} \leq \varepsilon \). Then choose \( y \in \mathbb{Q} \) such that
\[ M - \frac{1}{2n} < y < M. \]
From this it follows that
\[ M < y + \frac{1}{2n} < M + \frac{1}{2n} < m \]
which shows \( y + \frac{1}{2n} \in \mathbb{Q}_m \) is greater than \( M \) violating the assumption that \( M \) is an upper bound for \( \mathbb{Q}_m \).
6. By item 4, we can choose \( q \in \mathbb{Q}_b \) to be as close to \( b \) as we choose and in particular \( q \) can be chosen to be in \( \mathbb{Q}_b \) with \( q > a \).

**Exercise 3.1.** A subset \( \alpha \subset \mathbb{Q} \) is called a cut (see [2, p. 17]) if:
1. \( \alpha \neq 0 \) and \( \alpha \neq Q \),
2. if \( p \in \alpha \) and \( q \in Q \) and \( q < p \), then \( q \in \alpha \),
3. if \( p \in \alpha \), then there exists \( r \in \alpha \) with \( r > p \).

Show \( \alpha \) is bounded from above. Then let \( m := \sup \alpha \) and show that 
\[
\alpha = Q_m := \{ y \in Q : y < m \}.
\]

In this way we see that we may identify \( \mathbb{R} \) with the cuts of \( Q \). This motivated
Dedekind's construction of the real numbers as described in Rudin.

**Definition 3.5.** A sequence \( \{q_n\}_{n=1}^{\infty} \subset \mathbb{R} \) converges to \( 0 \in \mathbb{R} \) if for all \( \varepsilon > 0 \) in \( \mathbb{R} \) there exists \( N \in \mathbb{N} \) such that \( |q_n| \leq \varepsilon \) for all \( n \geq N \). Alternatively put, for all \( M \in \mathbb{N} \) we have \( |q_n| \leq \frac{1}{M} \) for a.a. \( n \).

**Definition 3.6.** A sequence \( \{q_n\}_{n=1}^{\infty} \subset \mathbb{R} \) converges to \( q \in \mathbb{R} \) if \( |q - q_n| \to 0 \) as \( n \to \infty \), i.e. if for all \( N \in \mathbb{N} \), \( |q - q_n| \leq \frac{1}{N} \) for a.a. \( n \). As usual if \( \{q_n\}_{n=1}^{\infty} \) converges to \( q \) we will write \( q_n \to q \) as \( n \to \infty \) or \( q = \lim_{n \to \infty} q_n \).

**Definition 3.7.** A sequence \( \{q_n\}_{n=1}^{\infty} \subset \mathbb{R} \) is Cauchy if \( |q_n - q_m| \to 0 \) as \( m,n \to \infty \). More precisely we require for each \( \varepsilon > 0 \) in \( \mathbb{R} \) that \( |q_n - q_m| \leq \varepsilon \) for a.a. \( (m,n) \), i.e. there should exist \( N \in \mathbb{N} \) such that \( |q_n - q_m| \leq \varepsilon \) for all \( m,n \geq N \).

**Proposition 3.8.** If \( \mathbb{R} \) is a complete ordered field, then every subset \( S \subset \mathbb{R} \) which is bounded from below has a greatest lower bound, \( \inf(S) = \inf(S) \). In fact,
\[
\inf(S) = - \sup(-S).
\]

**Proof.** We let \( m := - \sup(-S) \). Then we have \( -s \leq -m \) for all \( s \in S \), i.e. \( s \geq m \) for all \( s \in S \) so that \( m \) is a lower bound for \( S \). Moreover if \( \varepsilon > 0 \) is given there exists \( s_t \in S \) such that \( -s_t \geq -m - \varepsilon \), i.e. \( s_t \leq m + \varepsilon \). This shows that any lower bound, \( k \) of \( S \) must satisfy, \( k \leq m + \varepsilon \) for all \( \varepsilon > 0 \), i.e. \( k \leq m \). This shows that \( m \) is the greatest lower bound for \( S \). ■

**Proposition 3.9 (\( Q \) is dense in \( \mathbb{R} \)).** For all \( b \in \mathbb{R} \), there exists \( q_n \in Q \) such that \( q_n \uparrow b \). Similarly there exists \( p_n \in Q \) such that \( q_n \downarrow b \).

**Proof.** Let \( q_1 \in Q \) such that \( b - 1 < q_1 < b \). If \( q_1 \leq q_2 \leq \cdots \leq q_n \) in \( Q_b \) have been chosen, let \( q_{n+1} \in Q_b \) be chosen so that
\[
\max\left(q_n, b - \frac{1}{n+1}\right) < q_{n+1} < b.
\]
We then have \( q_n \uparrow b \) as \( n \uparrow \infty \). The second assertion can be proved in much the same way as the first. Alternatively, let \( q_n \in Q \) such that \( q_n \uparrow -b \) and set \( p_n := -q_n \in Q \). Then \( p_n \downarrow b \). ■

The next few results are analogous to what you have already shown in the case \( \mathbb{R} \) is replaced by \( Q \). As the proofs are essentially identical to what you have already done they will be omitted.

**Proposition 3.10.** If \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} \) is a convergent sequence then it is Cauchy. If \( \{a_n\}_{n=1}^{\infty} \) is Cauchy sequence then it is bounded.

**Theorem 3.11 (Basic Limit Results).** Suppose that \( \{a_n\} \) and \( \{b_n\} \) are sequences of real numbers such that \( A := \lim_{n \to \infty} a_n \) and \( B := \lim_{n \to \infty} b_n \) exists in \( \mathbb{R} \). Then;
1. \( \lim_{n \to \infty} |a_n| = |A| \).
2. If \( A \neq 0 \) then \( \lim_{n \to \infty} \frac{1}{a_n} = \frac{1}{A} \).
3. \( \lim_{n \to \infty} (a_n + b_n) = A + B \).
4. \( \lim_{n \to \infty} (a_n b_n) = A \cdot B \).
5. If \( a_n \leq b_n \) for all \( n \), then \( A \leq B \).
6. If \( \{x_n\} \subset \mathbb{R} \) is another sequence such that \( a_n \leq x_n \leq b_n \) and \( A = B \), then \( \lim_{n \to \infty} x_n = A = B \).

**Exercise 3.2 (Sandwich Theorem).** Prove item 6. of Theorem 3.11, i.e. prove the sandwich theorem. Hint: start by showing; if \( a \leq x \leq b \) then \( |x| \leq \max(|a|, |b|) \).

### 3.1 Limsups, Liminfs and Extended Limits

**Notation 3.12** The extended real numbers is the set \( \tilde{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\} \), i.e. it is \( \mathbb{R} \) with two new points called \( \infty \) and \( -\infty \). We use the following conventions,
\[
\pm \infty \cdot 0 = 0, \pm \infty \cdot a = \pm \infty \text{ if } a \in \mathbb{R} \text{ with } a > 0, \pm \infty \cdot a = \mp \infty \text{ if } a \in \mathbb{R} \text{ with } a < 0, \pm \infty + a = \pm \infty \text{ for any } a \in \mathbb{R}, \infty + \infty = \infty \text{ and } -\infty - \infty = -\infty \text{ while } \infty - \infty \text{ is not defined. A sequence } a_n \in \tilde{\mathbb{R}} \text{ is said to converge to } \infty \text{ (} -\infty \text{) if for all } M \in \mathbb{R} \text{ there exists } m \in \mathbb{N} \text{ such that } a_n \geq M \text{ (} a_n \leq M \text{) for all } n \geq m.

**Theorem 3.13.** If \( \{a_n\} \subset \mathbb{R} \) is a bounded non-decreasing sequence the \( \lim_{n \to \infty} a_n = \inf_{n \in \mathbb{N}} a_n \) and if \( \{a_n\} \subset \mathbb{R} \) is a bounded non-increasing sequence the \( \lim_{n \to \infty} a_n = \sup_{n \in \mathbb{N}} a_n \).

**Lemma 3.14.** Suppose \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are convergent sequences in \( \tilde{\mathbb{R}} \), then:
1. If \( a_n \leq b_n \) for a.a. \( n \) then \( \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \).
2. If \( c \in \mathbb{R} \), \( \lim_{n \to \infty} (ca_n) = c \lim_{n \to \infty} a_n \).
3. If \( \{a_n + b_n\}_{n=1}^{\infty} \) is convergent and
\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n
\]
provided the right side is not of the form \( \infty - \infty \).
4. $\{a_nb_n\}_{n=1}^{\infty}$ is convergent and

$$\lim_{n \to \infty} (a_nb_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$$

(3.2)

provided the right hand side is not of the form $\pm \infty \cdot 0$ of $0 \cdot (\pm \infty)$.

Before going to the proof consider the simple example where $a_n = n$ and $b_n = -an$ with $a > 0$. Then

$$\lim(a_n + b_n) = \begin{cases} 
\infty & \text{if } \alpha < 1 \\
0 & \text{if } \alpha = 1 \\
-\infty & \text{if } \alpha > 1
\end{cases}$$

while

$$\lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = -\infty - \infty.$$

This shows that the requirement that the right side of Eq. (3.1) is not of form $\infty - \infty$ is necessary in Lemma 3.1. Similarly by considering the examples $a_n = n$ and $b_n = n^{-\alpha}$ with $\alpha > 0$ shows the necessity for assuming right hand side of Eq. (3.2) is not of the form $\pm \infty \cdot 0$.

**Proof.** The proofs of items 1. and 2. are left to the reader.

**Proof of Eq. (3.1).** Let $a := \lim_{n \to \infty} a_n$ and $b = \lim_{n \to \infty} b_n$. Case 1., suppose $b = \infty$, in which case we must assume $a > -\infty$. In this case, for every $M > 0$, there exists $N$ such that $b_n \geq M$ and $a_n \geq a - 1$ for all $n \geq N$ and this implies

$$a_n + b_n \geq M + a - 1$$

for all $n \geq N$. Since $M$ is arbitrary it follows that $a_n + b_n \to \infty$ as $n \to \infty$. The cases where $b = -\infty$ or $a = \pm \infty$ are handled similarly. Case 2. If $a, b \in \mathbb{R}$, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a - a_n| \leq \varepsilon \text{ and } |b - b_n| \leq \varepsilon \text{ for all } n \geq N.$$

Therefore,

$$|a + b - (a_n + b_n)| = |a - a_n + b - b_n| \leq |a - a_n| + |b - b_n| \leq 2\varepsilon$$

for all $n \geq N$. Since $n$ is arbitrary, it follows that $\lim_{n \to \infty} (a_n + b_n) = a + b$.

**Proof of Eq. (3.2).** It will be left to the reader to prove the case where $\lim a_n$ and $\lim b_n$ exist in $\mathbb{R}$. I will only consider the case where $a = \lim_{n \to \infty} a_n \neq 0$ and $\lim_{n \to \infty} b_n = \infty$ here. Let us also suppose that $a > 0$ (the case $a < 0$ is handled similarly) and let $\alpha := \min \left(\frac{1}{a}, 1\right)$. Given any $M < \infty$, there exists $N \in \mathbb{N}$ such that $a_n \geq \alpha$ and $b_n \geq M$ for all $n \geq N$ and for this choice of $N$, $a_nb_n \geq M\alpha$ for all $n \geq N$. Since $\alpha > 0$ is fixed and $M$ is arbitrary it follows that $\lim_{n \to \infty} (a_n b_n) = \infty$ as desired. \[ \Box \]

For any subset $A \subset \mathbb{R}$, let $\sup A$ and $\inf A$ denote the least upper bound and greatest lower bound of $A$ respectively. The convention being that $\sup A = \infty$ if $\infty \in A$ or $A$ is not bounded from above and $\inf A = -\infty$ if $-\infty \in A$ or $A$ is not bounded from below. We will also use the conventions that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

**Notation 3.15** Suppose that $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a sequence of numbers. Then

$$\liminf_{n \to \infty} x_n = \liminf_{n \to \infty} \{x_k : k \geq n\} \text{ and }$$

(3.3)

$$\limsup_{n \to \infty} x_n = \limsup_{n \to \infty} \{x_k : k \geq n\}.$$ 

(3.4)

We will also write $\lim\inf$ for $\liminf$ and $\lim\sup$ for $\limsup$.

**Remark 3.16.** Notice that if $a_k := \inf \{x_k : k \geq n\}$ and $b_k := \sup \{x_k : k \geq n\}$, then $\{a_k\}$ is an increasing sequence while $\{b_k\}$ is a decreasing sequence. Therefore the limits in Eq. (3.3) and Eq. (3.4) always exist in $\mathbb{R}$ and

$$\lim\inf_{n \to \infty} x_n = \sup\inf_{n \to \infty} \{x_k : k \geq n\} \text{ and }$$

(3.5)

$$\lim\sup_{n \to \infty} x_n = \inf\sup_{n \to \infty} \{x_k : k \geq n\}.$$ 

(3.6)

The following proposition contains some basic properties of liminf and limsup.

**Proposition 3.17.** Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers. Then

1. $\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n$.
2. $\lim_{n \to \infty} a_n$ exists in $\mathbb{R}$ if

$$\lim\inf_{n \to \infty} a_n = \lim\sup_{n \to \infty} a_n \in \mathbb{R}.$$

3. There is a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{k \to \infty} a_{n_k} = \lim\sup_{n \to \infty} a_n$. Similarly, there is a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{k \to \infty} a_{n_k} = \lim\inf_{n \to \infty} a_n$.

4. $\lim\sup_{n \to \infty} (a_n + b_n) \leq \lim\sup_{n \to \infty} a_n + \lim\sup_{n \to \infty} b_n$ whenever the right side of this equation is not of the form $\infty - \infty$.

5. If $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$, then

$$\lim\sup_{n \to \infty} (a_n b_n) \leq \lim\sup_{n \to \infty} a_n \cdot \lim\sup_{n \to \infty} b_n.$$ 

(3.6)
The proof for the case $A$ and therefore by the Sandwich theorem, $\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n$.

2. Now suppose that $\liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = a \in \mathbb{R}$. Then for all $\varepsilon > 0$, there is an integer $N$ such that

$$a - \varepsilon \leq \inf\{a_k : k \geq N\} \leq \sup\{a_k : k \geq N\} \leq a + \varepsilon,$$

i.e.

$$a - \varepsilon \leq a_k \leq a + \varepsilon \text{ for all } k \geq N.$$ 

Hence by the definition of the limit, $\lim_{k \to \infty} a_k = a$. If $\liminf_{n \to \infty} a_n = \infty$, then we know for all $M \in (0, \infty)$ there is an integer $N$ such that

$$M \leq \inf\{a_k : k \geq N\}$$

and hence $\lim_{n \to \infty} a_n = \infty$. The case where $\limsup_{n \to \infty} a_n = -\infty$ is handled similarly.

Conversely, suppose that $\lim_{n \to \infty} a_n = A \in \mathbb{R}$ exists. If $A \in \mathbb{R}$, then for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|A - a_n| \leq \varepsilon$ for all $n \geq N(\varepsilon)$, i.e.

$$A - \varepsilon \leq a_n \leq A + \varepsilon \text{ for all } n \geq N(\varepsilon).$$

From this we learn that

$$A - \varepsilon \leq \lim \inf_{n \to \infty} a_n \leq \lim \sup_{n \to \infty} a_n \leq A + \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, it follows that

$$A \leq \lim \inf_{n \to \infty} a_n \leq \lim \sup_{n \to \infty} a_n \leq A,$$

i.e. that $A = \lim \inf_{n \to \infty} a_n = \lim \sup_{n \to \infty} a_n$. If $A = \infty$, then for all $M > 0$ there exists $N = N(M)$ such that $a_n \geq M$ for all $n \geq N$. This show that $\liminf_{n \to \infty} a_n \geq M$ and since $M$ is arbitrary it follows that

$$\infty \leq \lim \inf_{n \to \infty} a_n \leq \lim \sup_{n \to \infty} a_n.$$ 

The proof for the case $A = -\infty$ is analogous to the $A = \infty$ case. 

**Theorem 3.18 (\(\mathbb{R}\) is complete).** If $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is a Cauchy sequence, then $\lim_{n \to \infty} a_n = \lim \sup_{n \to \infty} a_n = \lim \inf_{n \to \infty} a_n$ all exist in $\mathbb{R}$.

**Proof.** Assume that $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is Cauchy and let $a := \liminf_{n \to \infty} a_n$ and $b := \limsup_{n \to \infty} a_n$. It suffices to show $a = b$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_m - a_n| \leq \varepsilon \text{ for all } m, n \geq N.$$ 

Thus for or $m, n, k \geq N$ we have $a_m \leq a_n + \varepsilon$ for all $m, n \geq k$ and therefore,

$$b \leq \sup_{m \geq k} a_m \leq a_n + \varepsilon \text{ for all } n \geq k$$

and hence

$$b \leq \sup_{m \geq k} a_m \leq \inf_{n \geq k} a_n + \varepsilon \leq a + \varepsilon.$$ 

As $\varepsilon > 0$ is arbitrary, this inequality shows $b \leq a$. 

\[ \blacksquare \]

### 3.2 The Decimal Representation of a Real Number

**Lemma 3.19 (Geometric Series).** Let $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{Q}$, $m, n \in \mathbb{Z}$ and $S := \sum_{k=n}^{m} \alpha^k$. Then

$$S = \begin{cases} 
  m-n+1 & \text{if } \alpha = 1, \\
  \frac{1}{\alpha-1} \left( \alpha^m - \alpha^n \right) & \text{if } \alpha \neq 1.
\end{cases}$$

**Proof.** When $\alpha = 1$,

$$S = \sum_{k=n}^{m} 1^k = m - n + 1.$$ 

If $\alpha \neq 1$, then

$$\alpha S - S = \alpha^{m+1} - \alpha^n.$$ 

Solving for $S$ gives

$$S = \sum_{k=n}^{m} \alpha^k = \frac{\alpha^{m+1} - \alpha^n}{\alpha - 1} \quad \text{if } \alpha \neq 1. \quad (3.7)$$

\[ \blacksquare \]

Taking $\alpha = 10^{-1}$ in Eq. (3.7) implies

$$\sum_{k=n}^{m} 10^{-k} = \frac{10^{-m+1} - 10^{-n}}{10^{-1} - 1} = \frac{1}{10^{n-1}} - \frac{1 - 10^{-(m-n+1)}}{9}$$

and in particular, for all $M \geq n$,

$$\lim_{m \to \infty} \sum_{k=n}^{m} 10^{-k} = \frac{1}{9} \cdot 10^{-n} \geq \sum_{k=n}^{M} 10^{-k}. $$
Definition 3.20 (Decimal Numbers). Let $\mathbb{D}$ denote those sequences $\alpha \in \{0, 1, 2, \ldots, 9\}^\mathbb{N}$ with the following properties:

1. there exists $N \in \mathbb{N}$ such that $\alpha_{-n} = 0$ for all $n \geq N$ and
2. $\alpha_n \neq 0$ for some $n \in \mathbb{Z}$.

A decimal number is then an expression of the form

$$\alpha_N \alpha_{N+1} \ldots \alpha_0 \alpha_1 \alpha_2 \alpha_3 \ldots$$

For example

$$52 + \sqrt{2} \approx 53.41421356237309504880168872420969807856967187537694807\ldots$$

To every decimal number $\alpha \in \mathbb{D}$ is the sequence $a_n = a_n(\alpha)$ defined for $n \in \mathbb{N}$ by

$$a_n := \sum_{k=-\infty}^{n} \alpha_k 10^{-k}.$$ (a finite sum).

Since for $m > n$,

$$|a_m - a_n| = \left| \sum_{k=n+1}^{m} \alpha_k 10^{-k} \right| \leq \sum_{k=n+1}^{m} 10^{-k} \leq 9 \cdot \frac{1}{10^n} = \frac{1}{10^n},$$

it follows that

$$|a_m - a_n| \leq \frac{1}{10^{\min(m,n)}} \to 0 \text{ as } m, n \to \infty$$

which shows $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Thus to every decimal number we may associate the real number

$$a(\alpha) := \lim_{n \to \infty} a_n.$$

Theorem 3.21. If $x \geq 0$ is a real number, there exists $\alpha \in \mathbb{D}$ such that $x = a(\alpha)$, i.e. all real numbers can be represented in decimal form.

Proof. If $x = 0$, we can take $\alpha_n = 0$ for all $n$ so that $0 = a(\alpha)$. So suppose that $x > 0$ and let $p := \min(\{n \in \mathbb{N} : x < n\})$. Set $m = p - 1$, then

$$m \leq x < m + 1.$$ We then define $\alpha_k$ for $k \leq 0$ so that $m = \alpha_{-N} \ldots \alpha_0$. We now construct $\alpha_k$ for $k \geq 1$. For $k = 1$ we write

$$[m, m+1) = \sum_{l=0}^{9} [m + \frac{l}{10}, m + \frac{l+1}{10})$$

and then choose $\alpha_1 = l$ if $x \in [m + \frac{l}{10}, m + \frac{l+1}{10})$. We then construct $\alpha_2$ using,

3.3 (Optional) Proof of Uniqueness in Theorem 3.3

$$[m + \frac{\alpha_1}{10}, m + \frac{\alpha_1 + 1}{10}) \sum_{l=0}^{9} [m + \frac{\alpha_1}{10} + \frac{l}{100}, m + \frac{\alpha_1 + 1}{10} + \frac{l+1}{100})$$

and set $\alpha_2 = l$ for $x \in [m + \frac{\alpha_1}{10} + \frac{l}{100}, m + \frac{\alpha_1 + l+1}{100})$. Continuing this way inductively we construct $(\alpha_k)_{k=1}^{\infty}$ such that

$$x \in [m + \sum_{j=1}^{k} \frac{\alpha_j}{10^j}, m + \sum_{j=1}^{k} \frac{\alpha_j}{10^j} + \frac{\alpha_k}{10^k}).$$

It is now easy to see that $x = a(\alpha)$.

Remark 3.22. The representation of $x \geq 0$ as a decimal number may not be unique. For example,

$$0.999 = \sum_{k=1}^{\infty} \frac{9}{10^k} = \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} = 0.9999999999\ldots$$

On the other hand if we agree to not allow a tail of repeated 9’s as an element of $\mathbb{D}$, then the representation would be unique.

3.3 (Optional) Proof of Uniqueness in Theorem 3.3

Theorem 3.23 (Real numbers are unique). Suppose that $\mathbb{F}$ and $\mathbb{G}$ are two complete ordered fields. Then there is a unique order preserving isomorphism, $\varphi : \mathbb{F} \to \mathbb{G}$.

(Sketch). Suppose that $\varphi : \mathbb{F} \to \mathbb{G}$ is an order preserving homomorphism. The usual arguments show that any homomorphism, $\varphi : \mathbb{F} \to \mathbb{G}$ must satisfy $\varphi(q1_p) = q1_G$. We know that $\{q \cdot 1_F : q \in \mathbb{Q}\}$ and $\{q \cdot 1_G : q \in \mathbb{Q}\}$ are dense copies of $\mathbb{Q}$ inside of $\mathbb{F}$ and $\mathbb{G}$ respectively. Now for general $a \in \mathbb{F}$ choose $q_n, p_n \in \mathbb{Q}$. That $q_n1_F \uparrow a$ and $p_n1_F \downarrow a$. Since $\varphi$ is order preserving we must have $q_n1_G \leq \varphi(q_n1_F)$ and $p_n1_G \geq \varphi(p_n1_F)$. Since $\varphi(q_n1_F)$ is increasing and $\varphi(p_n1_F)$ is decreasing, moreover, since $p_n - q_n \to 0$ we must have $\lim_{n \to \infty} \varphi(q_n1_F) = \lim_{n \to \infty} \varphi(p_n1_F)$. Since $\varphi(q_n1_F) \leq \varphi(a) \leq \varphi(p_n1_F)$ for all $n$ it then follows that $\varphi(a) = \lim_{n \to \infty} q_n1_G = \lim_{n \to \infty} p_n1_G$ and we have shown $\varphi$ is uniquely determined.

For the converse, if $q_n \in \mathbb{Q}$ we know that

$$|q_n1_F - q_m1_F| = |q_n - q_m|1_F$$

and

$$|q_m1_G - q_n1_G| = |q_n - q_m|1_G.$$

Thus if $\{q_n1_F\}_{n=1}^{\infty}$ is convergent in $\mathbb{F}$ iff $\{q_n1_G\}_{n=1}^{\infty}$ is convergent in $\mathbb{G}$. Thus for any $a \in \mathbb{F}$ we choose $q_n \in \mathbb{Q}$ such that $q_n1_F \uparrow a$ and then define $\varphi(a) :=$
The other properties of $\varphi$ are proved similarly.
Complex Numbers

**Definition 4.1 (Complex Numbers).** Let $\mathbb{C} = \mathbb{R}^2$ equipped with multiplication rule

$$(a, b)(c, d) \equiv (ac - bd, bc + ad)$$

and the usual rule for vector addition. As is standard we will write $0 = (0, 0)$, $1 = (1, 0)$ and $i = (0, 1)$ so that every element $z$ of $\mathbb{C}$ may be written as $z = x + yi$ which in the future will be written simply as $z = x + iy$. If $z = x + iy$, let $\text{Re} z = x$ and $\text{Im} z = y$.

Writing $z = a + ib$ and $w = c + id$, the multiplication rule in Eq. (4.1) becomes

$$(a + ib)(c + id) \equiv (ac - bd) + i(bc + ad)$$

and in particular $1^2 = 1$ and $i^2 = -1$.

**Proposition 4.2.** The complex numbers $\mathbb{C}$ with the above multiplication rule satisfies the usual definitions of a field – see Definition 2.1. For example $z$ and in particular $1$ has a multiplicative inverse given by

$$z^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}.$$  

Moreover $\mathbb{C}$ contains $\mathbb{R}$ as sub-field under the identification

$$\mathbb{R} \ni a \rightarrow a1 + 0i = (a, 0) \in \mathbb{C}.$$  

**Proof.** Suppose $z = a + ib \neq 0$, we wish to find $w = c + id$ such that $zw = 1$ and this happens by Eq. (4.2) iff

$$ac - bd = 1$$  

and

$$bc + ad = 0.$$  

Solving these equations as follows

$$a(1.4) + b(1.5) \implies (a^2 + b^2) c = a \implies \text{Re} w = c = \frac{a}{a^2 + b^2}$$

$$-b(1.4) + a(1.5) \implies (a^2 + b^2) d = -b \implies \text{Im} w = d = -\frac{b}{a^2 + b^2}.$$

gives implies the result in Eq. (4.3).

Probably the most painful thing to check directly is the associative law, namely that $[z_1 z_2] z_3 = z_1 [z_2 z_3]$ for all $z_1, z_2, z_3 \in \mathbb{C}$. This is equivalent to showing for all $a, b, u, v, x, y \in \mathbb{R}$ that

$$[(a + ib)(u + iv)](x + iy) = (a + ib)[(u + iv)(x + iy)].$$

We do this by working out both sides as follows;

$$LHS = [(au - bv) + i(au + bv)](x + iy)$$

$$= (au - bv)x - (av + bu)y + i[(av + bu)x + (au - bv)y];$$

$$RHS = (a + ib)[(ux - vy) + i(uy + vx)]$$

$$= a(ux - vy) - b(uy + vx) + i[b(ux - vy) + a(uy + vx)].$$

The reader should now easily see that both of these expressions are in fact equal. The remaining axioms of a field are checked similarly. □

**Notation 4.3** We will write $1/z$ for $z^{-1}$ and $w/z$ to mean $z^{-1} \cdot w$.

**Notation 4.4 (Conjugation and Modulus)** If $z = a + ib$ with $a, b \in \mathbb{R}$ let $ar{z} = a - ib$ and

$$|z|^2 \equiv z\bar{z} = a^2 + b^2.$$  

Notice that

$$\text{Re} z = \frac{1}{2}(z + \bar{z}) \text{ and } \text{Im} z = \frac{1}{2i}(z - \bar{z}).$$

**Proposition 4.5.** Complex conjugation and the modulus operators satisfy:

1. $\bar{\bar{z}} = z$,
2. $\bar{zw} = \bar{z}\bar{w}$ and $\bar{z} + \bar{w} = \bar{z + w}$,
3. $|\bar{z}| = |z|$,
4. $|zw| = |z||w|$ and in particular $|z^n| = |z|^n$ for all $n \in \mathbb{N}$,
5. $|\text{Re} z| \leq |z|$ and $|\text{Im} z| \leq |z|$
6. $|z + w| \leq |z| + |w|$.
7. $z = 0$ iff $|z| = 0$. 
8. If \( z \neq 0 \) then 
\[
    z^{-1} := \frac{\bar{z}}{|z|^2}
\]
(also written as \( \frac{1}{z} \)) is the inverse of \( z \).

9. \( |z^{-1}| = |z|^{-1} \) and more generally \( |z^n| = |z|^n \) for all \( n \in \mathbb{Z} \).

**Proof.** 1. and 3. are geometrically obvious as well as easily verified.
2. Say \( z = a + ib \) and \( w = c + id \), then \( \bar{z}w \) is the same as \( zw \) with \( b \) replaced by \( -b \) and \( d \) replaced by \( -d \), and looking at Eq. (4.2) we see that
\[
    \bar{z}w = (ac - bd) - i(bc + ad) = \bar{zw}.\]

4. \( |zw|^2 = zw\bar{w} = |z|^2 |w|^2 \) as real numbers and hence \( |zw| = |z| |w| \).

5. Geometrically obvious or also follows from
\[
    |z| = \sqrt{\text{Re} z^2 + \text{Im} z^2}.\]

6. This is the triangle inequality which may be understood geometrically or by the computation
\[
    |z + w|^2 = (z + w)(\bar{z} + \bar{w}) = |z|^2 + |w|^2 + w\bar{z} + \bar{w}z
\]
\[
    = |z|^2 + |w|^2 + w\bar{z} + \bar{w}z
\]
\[
    = |z|^2 + |w|^2 + 2 \text{Re}(w\bar{z}) \leq |z|^2 + |w|^2 + 2|z||w|
\]
\[
    = (|z| + |w|)^2.\]

7. Obvious.

8. Follows from Eq. (4.3). Alternatively if \( \rho = \rho_1 + i\rho_2 > 0 \) is a real number then \( \rho^{-1} = \rho_1^{-1} + i\rho_2^{-1} \) as is easily verified since \( \mathbb{R} \) is a sub-field of \( \mathbb{C} \). Thus since \( \bar{z}z = |z|^2 \) we find
\[
    \frac{1}{|z|^2} \bar{z}z = \frac{1}{|z|^2} |z|^2 = 1 \implies z^{-1} = \frac{1}{|z|^2} \bar{z} = \frac{\text{Re} z}{|z|^2} - i \frac{\text{Im} z}{|z|^2}.\]

\[
    9. |z^{-1}| = \left| \frac{\bar{z}}{|z|^2} \right| = \left| \frac{1}{|z|^2} \right| |\bar{z}| = \frac{1}{|z|}.\]

**Lemma 4.6.** For complex number \( u, v, w, z \in \mathbb{C} \) with \( v \neq 0 \neq z \), we have
\[
    \frac{1}{\frac{u}{v}} = \frac{v}{u}, \quad \text{i.e. } \frac{v}{u} = (uv)^{-1}
\]
\[
    \frac{u}{v} \quad \text{and} \quad \frac{w}{z} = \frac{uv}{vz}
\]
\[
    \frac{u}{v} + \frac{w}{z} = \frac{uz + vw}{vz}.\]

**Proof.** For the first item, it suffices to check that
\[
    (uv) \left( u^{-1}v^{-1} \right) = u^{-1}uvv^{-1} = 1 \cdot 1 = 1.
\]

The rest follow using
\[
    \frac{u}{v} \quad \text{and} \quad \frac{w}{z} = \frac{uv}{vz}.
\]

\[
    \frac{u}{v} + \frac{w}{z} = \frac{uz + vw}{vz}.
\]

**Definition 4.7.** A sequence \( \{z_n\}_{n=1}^{\infty} \subset \mathbb{C} \) is **Cauchy** if \( |z_n - z_m| \to 0 \) as \( m, n \to \infty \) and is **convergent** to \( z \in \mathbb{C} \) if \( |z_n - z| \to 0 \) as \( n \to \infty \). As usual if \( \{z_n\}_{n=1}^{\infty} \) converges to \( z \) we will write \( z_n \to z \) as \( n \to \infty \) or \( z = \lim_{n \to \infty} z_n \).

**Theorem 4.8.** The complex numbers are complete, i.e. all Cauchy sequences are convergent.

**Proof.** This follows from the completeness of real numbers and the easily proved observations that if \( z_n = a_n + ib_n \in \mathbb{C} \), then
1. \( \{z_n\}_{n=1}^{\infty} \subset \mathbb{C} \) is Cauchy iff \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} \) and \( \{b_n\}_{n=1}^{\infty} \subset \mathbb{R} \)
2. \( z_n \to z = a + ib \) as \( n \to \infty \) iff \( a_n \to a \) and \( b_n \to b \) as \( n \to \infty \).
\[ M_z w = \begin{pmatrix} ac - bd \\ bc + ad \end{pmatrix} = \begin{pmatrix} a - b \\ b \ a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \]

so that

\[ M_z = \begin{pmatrix} a - b \\ b \ a \end{pmatrix} = aI + bJ \]

where

\[ J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

We now have the following simple observations;

1. \( J^2 = -I \) and \( J^* = -J \),
2. \( M_z M_w = M_w M_z \) because \( J \) and \( I \) commute,
3. we have

\[ M_z M_w = (aI + bJ) (cI + dJ) = (ac - bd) I + (ad + bc) J = M_{zw}, \]

4. the associativity of complex multiplication follows from the associativity properties of matrix multiplication,
5. \( M_z^* = aI - bJ = M_{\bar{z}} \) and in particular
6. \( M_{\bar{z}} = (M_z M_w)^* = M_w^* M_z^* = M_{\bar{w}} M_{\bar{z}} = M_{\bar{w}\bar{z}}, \)
7. \( M_{\bar{z}}^* M_z = M_{\bar{z} z} = M_{|z|^2} = \det(M_z), \)
8. \( |wz| = \det(M_{wz}) = \det(M_w M_z) = \det(M_w) \det(M_z) = |w| |z|, \)
9. \( M_z \) is invertible iff \( \det(M_z) \neq 0 \) which happens iff \( |z|^2 \neq 0 \) and in this case we know from basic linear algebra that

\[ M_z^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{|z|^2} M_z^* = M_{\frac{1}{|z|^2} \bar{z}}, \]

10. With this notation we have \( M_z M_w = M_{zw} \) and since \( I \) and \( J \) commute it follows that \( zw = wz \). Moreover, since matrix multiplication is associative so is complex multiplication. Also notice that \( M_z \) is invertible iff \( \det(M_z) = a^2 + b^2 = |z|^2 \neq 0 \) in which case

\[ M_z^{-1} = \frac{1}{|z|^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = M_{\bar{z}/|z|^2}; \]

as we have already seen above.
5

Sums

5.1 Sums of positive functions

In this and the next few sections, let \( X \) and \( Y \) be two sets. We will write \( \alpha \subset \subset X \) to denote that \( \alpha \) is a finite subset of \( X \) and write \( 2^X \) for those \( \alpha \subset \subset X \).

**Definition 5.1.** Suppose that \( a : X \to [0, \infty] \) is a function and \( F \subset X \) is a subset, then

\[
\sum_F a = \sum_{x \in F} a(x) := \sup \left\{ \sum_{x \in \alpha} a(x) : \alpha \subset \subset F \right\}.
\]

**Remark 5.2.** Suppose that \( X = \mathbb{N} = \{1, 2, 3, \ldots\} \) and \( a : X \to [0, \infty] \), then

\[
\sum a = \sum_{n=1}^{\infty} a(n) := \lim_{N \to \infty} \sum_{n=1}^{N} a(n).
\]

Indeed for all \( N \), \( \sum_{n=1}^{N} a(n) \leq \sum_{n=1}^{\infty} a \), and thus passing to the limit we learn that

\[
\sum_{n=1}^{\infty} a(n) \leq \sum a.
\]

Conversely, if \( \alpha \subset \subset \mathbb{N} \), then for all \( N \) large enough so that \( \alpha \subset \{1, 2, \ldots, N\} \), we have \( \sum_{\alpha} a \leq \sum_{n=1}^{N} a(n) \) which upon passing to the limit implies that

\[
\sum_{\alpha} a \leq \sum_{n=1}^{\infty} a(n).
\]

Taking the supremum over \( \alpha \) in the previous equation shows

\[
\sum a \leq \sum_{n=1}^{\infty} a(n).
\]

**Remark 5.3.** Suppose \( a : X \to [0, \infty] \) and \( \sum_X a < \infty \), then \( \{x \in X : a(x) > 0\} \) is at most countable. To see this first notice that for any \( \varepsilon > 0 \), the set \( \{x : a(x) \geq \varepsilon\} \) must be finite for otherwise \( \sum_X a = \infty \). Thus

\[
\{x \in X : a(x) > 0\} = \bigcup_{k=1}^{\infty} \{x : a(x) \geq 1/k\}
\]

which shows that \( \{x \in X : a(x) > 0\} \) is a countable union of finite sets and thus countable by Lemma 6.6.

**Lemma 5.4.** Suppose that \( a, b : X \to [0, \infty] \) are two functions, then

\[
\sum_X (a + b) = \sum_X a + \sum_X b \quad \text{and} \quad \sum_X \lambda a = \lambda \sum_X a
\]

for all \( \lambda \geq 0 \).

I will only prove the first assertion, the second being easy. Let \( \alpha \subset \subset X \) be a finite set, then

\[
\sum_{\alpha} (a + b) = \sum_{\alpha} a + \sum_{\alpha} b \leq \sum_X a + \sum_X b
\]

which after taking sups over \( \alpha \) shows that

\[
\sum_X (a + b) \leq \sum_X a + \sum_X b.
\]

Similarly, if \( \alpha, \beta \subset \subset X \), then

\[
\sum_{\alpha} a + \sum_{\beta} b \leq \sum_{\alpha \cup \beta} a + \sum_{\alpha \cup \beta} b = \sum_{\alpha \cup \beta} (a + b) \leq \sum_X (a + b).
\]

Taking sups over \( \alpha \) and \( \beta \) then shows that

\[
\sum_X a + \sum_X b \leq \sum (a + b).
\]

**Lemma 5.5.** Let \( X \) and \( Y \) be sets, \( R \subset X \times Y \) and suppose that \( a : R \to \mathbb{R} \) is a function. Let \( \sigma R := \{y \in Y : (x, y) \in R\} \) and \( R_y := \{x \in X : (x, y) \in R\} \). Then
Sums

(Recall the conventions: \( \sup_{(x,y) \in R} a(x,y) \) for all \( (x,y) \in R \) and therefore that Equations (5.1) and (5.2) show that \( \inf_{(x,y) \in R} a(x,y) = \inf_{y \in Y} \sup_{x \in X} a(x,y) \).

\[ \text{Proof.} \] Let \( M = \sup_{(x,y) \in R} a(x,y) \), \( N_x := \sup_{y \in Y} a(x,y) \). Then \( a(x,y) \leq M \) for all \( (x,y) \in R \) implies \( N_x = \sup_{y \in Y} a(x,y) \leq M \) and therefore
\[
\sup_{x \in X} \inf_{y \in Y} a(x,y) = \sup_{x \in X} a(x,y) \leq M \tag{5.1}
\]

Similarly for any \( (x,y) \in R \),
\[ a(x,y) \leq N_x \leq \sup_{x \in X} N_x = \sup_{x \in X} \inf_{y \in Y} a(x,y) \]
and therefore
\[ M = \sup_{(x,y) \in R} a(x,y) \leq \sup_{x \in X} \inf_{y \in Y} a(x,y) \tag{5.2} \]

Equations (5.1) and (5.2) show that
\[ \sup_{(x,y) \in R} a(x,y) = \sup_{x \in X} \inf_{y \in Y} a(x,y) \]

The assertions involving infimums are proved analogously or follow from what we have just proved applied to the function \(-a\).  

\[ \text{Fig. 5.1. The } x \text{ and } y \text{ – slices of a set } R \subset X \times Y. \]

Theorem 5.6 (Monotone Convergence Theorem for Sums). Suppose that \( f_n : X \to [0, \infty] \) is an increasing sequence of functions and
\[ f(x) := \lim_{n \to \infty} f_n(x) = \sup_n f_n(x). \]

Then
\[ \lim_{n \to \infty} \sum_X f_n = \sum_X f. \]

\[ \text{Proof.} \] We will give two proofs.

First proof. Let \( 2^X := \{ A \subset X : A \subset \subset X \} \).

Then
\[
\lim_{n \to \infty} \sum_X f_n = \sup_n \sum_X f_n = \sup_n \sum_{\alpha \in 2^X} \sum_{n} f_n = \sup_{\alpha \in 2^X} \sum_{n} \lim_{n \to \infty} f_n = \sup_{\alpha \in 2^X} \sum_{n} f = \sum_X f.
\]

Second Proof. Let \( S_n = \sum_X f_n \) and \( S = \sum_X f \). Since \( f_n \leq f_m \leq f \) for all \( n \leq m \), it follows that
\[ S_n \leq S_m \leq S \]
which shows that \( \lim_{n \to \infty} S_n \) exists and is less that \( S \), i.e.
\[ A := \lim_{n \to \infty} \sum_X f_n \leq \sum_X f. \tag{5.3} \]

Noting that \( \sum_{n} f_n \leq \sum_X f_n = S_n \leq A \) for all \( \alpha \subset \subset X \) and in particular,
\[ \sum_{n} f_n \leq A \] for all \( n \) and \( \alpha \subset \subset X \).

Letting \( n \) tend to infinity in this equation shows that
\[ \sum_{\alpha} f \leq A \] for all \( \alpha \subset \subset X \)
and then taking the sup over all \( \alpha \subset \subset X \) gives
\[ \sum_{X} f \leq A = \lim_{n \to \infty} \sum_{X} f_n \tag{5.4} \]
which combined with Eq. (5.3) proves the theorem.
Lemma 5.7 (Fatou’s Lemma for Sums). Suppose that \( f_n : X \to [0, \infty] \) is a sequence of functions, then
\[
\sum_X \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \sum_X f_n.
\]

Proof. Define \( g_k := \inf_{n \geq k} f_n \) so that \( g_k \uparrow \liminf_{n \to \infty} f_n \) as \( k \to \infty \). Since \( g_k \leq f_n \) for all \( n \geq k \),
\[
\sum_X g_k \leq \sum_X f_n \text{ for all } n \geq k
\]
and therefore
\[
\sum_X g_k \leq \liminf_{n \to \infty} \sum_X f_n \text{ for all } k.
\]
We may now use the monotone convergence theorem to let \( k \to \infty \) to find
\[
\sum_X \liminf_{n \to \infty} f_n = \sum_X \liminf_{k \to \infty} \sum_X g_k \leq \liminf_{n \to \infty} \sum_X f_n.
\]

Remark 5.8. If \( A = \sum_X a < \infty \), then for all \( \varepsilon > 0 \) there exists \( \alpha_\varepsilon \subset X \) such that
\[
A \geq \sum_\alpha a \geq A - \varepsilon
\]
for all \( \alpha \subset X \) containing \( \alpha_\varepsilon \) or equivalently,
\[
\left| A - \sum_\alpha a \right| \leq \varepsilon \tag{5.5}
\]
for all \( \alpha \subset X \) containing \( \alpha_\varepsilon \). Indeed, choose \( \alpha_\varepsilon \) so that \( \sum_{\alpha_\varepsilon} a \geq A - \varepsilon \).

5.2 Sums of complex functions

Definition 5.9. Suppose that \( a : X \to \mathbb{C} \) is a function, we say that
\[
\sum_X a = \sum_{x \in X} a(x)
\]
exists and is equal to \( A \in \mathbb{C} \), if for all \( \varepsilon > 0 \) there is a finite subset \( \alpha_\varepsilon \subset X \) such that for all \( \alpha \subset X \) containing \( \alpha_\varepsilon \) we have
\[
\left| A - \sum_{\alpha} a \right| \leq \varepsilon.
\]

The following lemma is left as an exercise to the reader.

Lemma 5.10. Suppose that \( a, b : X \to \mathbb{C} \) are two functions such that \( \sum_X a \) and \( \sum_X b \) exist, then \( \sum_X (a + \lambda b) \) exists for all \( \lambda \in \mathbb{C} \) and
\[
\sum_X (a + \lambda b) = \sum_X a + \lambda \sum_X b.
\]

Definition 5.11 (Summable). We call a function \( a : X \to \mathbb{C} \) summable if
\[
\sum_X |a| < \infty.
\]

Proposition 5.12. Let \( a : X \to \mathbb{C} \) be a function, then \( \sum_X a \) exists iff \( \sum_X |a| < \infty \), i.e. iff \( a \) is summable. Moreover if \( a \) is summable, then
\[
\left| \sum_X a \right| \leq \sum_X |a|.
\]

Proof. If \( \sum_X |a| < \infty \), then \( \sum_X (\text{Re} a)^\pm < \infty \) and \( \sum_X (\text{Im} a)^\pm < \infty \) and hence by Remark 5.8 these sums exist in the sense of Definition 5.9. Therefore by Lemma 5.10 \( \sum_X a \) exists and
\[
\sum_X a = \sum_X (\text{Re} a)^+ - \sum_X (\text{Re} a)^- + i \left( \sum_X (\text{Im} a)^+ - \sum_X (\text{Im} a)^- \right).
\]

Conversely, if \( \sum_X |a| = \infty \) then, because \( |a| \leq |\text{Re} a| + |\text{Im} a| \), we must have
\[
\sum_X |\text{Re} a| = \infty \text{ or } \sum_X |\text{Im} a| = \infty.
\]

Thus it suffices to consider the case where \( a : X \to \mathbb{R} \) is a real function. Write \( a = a^+ - a^- \) where
\[
a^+(x) = \max(a(x), 0) \text{ and } a^-(x) = \max(-a(x), 0). \tag{5.6}
\]

Then \( |a| = a^+ + a^- \) and
\[
\infty = \sum_X |a| = \sum_X a^+ + \sum_X a^-
\]
which shows that either \( \sum_X a^+ = \infty \) or \( \sum_X a^- = \infty \). Suppose, with out loss of generality, that \( \sum_X a^+ = \infty \). Let \( X' := \{ x \in X : a(x) \geq 0 \} \), then we know that \( \sum_{X'} a = \infty \) which means there are finite subsets \( \alpha_n \subset X' \subset X \) such that \( \sum_{\alpha_n} a \geq n \) for all \( n \). Thus if \( \alpha \subset X \) is any finite set, it follows that
lim_{n \to \infty} \sum_{\alpha \in \Omega} a = \infty, and therefore \( \sum_X a \) cannot exist as a number in \( \mathbb{R} \). Finally if \( a \) is summable, write \( \sum_X a = \rho e^{i\theta} \) with \( \rho \geq 0 \) and \( \theta \in \mathbb{R} \), then

\[
\left| \sum_X a \right| = \rho = e^{-i\theta} \sum_X a = \sum_X e^{-i\theta} a
\]

\[
= \sum_X \text{Re} \left[ e^{-i\theta} a \right] \leq \sum_X (\text{Re} \left[ e^{-i\theta} a \right])^+ \leq \sum_X \left| e^{-i\theta} a \right| \leq \sum_X |a|.
\]

Alternatively, this may be proved by approximating \( \sum_X a \) by a finite sum and then using the triangle inequality of \( |\cdot| \).

Remark 5.13. Suppose that \( X = \mathbb{N} \) and \( a : \mathbb{N} \to \mathbb{C} \) is a sequence, then it is not necessarily true that

\[
\sum_{n=1}^{\infty} a(n) = \sum_{n \in \mathbb{N}} a(n).
\]

This is because

\[
\sum_{n=1}^{\infty} a(n) = \lim_{N \to \infty} \sum_{n=1}^{N} a(n)
\]

depends on the ordering of the sequence \( a \) where as \( \sum_{n \in \mathbb{N}} a(n) \) does not. For example, take \( a(n) = (-1)^n/n \) then \( \sum_{n \in \mathbb{N}} |a(n)| = \infty \) i.e. \( \sum_{n \in \mathbb{N}} a(n) \) does not exist while \( \sum_{n=1}^{\infty} a(n) \) does exist. On the other hand, if

\[
\sum_{n \in \mathbb{N}} |a(n)| = \sum_{n=1}^{\infty} |a(n)| < \infty
\]

then Eq. (5.7) is valid.

Theorem 5.14 (Dominated Convergence Theorem for Sums). Suppose that \( f_n : X \to \mathbb{C} \) is a sequence of functions on \( X \) such that \( f_n(x) = \lim_{n \to \infty} f_n(x) \in \mathbb{C} \) exists for all \( x \in X \). Further assume there is a dominating function \( g : X \to [0, \infty) \) such that

\[
|f_n(x)| \leq g(x) \text{ for all } x \in X \text{ and } n \in \mathbb{N}
\]

and that \( g \) is summable. Then

\[
\lim_{n \to \infty} \sum_{x \in X} f_n(x) = \sum_{x \in X} f(x).
\]

Proof. Notice that \( |f| = \lim |f_n| \leq g \) so that \( f \) is summable. By considering the real and imaginary parts of \( f \) separately, it suffices to prove the theorem in the case where \( f \) is real. By Fatou’s Lemma,

\[
\sum_X (g \pm f) = \sum_X \lim \inf_{n \to \infty} (g \pm f_n) \leq \lim \inf_{n \to \infty} \sum_X (g \pm f_n)
\]

\[
= \sum_X g \pm \lim \inf_{n \to \infty} \left( \pm \sum_X f_n \right).
\]

Since \( \lim \inf_{n \to \infty} (-a_n) = -\lim \sup_{n \to \infty} a_n \), we have shown,

\[
\sum_X g \pm \sum_X f_n \leq \sum_X g + \left\{ \lim \inf_{n \to \infty} \sum_X f_n \pm \lim \sup_{n \to \infty} \sum_X f_n \right\}
\]

and therefore

\[
\lim \sup_{n \to \infty} \sum_X f_n \leq \sum_X f \leq \lim \inf_{n \to \infty} \sum_X f_n.
\]

This shows that \( \lim_{n \to \infty} \sum_X f_n \) exists and is equal to \( \sum_X f \).

Proof. (Second Proof.) Passing to the limit in Eq. (5.8) shows that \( |f| \leq g \) and in particular that \( f \) is summable. Given \( \varepsilon > 0 \), let \( \alpha \subset \subset X \) such that \( \sum_{X \setminus \alpha} g \leq \varepsilon \).

Then for \( \beta \subset \subset X \) such that \( \alpha \subset \beta \),

\[
\left| \sum_{\beta} f - \sum_{\beta} f_n \right| = \left| \sum_{\beta} (f - f_n) \right| \leq \sum_{\beta} |f - f_n| = \sum_{\alpha} |f - f_n| + \sum_{\beta \setminus \alpha} |f - f_n| \leq \sum_{\alpha} |f - f_n| + 2\varepsilon.
\]

and hence that

\[
\left| \sum_{\beta} f - \sum_{\beta} f_n \right| \leq \sum_{\alpha} |f - f_n| + 2\varepsilon.
\]

Since this last equation is true for all such \( \beta \subset \subset X \), we learn that
\[ \left| \sum_{X} f - \sum_{X} f_n \right| \leq \sum_{\alpha} |f - f_n| + 2\varepsilon \]

which then implies that
\[ \limsup_{n \to \infty} \left| \sum_{X} f - \sum_{X} f_n \right| \leq \limsup_{n \to \infty} \sum_{\alpha} |f - f_n| + 2\varepsilon \]

\[ = 2\varepsilon. \]

Because \( \varepsilon > 0 \) is arbitrary we conclude that
\[ \limsup_{n \to \infty} \left| \sum_{X} f - \sum_{X} f_n \right| = 0. \]

which is the same as Eq. (5.9). \( \square \)

**Remark 5.15.** Theorem 5.14 may easily be generalized as follows. Suppose \( f_n, g_n, g \) are summable functions on \( X \) such that \( f_n \to f \) and \( g_n \to g \) pointwise, \( |f_n| \leq g_n \) and \( \sum_X g_n \to \sum_X g \) as \( n \to \infty \). Then \( f \) is summable and Eq. (5.9) still holds. For the proof we use Fatou’s Lemma to again conclude
\[ \sum_X (g \pm f) = \liminf_{n \to \infty} (g_n \pm f_n) \leq \liminf_{n \to \infty} \sum_X (g_n \pm f_n) \]

\[ = \sum_X g \mp \liminf_{n \to \infty} \left( \pm \sum_X f_n \right) \]

and then proceed exactly as in the first proof of Theorem 5.14.

### 5.3 Iterated sums and the Fubini and Tonelli Theorems

Let \( X \) and \( Y \) be two sets. The proof of the following lemma is left to the reader.

**Lemma 5.16.** Suppose that \( a : X \to \mathbb{C} \) is function and \( F \subset X \) is a subset such that \( a(x) = 0 \) for all \( x \notin F \). Then \( \sum_F a \) exists iff \( \sum_X a \) exists and when the sums exists,
\[ \sum_X a = \sum_F a. \]

**Theorem 5.17 (Tonelli’s Theorem for Sums).** Suppose that \( a : X \times Y \to [0, \infty] \), then
\[ \sum_{X \times Y} a = \sum_X \sum_Y a = \sum_Y \sum_X a. \]

**Proof.** It suffices to show, by symmetry, that
\[ \sum_{X \times Y} a = \sum_X \sum_Y a \]

Let \( A \subset X \times Y \). Then for any \( \alpha \subset X \) and \( \beta \subset Y \) such that \( A \subset \alpha \times \beta \), we have
\[ \sum_{A} a \leq \sum_{\alpha \times \beta} a = \sum_{\alpha} \sum_{\beta} a = \sum_X \sum_Y a, \]

i.e. \( \sum_A a \leq \sum_X \sum_Y a \). Taking the sup over \( A \) in this last equation shows
\[ \sum_{X \times Y} a \leq \sum_X \sum_Y a. \]

For the reverse inequality, for each \( x \in X \) choose \( \beta_x \subset \subset Y \) such that \( \beta_x \uparrow Y \) as \( n \uparrow \infty \) and
\[ \sum_{y \in Y} a(x, y) = \lim_{n \to \infty} \sum_{y \in \beta_n} a(x, y). \]

If \( \alpha \subset X \) is a given finite subset of \( X \), then
\[ \sum_{x \in \alpha} a(x, y) = \lim_{n \to \infty} \sum_{y \in \beta_n} a(x, y) \quad \text{for all } x \in \alpha \]

where \( \beta_n := \cup_{x \in \alpha} \beta_x \subset \subset Y \). Hence
\[ \sum_{x \in \alpha} \sum_{y \in Y} a(x, y) = \sum_{x \in \alpha} \lim_{n \to \infty} \sum_{y \in \beta_n} a(x, y) = \lim_{n \to \infty} \sum_{x \in \alpha} \sum_{y \in \beta_n} a(x, y) \]

\[ = \lim_{n \to \infty} \sum_{(x, y) \in \alpha \times \beta_n} a(x, y) \leq \sum_{X \times Y} a. \]

Since \( \alpha \) is arbitrary, it follows that
\[ \sum_{x \in X} \sum_{y \in Y} a(x, y) = \sup_{\alpha \subset \subset X} \sum_{x \in \alpha} \sum_{y \in Y} a(x, y) \leq \sum_{X \times Y} a \]

which completes the proof. \( \square \)

**Theorem 5.18 (Fubini’s Theorem for Sums).** Now suppose that \( a : X \times Y \to \mathbb{C} \) is a summable function, i.e. by Theorem 5.17 any one of the following equivalent conditions hold:

1. \( \sum_{X \times Y} |a| < \infty \),
2. \( \sum_X \sum_Y |a| < \infty \) or
Lemma 5.20 (Properties of \( \inf \) and \( \sup \)). We have:

1. \( (\bigcup_{n} A_n)^c = \bigcap_{n} A_n^c \).
2. \( \{ A_n \text{ i.o.} \}^c = \{ A_n^c \text{ a.a.} \} \).
3. \( \limsup_{n \to \infty} A_n = \{ x \in X : \sum_{n=1}^{\infty} 1_{A_n} (x) = \infty \} \).
4. \( \liminf_{n \to \infty} A_n = \{ x \in X : \sum_{n=1}^{\infty} 1_{A_n}^c (x) < \infty \} \).
5. \( \sup_{k \geq n} 1_{A_k} (x) = 1_{\bigcup_{k \geq n} A_k} \).
6. \( \inf_{k \geq n} 1_{A_k} (x) = 1_{\bigcap_{k \geq n} A_k} \).
7. \( \limsup_{n \to \infty} A_n = \limsup_{n \to \infty} \sup_{k \geq n} 1_{A_k} \), and
8. \( \liminf_{n \to \infty} A_n = \liminf_{n \to \infty} \inf_{k \geq n} 1_{A_k} \).

Proof. These results follow fairly directly from the definitions and so the proof is left to the reader. (The reader should definitely provide a proof for herself.)

5.4 Supremum and Infiniums of sets

Definition 5.19. Given a set \( A \subset X \), let

\[
1_{A} (x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A
\end{cases}
\]

be the indicator function of \( A \).

Lemma 5.20 (Properties of \( \inf \) and \( \sup \)). We have:

1. \( (\bigcup_{n} A_n)^c = \bigcap_{n} A_n^c \).
2. \( \{ A_n \text{ i.o.} \}^c = \{ A_n^c \text{ a.a.} \} \).
3. \( \limsup_{n \to \infty} A_n = \{ x \in X : \sum_{n=1}^{\infty} 1_{A_n} (x) = \infty \} \).
4. \( \liminf_{n \to \infty} A_n = \{ x \in X : \sum_{n=1}^{\infty} 1_{A_n}^c (x) < \infty \} \).
5. \( \sup_{k \geq n} 1_{A_k} (x) = 1_{\bigcup_{k \geq n} A_k} \).
6. \( \inf_{k \geq n} 1_{A_k} (x) = 1_{\bigcap_{k \geq n} A_k} \).
7. \( \limsup_{n \to \infty} A_n = \limsup_{n \to \infty} \sup_{k \geq n} 1_{A_k} \), and
8. \( \liminf_{n \to \infty} A_n = \liminf_{n \to \infty} \inf_{k \geq n} 1_{A_k} \).

Proof. These results follow fairly directly from the definitions and so the proof is left to the reader. (The reader should definitely provide a proof for herself.)

5.5 \( \ell^p \) – spaces, Minkowski and Holder Inequalities

In this chapter, let \( \mu : X \to (0, \infty) \) be a given function. Let \( F \) denote either \( \mathbb{R} \) or \( \mathbb{C} \). For \( p \in (0, \infty) \) and \( f : X \to F \), let

\[
\| f \|_p := \left( \sum_{x \in X} |f(x)|^p \mu(x) \right)^{1/p}
\]

and for \( p = \infty \) let

\[
\| f \|_\infty = \sup \{|f(x)| : x \in X\}.
\]

Also, for \( p > 0 \), let

\[
\ell^p (\mu) = \{ f : X \to F : \| f \|_p < \infty \}.
\]

In the case where \( \mu(x) = 1 \) for all \( x \in X \) we will simply write \( \ell^p (X) \) for \( \ell^p (\mu) \).

Definition 5.21. A norm on a vector space \( Z \) is a function \( \| \cdot \| : Z \to [0, \infty) \) such that

1. (Homogeneity) \( \| \lambda f \| = |\lambda| \| f \| \) for all \( \lambda \in \mathbb{F} \) and \( f \in Z \).
2. (Triangle inequality) \( \| f + g \| \leq \| f \| + \| g \| \) for all \( f, g \in Z \).
3. (Positive definite) \( \| f \| = 0 \) implies \( f = 0 \).

A function \( p : Z \to [0, \infty) \) satisfying properties 1. and 2. but not necessarily 3. above will be called a semi-norm on \( Z \).

A pair \( (Z, \| \cdot \|) \) where \( Z \) is a vector space and \( \| \cdot \| \) is a norm on \( Z \) is called a normed vector space.

The rest of this section is devoted to the proof of the following theorem.

Theorem 5.22. For \( p \in [1, \infty) \), \( (\ell^p (\mu), \| \cdot \|_p) \) is a normed vector space.

Proof. The only difficulty is the proof of the triangle inequality which is the content of Minkowski’s Inequality proved in Theorem 5.28 below.

Proposition 5.23. Let \( f : [0, \infty) \to [0, \infty) \) be a continuous strictly increasing function such that \( f(0) = 0 \) (for simplicity) and \( \lim_{s \to \infty} f(s) = \infty \). Let \( g = f^{-1} \) and for \( s, t \geq 0 \) let

\[
F(s) = \int_0^s f(s') ds' \quad \text{and} \quad G(t) = \int_0^t g(t') dt'.
\]

Then for all \( s, t \geq 0 \),

\[
st \leq F(s) + G(t)
\]

and equality holds iff \( t = f(s) \).

Proof. Let

\[
A_s := \{ (\sigma, \tau) : 0 \leq \tau \leq f(\sigma) \text{ for } 0 \leq \sigma \leq s \} \quad \text{and} \quad B_t := \{ (\sigma, \tau) : 0 \leq \sigma \leq g(\tau) \text{ for } 0 \leq \tau \leq t \}
\]

then as one sees from Figure 5.2 \([0, s] \times [0, t] \subset A_s \cup B_t \). (In the figure: \( s = 3 \), \( t = 1 \), \( A_3 \) is the region under \( t = f(s) \) for \( 0 \leq s \leq 3 \) and \( B_1 \) is the region to the
left of the curve \( s = g(t) \) for \( 0 \leq t \leq 1 \). Hence if \( m \) denotes the area of a region in the plane, then

\[
st = m([0, s] \times [0, t]) \leq m(A_s) + m(B_t) = F(s) + G(t).
\]

As it stands, this proof is a bit on the intuitive side. However, it will become rigorous if one takes \( m \) to be “Lebesgue measure” on the plane which will be introduced later.

We can also give a calculus proof of this theorem under the additional assumption that \( f \) is \( C^1 \). (This restricted version of the theorem is all we need in this section.) To do this fix \( t \geq 0 \) and let

\[
h(s) = st - F(s) = \int_0^s (t - f(\sigma))d\sigma.
\]

If \( \sigma > g(t) = f^{-1}(t) \), then \( t - f(\sigma) < 0 \) and hence if \( s > g(t) \), we have

\[
h(s) = \int_0^s (t - f(\sigma))d\sigma = \int_0^{g(t)} (t - f(\sigma))d\sigma + \int_{g(t)}^s (t - f(\sigma))d\sigma
\]

\[
\leq \int_0^{g(t)} (t - f(\sigma))d\sigma = h(g(t)).
\]

Combining this with \( h(0) = 0 \) we see that \( h(s) \) takes its maximum at some point \( s \in (0, g(t)] \) and hence at a point where \( 0 = h'(s) = t - f(s) \). The only solution to this equation is \( s = g(t) \) and we have thus shown

\[
st - F(s) = h(s) \leq \int_0^{g(t)} (t - f(\sigma))d\sigma = h(g(t))
\]

with equality when \( s = g(t) \). To finish the proof we must show \( \int_0^{g(t)} (t - f(\sigma))d\sigma = G(t) \). This is verified by making the change of variables \( \sigma = g(\tau) \) and then integrating by parts as follows:

\[
\int_0^{g(t)} (t - f(\sigma))d\sigma = \int_0^t (t - f(g(\tau)))g'(\tau)d\tau = \int_0^t (t - \tau)g'(\tau)d\tau
\]

\[
= \int_0^t g(\tau)d\tau = G(t).
\]

**Lemma 5.25.** Let \( p \in (1, \infty) \) and \( q := \frac{p}{p-1} \in (1, \infty) \) be the conjugate exponent. Then

\[
st \leq \frac{sp}{p} + \frac{t^q}{q} \quad \text{for all } s, t \geq 0
\]

with equality if and only if \( t^q = s^p \). (See Example ?? below for a generalization of the inequality in Eq. [5.17].)

Proof. Let \( F(s) = \frac{sp}{p} \) for \( p > 1 \). Then \( f(s) = s^{p-1} = t \) and \( g(t) = t^{1-p} = t^{q-1} \), wherein we have used \( q - 1 = p/(p-1) - 1 = 1/(p-1) \). Therefore \( G(t) = t^q/q \) and hence by Proposition 5.23

\[
st \leq \frac{sp}{p} + \frac{t^q}{q}
\]

with equality if \( t = s^{p-1} \), i.e. \( t^q = s^{q(p-1)} = s^p \).

** For those who do not want to use Proposition 5.23 here is a direct calculus proof. Fix \( t > 0 \) and let

\[
h(s) := st - \frac{sp}{p}.
\]

Then \( h(0) = 0 \), \( \lim_{s \to \infty} h(s) = -\infty \) and \( h'(s) = t - s^{p-1} \) which equals zero iff \( s = t^{p-1} \). Since

\[
h\left(t\frac{1}{p-1}\right) = t\frac{1}{p-1} - \frac{tp}{p} = t\frac{1}{p-1} - \frac{tp}{p} = t^q\left(1 - \frac{1}{p}\right) = \frac{t^q}{q},
\]

it follows from the first derivative test that

\[
\max h = \max \left\{ h(0), h\left(t\frac{1}{p-1}\right) \right\} = \max \left\{ 0, \frac{t^q}{q} \right\} = \frac{t^q}{q}.
\]

**Definition 5.24.** The conjugate exponent \( q \in [1, \infty] \) to \( p \in [1, \infty] \) is \( q := \frac{p}{p-1} \) with the conventions that \( q = \infty \) if \( p = 1 \) and \( q = 1 \) if \( p = \infty \). Notice that \( q \) is characterized by any of the following identities:

\[
\frac{1}{p} + \frac{1}{q} = 1, \quad 1 + \frac{q}{p} = q, \quad p - \frac{p}{q} = 1 \quad \text{and} \quad q(p-1) = p.
\]

\[
(5.10)
\]
So we have shown

\[ st - \frac{sp}{p} \leq \frac{t^q}{q} \] with equality iff \( t = s^{p-1} \).

**Theorem 5.26 (Hölder’s inequality).** Let \( p, q \in [1, \infty] \) be conjugate exponents. For all \( f, g : X \to \mathbb{F} \),

\[ \|fg\|_1 \leq \|f\|_p \cdot \|g\|_q. \] (5.12)

If \( p \in (1, \infty) \) and \( f \) and \( g \) are not identically zero, then equality holds in Eq. (5.12) iff

\[ \left( \frac{|f|}{\|f\|_p} \right)^p = \left( \frac{|g|}{\|g\|_q} \right)^q. \] (5.13)

**Proof.** The proof of Eq. (5.12) for \( p \in \{1, \infty\} \) is easy and will be left to the reader. The cases where \( \|f\|_p = 0 \) or \( \|g\|_q = 0 \) or \( \infty \) are easily dealt with and are also left to the reader. So we will assume that \( p \in (1, \infty) \) and \( 0 < \|f\|_p, \|g\|_q < \infty \). Letting \( s = \|f(x)\|/\|f\|_p \) and \( t = |g|/\|g\|_q \) in Lemma 5.25 implies

\[ \frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \left[ \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \right] \frac{|g(x)|^q}{\|g\|_q^q}, \]

with equality iff

\[ \frac{|f(x)|^p}{\|f\|_p^p} = s^p = t^q = \frac{|g(x)|^q}{\|g\|_q^q}. \] (5.14)

Multiplying this equation by \( \mu(x) \) and then summing on \( x \) gives

\[ \frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1 \]

with equality iff Eq. (5.14) holds for all \( x \in X \), i.e. iff Eq. (5.13) holds. \( \blacksquare \)

**Definition 5.27.** For a complex number \( \lambda \in \mathbb{C} \), let

\[ \text{sgn}(\lambda) = \begin{cases} \frac{\lambda}{|\lambda|} & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0. \end{cases} \]

For \( \lambda, \mu \in \mathbb{C} \) we will write \( \text{sgn}(\lambda) \equiv \text{sgn}(\mu) \) if \( \text{sgn}(\lambda) = \text{sgn}(\mu) \) or \( \lambda \mu = 0 \).

**Theorem 5.28 (Minkowski’s Inequality).** If \( 1 \leq p \leq \infty \) and \( f, g \in \ell^p(\mu) \) then

\[ \|f + g\|_p \leq \|f\|_p + \|g\|_p. \] (5.15)

Moreover, assuming \( f \) and \( g \) are not identically zero, equality holds in Eq. (5.15) iff

\[ \text{sgn}(f) = \text{sgn}(g) \quad \text{when } p = 1 \] and

\[ f = cg \text{ for some } c > 0 \text{ when } p \in (1, \infty). \]

**Proof.** For \( p = 1 \),

\[ \|f + g\|_1 = \sum_x |f + g|^{\mu} \leq \sum_x (|f|^{\mu} + |g|^{\mu}) = \sum_x |f|^{\mu} + \sum_x |g|^{\mu} \]

with equality iff

\[ |f| + |g| = |f + g| \iff \text{sgn}(f) = \text{sgn}(g). \]

For \( p = \infty \),

\[ \|f + g\|_\infty = \sup_x |f + g| \leq \sup_x (|f| + |g|) \]

\[ \leq \sup_x |f| + \sup_x |g| = \|f\|_\infty + \|g\|_\infty. \]

Now assume that \( p \in (1, \infty) \). Since

\[ |f + g|^p \leq (2 \max(|f|, |g|))^p = 2^p \max(|f|^p, |g|^p) \leq 2^p (|f|^p + |g|^p) \]

it follows that

\[ \|f + g\|_p^p \leq 2^p (\|f\|_p^p + \|g\|_p^p) < \infty. \]

Eq. (5.15) is easily verified if \( \|f + g\|_p = 0 \), so we may assume \( \|f + g\|_p > 0 \). Multiplying the inequality,

\[ |f + g|^{p-1} \leq |f|^{p-1} + |g|^{p-1} \]

by \( \mu \), then summing on \( x \) and applying Hölder’s inequality on each term gives

\[ \sum_x |f + g|^p \mu \leq \sum_x |f|^{p-1} \mu + \sum_x |g|^{p-1} \mu \]

\[ \leq (\|f\|_p^p + \|g\|_p^p) \|f + g|^{p-1} \|_q. \] (5.17)

Since \( q(p-1) = p \), as in Eq. (5.10),

\[ \|f + g|^{p-1} \|_q = \sum_x (|f + g|^{p-1})^q \mu = \sum_x |f + g|^p \mu = \|f + g\|_p^p. \] (5.18)

Combining Eqs. (5.17) and (5.18) shows
\[ \| f + g \|_p^p \leq (\| f \|_p + \| g \|_p) \| f + g \|_p^{p/q} \] (5.19)

and solving this equation for \( \| f + g \|_p \) (making use of Eq. (5.10)) implies Eq. (5.15). Now suppose that \( f \) and \( g \) are not identically zero and \( p \in (1, \infty) \). Equality holds in Eq. (5.15) iff equality holds in Eq. (5.17) and Eq. (5.16). The latter happens iff

\[
\operatorname{sgn}(f) \doteq \operatorname{sgn}(g) \quad \text{and} \quad \left( \frac{\| f \|_p}{\| f + g \|_p} \right)^p = \left( \frac{|f + g|^p}{\| f + g \|_p} \right)^p = \left( \frac{|g|}{\| g \|_p} \right)^p.
\] (5.20)

wherein we have used

\[
\left( \frac{\| f + g \|_{p-1}}{\| f + g \|_p} \right)^q = \frac{|f + g|^p}{\| f + g \|_p}.
\]

Finally Eq. (5.20) is equivalent to \( |f| = c |g| \) with \( c = (\| f \|_p / \| g \|_p) > 0 \) and this equality along with \( \operatorname{sgn}(f) \doteq \operatorname{sgn}(g) \) implies \( f = cg \). □

5.6 Exercises

**Exercise 5.1.** Now suppose for each \( n \in \mathbb{N} := \{1, 2, \ldots\} \) that \( f_n : X \rightarrow \mathbb{R} \) is a function. Let

\[ D := \{ x \in X : \lim_{n \rightarrow \infty} f_n(x) = +\infty \} \]

show that

\[ D = \bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \{ x \in X : f_n(x) \geq M \}. \] (5.21)

**Exercise 5.2.** Let \( f_n : X \rightarrow \mathbb{R} \) be as in the last problem. Let

\[ C := \{ x \in X : \lim_{n \rightarrow \infty} f_n(x) \text{ exists in } \mathbb{R} \}. \]

Find an expression for \( C \) similar to the expression for \( D \) in (5.21). (Hint: use the Cauchy criteria for convergence.)

5.6.1 Limit Problems

**Exercise 5.3.** Show \( \lim \inf_{n \rightarrow \infty} (-a_n) = -\lim \sup_{n \rightarrow \infty} a_n \).

**Exercise 5.4.** Suppose that \( \lim \sup_{n \rightarrow \infty} a_n = M \in \mathbb{R} \), show that there is a subsequence \( \{ a_{n_k} \}_{k=1}^{\infty} \) of \( \{ a_n \}_{n=1}^{\infty} \) such that \( \lim_{k \rightarrow \infty} a_{n_k} = M \).

**Exercise 5.5.** Show that

\[ \lim \sup_{n \rightarrow \infty} (a_n + b_n) \leq \lim \sup_{n \rightarrow \infty} a_n + \lim \sup_{n \rightarrow \infty} b_n \] (5.22)

provided that the right side of Eq. (5.22) is well defined, i.e. no \( \infty - \infty \) or \( -\infty + \infty \) type expressions. (It is OK to have \( \infty + \infty = \infty \) or \( -\infty - \infty = -\infty \), etc.)

**Exercise 5.6.** Suppose that \( a_n \geq 0 \) and \( b_n \geq 0 \) for all \( n \in \mathbb{N} \). Show

\[ \lim \sup_{n \rightarrow \infty} (a_n b_n) \leq \lim \sup_{n \rightarrow \infty} a_n \cdot \lim \sup_{n \rightarrow \infty} b_n, \] (5.23)

provided the right hand side of (5.23) is not of the form \( 0 \cdot \infty \) or \( \infty \cdot 0 \).

**Exercise 5.7.** Prove Lemma 5.10.

**Exercise 5.8.** Prove Lemma 5.16.

5.6.2 Monotone and Dominated Convergence Theorem Problems

**Exercise 5.9.** Let \( M < \infty \), show there are polynomials \( p_n(t) \) and \( q_n(t) \) for \( n \in \mathbb{N} \) such that

\[ \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq M} \left| \sqrt{t} - q_n(t) \right| = 0 \] (5.24)

and

\[ \lim_{n \rightarrow \infty} \sup_{|t| \leq M} \left| |t| - p_n(t) \right| = 0 \] (5.25)

using the following outline.

1. Let \( f(x) = \sqrt{1-x} \) for \( |x| \leq 1 \) and use Taylor’s theorem with integral remainder (see Eq. ?? of Appendix ??), or analytic function theory if you know it, to show there are constants \( c_i \) with \( c_i > 0 \) for \( n \in \mathbb{N} \) such that
   \[ \sqrt{1-x} = 1 - \sum_{n=1}^{\infty} c_n x^n \text{ for all } |x| < 1. \] (5.26)

2. Let \( \tilde{q}_m(x) := 1 - \sum_{n=1}^{m} c_n x^n \). Use (5.26) to show \( \sum_{n=1}^{\infty} c_n = 1 \) and conclude from this that
   \[ \lim_{m \rightarrow \infty} \sup_{|x| \leq 1} \left| \sqrt{1-x} - \tilde{q}_m(x) \right| = 0. \] (5.27)

3. Conclude that \( q_n(t) := \sqrt{M} \tilde{q}_m(1/t - M) \) and \( p_n(t) := q_n(t^2) \) for \( n \in \mathbb{N} \) are polynomials verifying Eqs. (5.24) and (5.25) respectively.

\[ 1 \text{ in fact } c_m := \frac{(2m-1)!!}{2^m m!}, \text{ but this is not needed. } \]
**Notation 5.29** For \( u_0 \in \mathbb{R}^n \) and \( \delta > 0 \), let \( B_{u_0}(\delta) := \{ x \in \mathbb{R}^n : |x - u_0| < \delta \} \) be the ball in \( \mathbb{R}^n \) centered at \( u_0 \) with radius \( \delta \).

**Exercise 5.10.** Suppose \( U \subset \mathbb{R}^n \) is a set and \( u_0 \in U \) is a point such that \( U \cap (B_{u_0}(\delta) \setminus \{ u_0 \}) \neq \emptyset \) for all \( \delta > 0 \). Let \( G : U \setminus \{ u_0 \} \to \mathbb{C} \) be a function on \( U \setminus \{ u_0 \} \). Show that \( \lim_{u \to u_0} G(u) \) exists and is equal to \( \lambda \in \mathbb{C}^2 \) iff for all sequences \( \{ u_n \}_{n=1}^{\infty} \subset U \setminus \{ u_0 \} \) which converge to \( u_0 \) (i.e. \( \lim_{n \to \infty} u_n = u_0 \)) we have \( \lim_{n \to \infty} G(u_n) = \lambda \).

**Exercise 5.11.** Suppose that \( Y \) is a set, \( U \subset \mathbb{R}^n \) is a set, and \( f : U \times Y \to \mathbb{C} \) is a function satisfying:

1. For each \( y \in Y \), the function \( u \to f(u, y) \) is continuous on \( U \).
2. There is a summable function \( g : Y \to [0, \infty) \) such that
   \[ |f(u, y)| \leq g(y) \text{ for all } y \in Y \text{ and } u \in U. \]

Show that
   \[ F(u) := \sum_{y \in Y} f(u, y) \]  
   (5.28)

is a continuous function for \( u \in U \).

**Exercise 5.12.** Suppose that \( Y \) is a set, \( J = (a, b) \subset \mathbb{R} \) is an interval, and \( f : J \times Y \to \mathbb{C} \) is a function satisfying:

1. For each \( y \in Y \), the function \( u \to f(u, y) \) is differentiable on \( J \).
2. There is a summable function \( g : Y \to [0, \infty) \) such that
   \[ \left| \frac{\partial}{\partial u} f(u, y) \right| \leq g(y) \text{ for all } y \in Y \text{ and } u \in J. \]

3. There is a \( u_0 \in J \) such that \( \sum_{y \in Y} |f(u_0, y)| < \infty \).

Show:

a) for all \( u \in J \) that \( \sum_{y \in Y} |f(u, y)| < \infty \).

b) Let \( F(u) := \sum_{y \in Y} f(u, y) \), show \( F \) is differentiable on \( J \) and that
   \[ \hat{F}(u) = \sum_{y \in Y} \frac{\partial}{\partial u} f(u, y). \]

\[ \text{More explicitly, } \lim_{u \to u_0} G(u) = \lambda \text{ means for every } \epsilon > 0 \text{ there exists a } \delta > 0 \text{ such that} \]

\[ |G(u) - \lambda| < \epsilon \text{ whenever } u \in U \cap (B_{u_0}(\delta) \setminus \{ u_0 \}). \]

\[ \text{To say } g := f(\cdot, y) \text{ is continuous on } U \text{ means that } g : U \to \mathbb{C} \text{ is continuous relative to the metric on } \mathbb{R}^n \text{ restricted to } U. \]

(Hint: Use the mean value theorem.)

**Exercise 5.13 (Differentiation of Power Series).** Suppose \( R > 0 \) and \( \{ a_n \}_{n=0}^{\infty} \) is a sequence of complex numbers such that \( \sum_{n=0}^{\infty} |a_n| r^n < \infty \) for all \( r \in (0, R) \). Show, using Exercise 5.12, \( f(x) := \sum_{n=0}^{\infty} a_n x^n \) is continuously differentiable for \( x \in (-R, R) \) and

\[ f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}. \]

**Exercise 5.14.** Show the functions

\[ e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad (5.29) \]

\[ \sin x := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \]

\[ \cos x := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \]

(5.30)

are infinitely differentiable and they satisfy

\[ \frac{d}{dx} e^x = e^x \text{ with } e^0 = 1 \]

\[ \frac{d}{dx} \sin x = \cos x \text{ with } \sin(0) = 0 \]

\[ \frac{d}{dx} \cos x = -\sin x \text{ with } \cos(0) = 1. \]

**Exercise 5.15.** Continue the notation of Exercise 5.14

1. Use the product and the chain rule to show,

\[ \frac{d}{dx} \left[ e^{-x}e^{(x+y)} \right] = 0 \]

and conclude from this, that \( e^{-x}e^{(x+y)} = e^y \) for all \( x, y \in \mathbb{R} \). In particular taking \( y = 0 \) this implies that \( e^{-x} = 1/e^x \) and hence that \( e^{(x+y)} = e^x e^y \). Use this result to show \( e^x \uparrow \infty \) as \( x \uparrow \infty \) and \( e^x \downarrow 0 \) as \( x \downarrow -\infty \).

**Remark:** since \( e^x \geq \sum_{n=0}^{N} \frac{x^n}{n!} \) when \( x \geq 0 \), it follows that \( \lim_{x \to \infty} \frac{e^n}{x^n} = 0 \) for any \( n \in \mathbb{N} \), i.e. \( e^x \) grows at a rate faster than any polynomial in \( x \) as \( x \to \infty \).

2. Use the product rule to show

\[ \frac{d}{dx} \left( \cos^2 x + \sin^2 x \right) = 0 \]

and use this to conclude that \( \cos^2 x + \sin^2 x = 1 \) for all \( x \in \mathbb{R} \).
Exercise 5.16. Let \( \{a_n\}_{n=-\infty}^{\infty} \) be a summable sequence of complex numbers, i.e. \( \sum_{n=-\infty}^{\infty} |a_n| < \infty \). For \( t \geq 0 \) and \( x \in \mathbb{R} \), define
\[
F(t,x) = \sum_{n=-\infty}^{\infty} a_ne^{-tn^2}e^{inx},
\]
where as usual \( e^{ix} = \cos(x) + i\sin(x) \), this is motivated by replacing \( x \) in Eq. (5.29) by \( ix \) and comparing the result to Eqs. (5.30) and (5.31).

1. \( F(t,x) \) is continuous for \( (t,x) \in [0,\infty) \times \mathbb{R} \). \textbf{Hint:} Let \( Y = \mathbb{Z} \) and \( u = (t,x) \) and use Exercise 5.11.

2. \( \partial F(t,x)/\partial t, \partial F(t,x)/\partial x \) and \( \partial^2 F(t,x)/\partial x^2 \) exist for \( t > 0 \) and \( x \in \mathbb{R} \). \textbf{Hint:} Let \( Y = \mathbb{Z} \) and \( u = t \) for computing \( \partial F(t,x)/\partial t \) and \( u = x \) for computing \( \partial F(t,x)/\partial x \) and \( \partial^2 F(t,x)/\partial x^2 \) via Exercise 5.12. In computing the \( t \) derivative, you should let \( \varepsilon > 0 \) and apply Exercise 5.12 with \( t = u > \varepsilon \) and then afterwards let \( \varepsilon \downarrow 0 \).

3. \( F \) satisfies the heat equation, namely
\[
\partial F(t,x)/\partial t = \partial^2 F(t,x)/\partial x^2 \text{ for } t > 0 \text{ and } x \in \mathbb{R}.
\]

5.6.3 \( \ell^p \) Exercises

Exercise 5.17. Generalize Proposition 5.23 as follows. Let \( a \in [-\infty,0] \) and \( f : \mathbb{R} \cap [a,\infty) \rightarrow [0,\infty) \) be a continuous strictly increasing function such that
\[
\lim_{s \rightarrow -\infty} f(s) = \infty, \quad f(a) = 0 \quad \text{if } a > -\infty \text{ or } \lim_{s \rightarrow -\infty} f(s) = 0 \quad \text{if } a = -\infty.
\]
Also let \( g = f^{-1} \), \( b = f(0) \geq 0 \),
\[
F(s) = \int_{0}^{s} f(s')ds' \quad \text{and} \quad G(t) = \int_{0}^{t} g(t')dt'.
\]

Then for all \( s,t \geq 0 \),
\[
st \leq F(s) + G(t \lor b) \leq F(s) + G(t)
\]
and equality holds iff \( t = f(s) \). In particular, taking \( f(s) = e^s \), prove Young’s inequality stating
\[
st \leq e^s + (t \lor 1) \ln (t \lor 1) - (t \lor 1) \leq e^s + t \ln t - t,
\]
where \( s \lor t := \min \{s,t\} \). \textbf{Hint:} Refer to Figures 5.3 and 5.4.

Exercise 5.18. Using differential calculus, prove the following inequalities

1. For \( y > 0 \), let \( g(x) := xy - e^x \) for \( x \in \mathbb{R} \). Use calculus to compute the maximum of \( g(x) \) and use this prove Young’s inequality;
\[
xy \leq e^x + y \ln y - y \text{ for } x \in \mathbb{R} \text{ and } y > 0.
\]

2. For \( p > 1 \) and \( y \geq 0 \), let \( g(x) := xy - x^p/p \) for \( x \geq 0 \). Again use calculus to compute the maximum of \( g(x) \) and show that your result gives the following inequality;
\[
xy \leq \frac{x^p}{p} + \frac{y^q}{q} \text{ for all } x,y \geq 0.
\]
where \( q = \frac{p}{p-1} \), i.e. \( \frac{1}{q} = 1 - \frac{1}{p} \).

3. Suppose now that \( u : [0,\infty) \rightarrow [0,\infty) \) is a \( C^1 \) - function such that: \( u(0) = 0 \), \( \lim_{x \rightarrow \infty} \frac{u(x)}{x} = \infty \), and \( u'(x) > 0 \) for all \( x > 0 \). Show
$$xy \leq u(x) + v(y) \text{ for all } x, y \geq 0,$$

where $v(y) = y(u')^{-1}(y) - u\left((u')^{-1}(y)\right)$. **Hint:** consider the function, $g(x) := xy - u(x)$. 
Set Operations and Countability

Let \( \mathbb{N} \) denote the positive integers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) be the non-negative integers and \( \mathbb{Z} = \mathbb{N}_0 \cup (-\mathbb{N}) \) – the positive and negative integers including 0, \( \mathbb{Q} \) the rational numbers, \( \mathbb{R} \) the real numbers (see Chapter 3 below), and \( \mathbb{C} \) the complex numbers. We will also use \( \mathbb{F} \) to stand for either of the fields \( \mathbb{R} \) or \( \mathbb{C} \).

**Notation 6.1** Given two sets \( X \) and \( Y \), let \( Y^X \) denote the collection of all functions \( f : X \to Y \). If \( X = \mathbb{N} \), we will say that \( f \in Y^\mathbb{N} \) is a sequence with values in \( Y \) and often write \( f_n \) for \( f(n) \) and express \( f \) as \( \{f_n\}_{n=1}^\infty \). If \( X = \{1,2,\ldots,N\} \), we will write \( Y^N \) in place of \( Y^{\{1,2,\ldots,N\}} \) and denote \( f \in Y^N \) by \( f = (f_1,f_2,\ldots,f_N) \) where \( f_n = f(n) \).

**Notation 6.2** More generally if \( \{X_\alpha : \alpha \in A\} \) is a collection of non-empty sets, let \( X_A = \prod_{\alpha \in A} X_\alpha \) and \( \pi_\alpha : X_A \to X_\alpha \) be the canonical projection map defined by \( \pi_\alpha(x) = x_\alpha \). If \( X_\alpha = X \) for some fixed space \( X \), then we will write \( \prod_{\alpha \in A} X_\alpha \) as \( X^A \) rather than \( X_A \).

Recall that an element \( x \in X_A \) is a “choice function,” i.e. an assignment \( x_\alpha := x(\alpha) \in X_\alpha \) for each \( \alpha \in A \). The axiom of choice (see Appendix ??) states that \( X_A \neq \emptyset \) provided that \( X_\alpha \neq \emptyset \) for each \( \alpha \in A \).

**Notation 6.3** Given a set \( X \), let \( 2^X \) denote the power set of \( X \) – the collection of all subsets of \( X \) including the empty set.

The reason for writing the power set of \( X \) as \( 2^X \) is that if we think of 2 meaning \( \{0,1\} \), then an element of \( a \in 2^X = \{0,1\}^X \) is completely determined by the set

\[ A := \{x \in X : a(x) = 1\} \subset X. \]

In this way elements in \( \{0,1\}^X \) are in one to one correspondence with subsets of \( X \).

For \( A \in 2^X \) let

\[ A^c := X \setminus A = \{x \in X : x \notin A\} \]

and more generally if \( A,B \subset X \) let

\[ B \setminus A := \{x \in B : x \notin A\} = A \cap B^c. \]

We also define the symmetric difference of \( A \) and \( B \) by

\[ A \triangle B := (B \setminus A) \cup (A \setminus B). \]

As usual if \( \{A_\alpha\}_{\alpha \in I} \) is an indexed collection of subsets of \( X \) we define the union and the intersection of this collection by

\[ \cup_{\alpha \in I} A_\alpha := \{x \in X : \exists \alpha \in I \text{ } x \in A_\alpha\} \quad \text{and} \quad \cap_{\alpha \in I} A_\alpha := \{x \in X : x \in A_\alpha \forall \alpha \in I\}. \]

**Notation 6.4** We will also write \( \prod_{\alpha \in I} A_\alpha \) for \( \cup_{\alpha \in I} A_\alpha \) in the case that \( \{A_\alpha\}_{\alpha \in I} \) are pairwise disjoint, i.e. \( A_\alpha \cap A_\beta = \emptyset \) if \( \alpha \neq \beta \).

Notice that \( \cup \) is closely related to \( \exists \) and \( \cap \) is closely related to \( \forall \). For example let \( \{A_n\}_{n=1}^\infty \) be a sequence of subsets from \( X \) and define

\[ A_n \text{ i.o.} := \{x \in X : \#\{n : x \in A_n\} = \infty\} \text{ and } \]
\[ A_n \text{ a.a.} := \{x \in X : x \in A_n \text{ for all } n \text{ sufficiently large}\}. \]

(One should read \( A_n \text{ i.o.} \) as \( A_n \) infinitely often and \( A_n \text{ a.a.} \) as \( A_n \) almost always.) Then \( x \in \{A_n \text{ i.o.}\} \) iff

\[ \forall N \in \mathbb{N} \exists n \geq N \exists x \in A_n \]

and this may be expressed as

\[ \{A_n \text{ i.o.}\} = \cap_{n=1}^\infty \cup_{n \geq N} A_n. \]

Similarly, \( x \in \{A_n \text{ a.a.}\} \) iff

\[ \exists N \in \mathbb{N} \forall n \geq N, \ x \in A_n \]

which may be written as

\[ \{A_n \text{ a.a.}\} = \cup_{n=1}^\infty \cap_{n \geq N} A_n. \]

**Definition 6.5.** A set \( X \) is said to be countable if is empty or there is an injective function \( f : X \to \mathbb{N} \), otherwise \( X \) is said to be uncountable.

**Lemma 6.6** (Basic Properties of Countable Sets).
1. If $A \subset X$ is a subset of a countable set $X$ then $A$ is countable.
2. Any infinite subset $A \subset \mathbb{N}$ is in one to one correspondence with $\mathbb{N}$.
3. A non-empty set $X$ is countable iff there exists a surjective map, $g : \mathbb{N} \to X$.
4. If $X$ and $Y$ are countable then $X \times Y$ is countable.
5. Suppose for each $m \in \mathbb{N}$ that $A_m$ is a countable subset of a set $X$, then $A = \cup_{m=1}^\infty A_m$ is countable. In short, the countable union of countable sets is still countable.
6. If $X$ is an infinite set and $Y$ is a set with at least two elements, then $Y^X$ is uncountable. In particular $2^X$ is uncountable for any infinite set $X$. 

**Proof.** 1. If $f : X \to \mathbb{N}$ is an injective map then so is the restriction, $f|_A$, of $f$ to the subset $A$.
2. Let $f(1) = \min A$ and define $f$ inductively by
$$f(n + 1) = \min (A \setminus \{ f(1), \ldots, f(n) \}).$$

Since $A$ is infinite the process continues indefinitely. The function $f : \mathbb{N} \to A$ defined this way is a bijection.
3. If $g : \mathbb{N} \to X$ is a surjective map, let
$$f(x) = \min g^{-1}(\{ x \}) = \min \{ n \in \mathbb{N} : f(n) = x \}.$$ 

Then $f : X \to \mathbb{N}$ is injective which combined with item 2. (taking $A = f(X)$) shows $X$ is countable. Conversely if $f : X \to \mathbb{N}$ is injective let $x_0 \in X$ be a fixed point and define $g : \mathbb{N} \to X$ by $g(n) = f^{-1}(n)$ for $n \in f(X)$ and $g(n) = x_0$ otherwise.
4. Let us first construct a bijection, $h$, from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. To do this put the elements of $\mathbb{N} \times \mathbb{N}$ into an array of the form
$$(1,1) (1,2) (1,3) \ldots \newline (2,1) (2,2) (2,3) \ldots \newline (3,1) (3,2) (3,3) \ldots \newline \vdots \quad \vdots \quad \vdots \ldots$$

and then “count” these elements by counting the sets $\{(i,j) : i+j=k\}$ one at a time. For example let $h(1) = (1,1)$, $h(2) = (2,1)$, $h(3) = (1,2)$, $h(4) = (3,1)$, $h(5) = (2,2)$, $h(6) = (1,3)$ and so on. If $f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$ are surjective functions, then the function $(f \times g) \circ h : \mathbb{N} \to X \times Y$ is surjective where $(f \times g)(m,n) := (f(m),g(n))$ for all $(m,n) \in \mathbb{N} \times \mathbb{N}$.
5. If $A = \emptyset$ then $A$ is countable by definition so we may assume $A \neq \emptyset$. With out loss of generality we may assume $A_1 \neq \emptyset$ and by replacing $A_m$ by $A_1$ if necessary we may also assume $A_m \neq \emptyset$ for all $m$. For each $m \in \mathbb{N}$ let $a_m : \mathbb{N} \to A_m$ be a surjective function and then define $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{m=1}^\infty A_m$ by
$$f(m,n) := a_m(n).$$

The function $f$ is surjective and hence so is the composition, $f \circ h : \mathbb{N} \to \bigcup_{m=1}^\infty A_m$, where $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is the bijection defined above.
6. Let us begin by showing $2^\mathbb{N} = \{ 0,1 \}^\mathbb{N}$ is uncountable. For sake of contradiction suppose $f : \mathbb{N} \to \{ 0,1 \}^\mathbb{N}$ is a surjection and write $f(n)$ as $(f_1(n), f_2(n), f_3(n), \ldots)$. Now define $a \in \{ 0,1 \}^\mathbb{N}$ by $a_n := 1 - f_n(n)$. By construction $f_n(n) \neq a_n$ for all $n$ and so $a \notin f(\mathbb{N})$. This contradicts the assumption that $f$ is surjective and shows $2^\mathbb{N}$ is uncountable. For the general case, since $Y_0^X \subset Y^X$ for any subset $Y_0 \subset Y$, if $Y_0^X$ is uncountable then so is $Y^X$. In this way we may assume $Y_0$ is a two point set which may as well be $Y_0 = \{ 0,1 \}$. Moreover, since $X$ is an infinite set we may find an injective map $i : \mathbb{N} \to X$ and use this to set up an injection, $i : 2^\mathbb{N} \to 2^X$ by setting $i(A) := \{ x_n : n \in \mathbb{N} \} \subset X$ for all $A \subset \mathbb{N}$. If $2^X$ were countable we could find a surjective map $f : 2^X \to \mathbb{N}$ in which case $f \circ i : 2^\mathbb{N} \to \mathbb{N}$ would be surjective as well. However this is impossible since we have already seen that $2^\mathbb{N}$ is uncountable.

**Corollary 6.7.** The set $(0,1) := \{ a \in \mathbb{R} : 0 < a < 1 \}$ is uncountable while $\mathbb{Q} \cap (0,1)$ is countable.

**Proof.** From Section 3.2 the set $\{ 0,1,2, \ldots \}^\mathbb{N}$ can be mapped injectively into $(0,1)$ and therefore it follows from Lemma 6.6 that $(0,1)$ is uncountable. For each $m \in \mathbb{N}$, let $A_m := \{ n \in \mathbb{N} \mid n < m \}$. Since $\mathbb{Q} \cap (0,1) = \cup_{m=1}^\infty A_m$ and $\#(A_m) < \infty$ for all $m$, another application of Lemma 6.6 shows $\mathbb{Q} \cap (0,1)$ is countable.

We end this section with some notation which will be used frequently in the sequel.

**Notation 6.8** If $f : X \to Y$ is a function and $\mathcal{E} \subset 2^Y$ let
$$f^{-1}(\mathcal{E}) := \{ f^{-1}(E) \mid E \in \mathcal{E} \}.$$ 

If $\mathcal{G} \subset 2^X$, let
$$f_* \mathcal{G} := \{ A \in 2^X \mid f^{-1}(A) \in \mathcal{G} \}.$$ 

**Definition 6.9.** Let $\mathcal{E} \subset 2^X$ be a collection of sets, $A \subset X$, $i_A : A \to X$ be the inclusion map ($i_A(x) = x$ for all $x \in A$) and
$$\mathcal{E}_A = i_A^{-1}(\mathcal{E}) = \{ A \cap E : E \in \mathcal{E} \}.$$ 

### 6.1 Exercises

Let $f : X \to Y$ be a function and $\{ A_i \}_{i \in I}$ be an indexed family of subsets of $Y$, verify the following assertions.
Exercise 6.1. $(\cap_{i \in I} A_i)^c = \cup_{i \in I} A_i^c$.

Exercise 6.2. Suppose that $B \subset Y$, show that $B \setminus (\cup_{i \in I} A_i) = \cap_{i \in I} (B \setminus A_i)$.

Exercise 6.3. $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$.

Exercise 6.4. $f^{-1}(\cap_{i \in I} A_i) = \cap_{i \in I} f^{-1}(A_i)$.

Exercise 6.5. Find a counterexample which shows that $f(C \cap D) = f(C) \cap f(D)$ need not hold.
Appendix: Constructing the Real Numbers

Let $\mathbb{C}$ denote the collection of Cauchy sequences $a = \{a_n\}_{n=1}^\infty \subseteq \mathbb{Q}$ and say $a, b \in \mathbb{C}$ are equivalent (write $a \sim b$) iff $\lim_{n \to \infty} |a_n - b_n| = 0$. (The reader should check that “$\sim$” is an equivalence relation.)

**Definition A.1.** A **real number** is an equivalence class, $\bar{a} := \{b \in \mathbb{C} : b \sim a\}$ associated to some element $a \in \mathbb{C}$. The collection of real numbers will be denoted by $\mathbb{R}$. For $q \in \mathbb{Q}$, let $i(q) = \bar{a}$ where $a$ is the constant sequence $a_n = q$ for all $n \in \mathbb{N}$. We will simply write $0$ for $i(0)$ and $1$ for $i(1)$.

**Exercise A.1.** Given $\bar{a}, \bar{b} \in \mathbb{R}$ show that the definitions

$$-\bar{a} = (-a), \bar{a} + \bar{b} := (a+b)$$

are well defined. Here $-a$, $a + b$ and $a \cdot b$ denote the sequences $\{-a_n\}_{n=1}^\infty$, $\{a_n + b_n\}_{n=1}^\infty$ and $\{a_n \cdot b_n\}_{n=1}^\infty$ respectively. Further verify that with these operations, $\mathbb{R}$ becomes a field and the map $i : \mathbb{Q} \rightarrow \mathbb{R}$ is injective homomorphism of fields. **Hint:** if $\bar{a} \neq 0$ show that $\bar{a}$ may be represented by a sequence $a \in \mathbb{C}$ with $|a_n| \geq \frac{1}{n}$ for all $n$ and some $N \in \mathbb{N}$. For this representative show the sequence $a^{-1} := \{a_n^{-1}\}_{n=1}^\infty \in \mathbb{C}$. The multiplicative inverse to $\bar{a}$ may now be constructed as: $\frac{1}{\bar{a}} = \bar{a}^{-1} := \{a_n^{-1}\}_{n=1}^\infty$.

**Definition A.2.** Let $\bar{a}, \bar{b} \in \mathbb{R}$. Then

1. $\bar{a} > 0$ if there exists an $N \in \mathbb{N}$ such that $a_n > \frac{1}{N}$ for a.a. $n$.
2. $\bar{a} \geq 0$ iff either $\bar{a} > 0$ or $\bar{a} = 0$. Equivalently (as the reader should verify), $\bar{a} \geq 0$ iff for all $N \in \mathbb{N}$, $|a_n| \geq \frac{1}{N}$ for a.a. $n$.
3. Write $\bar{a} > \bar{b}$ or $\bar{b} < \bar{a}$ if $\bar{a} - \bar{b} > 0$.
4. Write $\bar{a} \geq \bar{b}$ or $\bar{b} \leq \bar{a}$ if $\bar{a} - \bar{b} \geq 0$.

**Exercise A.2.** Show “$\geq$” make $\mathbb{R}$ into a linearly ordered field and the map $i : \mathbb{Q} \rightarrow \mathbb{R}$ preserves order. Namely if $\bar{a}, \bar{b} \in \mathbb{R}$ then

1. exactly one of the following relations hold: $\bar{a} < \bar{b}$ or $\bar{a} > \bar{b}$ or $\bar{a} = \bar{b}$.
2. If $\bar{a} \geq 0$ and $\bar{b} \geq 0$ then $\bar{a} + \bar{b} \geq 0$ and $\bar{a} \cdot \bar{b} \geq 0$.
3. If $q, r \in \mathbb{Q}$ then $q \leq r$ iff $i(q) \leq i(r)$.

The **absolute value** of a real number $\bar{a}$ is defined analogously to that of a rational number by

$$|\bar{a}| = \begin{cases} \bar{a} & \text{if } \bar{a} \geq 0 \\ -\bar{a} & \text{if } \bar{a} < 0 \end{cases}$$

Observe this definition is consistent with our previous definition of the absolute value on $\mathbb{Q}$, namely $i(|q|) = |i(q)|$. Also notice that $\bar{a} = 0$ (i.e. $a \sim 0$ where 0 denotes the constant sequence of all zeros) iff for all $N \in \mathbb{N}$, $|a_n| \leq \frac{1}{N}$ for a.a. $n$. This is equivalent to saying $|\bar{a}| \leq i\left(\frac{1}{N}\right)$ for all $N \in \mathbb{N}$ iff $\bar{a} = 0$.

**Definition A.3.** A sequence $\{\bar{a}_n\}_{n=1}^\infty \subseteq \mathbb{R}$ converges to $\bar{a} \in \mathbb{R}$ if $|\bar{a} - \bar{a}_n| \rightarrow 0$ as $n \rightarrow \infty$, i.e. if for all $N \in \mathbb{N}$, $|\bar{a} - \bar{a}_n| \leq i\left(\frac{1}{N}\right)$ for a.a. $n$. As before (for rational numbers) if $\{\bar{a}_n\}_{n=1}^\infty$ converges to $\bar{a}$ we will write $\bar{a}_n \rightarrow \bar{a}$ as $n \rightarrow \infty$ or $\bar{a} = \lim_{n \rightarrow \infty} \bar{a}_n$.

**Exercise A.3.** Given $\bar{a}, \bar{b} \in \mathbb{R}$ show

$$|\bar{a}\bar{b}| = |\bar{a}| \cdot |\bar{b}| \quad \text{and} \quad |\bar{a} + \bar{b}| \leq |\bar{a}| + |\bar{b}|.$$ 

The latter inequality being referred to as the **triangle inequality**.

By exercise A.3

$$|\bar{a}| = |\bar{a} - \bar{b} + \bar{b}| \leq |\bar{a} - \bar{b}| + |\bar{b}|$$

and hence

$$|\bar{a}| - |\bar{b}| \leq |\bar{a} - \bar{b}|$$

and by reversing the roles of $\bar{a}$ and $\bar{b}$ we also have

$$- (|\bar{b}| - |\bar{a}|) = |\bar{b}| - |\bar{a}| \leq |\bar{b} - \bar{a}| = |\bar{a} - \bar{b}|.$$ 

Therefore,

$$|\bar{a} - \bar{b}| \leq |\bar{a} - \bar{a}|$$

and consequently if $\{\bar{a}_n\}_{n=1}^\infty \subseteq \mathbb{R}$ converges to $\bar{a} \in \mathbb{R}$ then

$$|\bar{a}_n| - |\bar{a}| \leq |\bar{a}_n - \bar{a}| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$
Remark A.4. The field \( i(\mathbb{Q}) \) is dense in \( \mathbb{R} \) in the sense that if \( \bar{a} \in \mathbb{R} \) there exists \( \{q_n\}_{n=1}^{\infty} \subset \mathbb{Q} \) such that \( i(q_n) \to \bar{a} \) as \( n \to \infty \). Indeed, simply let \( q_n = a_n \) where \( a \) represents \( \bar{a} \). Since \( a \) is a Cauchy sequence, to any \( N \in \mathbb{N} \) there exists \( M \in \mathbb{N} \) such that

\[
-\frac{1}{N} \leq a_m - a_n \leq \frac{1}{N} \text{ for all } m, n \geq M
\]

and therefore

\[
-i \left( \frac{1}{N} \right) \leq i(a_m) - \bar{a} \leq i \left( \frac{1}{N} \right) \text{ for all } m \geq M.
\]

This shows

\[
|i(q_m) - \bar{a}| = |i(a_m) - \bar{a}| \leq i \left( \frac{1}{N} \right) \text{ for all } m \geq M
\]

and since \( N \) is arbitrary it follows that \( i(q_m) \to \bar{a} \) as \( m \to \infty \).

Definition A.5. A sequence \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} \) is Cauchy if \( |a_n - \bar{a}_m| \to 0 \) as \( m, n \to \infty \). More precisely we require for each \( N \in \mathbb{N} \) that \( |a_m - a_n| \leq i \left( \frac{1}{N} \right) \) for a.a. \( m, n \).

Exercise A.4. The analogues of the results in Exercises 1.3 and 1.4 hold with \( \mathbb{Q} \) replaced by \( \mathbb{R} \). (We now say a subset \( A \subset \mathbb{R} \) is bounded if there exists \( M \in \mathbb{N} \) such that \( |x| \leq i(M) \) for all \( x \in A \).)

For the purposes of real analysis the most important property of \( \mathbb{R} \) is that it is “complete.”

Theorem A.6. The ordered field \( \mathbb{R} \) is complete, i.e. all Cauchy sequences in \( \mathbb{R} \) are convergent.

Proof. Suppose that \( \{\bar{a}(m)\}_{m=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{R} \). By Remark A.4 we may choose \( q_m \in \mathbb{Q} \) such that

\[
|\bar{a}(m) - i(q_m)| \leq i \left( m^{-1} \right) \text{ for all } m \in \mathbb{N}.
\]

Given \( N \in \mathbb{N} \), choose \( M \in \mathbb{N} \) such that \( |\bar{a}(m) - \bar{a}(n)| \leq i \left( N^{-1} \right) \) for all \( m, n \geq M \). Then

\[
|i(q_m) - i(q_n)| \leq |i(q_m) - \bar{a}(m)| + |\bar{a}(m) - \bar{a}(n)| + |\bar{a}(n) - i(q_n)|
\]

\[
\leq i \left( m^{-1} \right) + i \left( n^{-1} \right) + i \left( N^{-1} \right)
\]

and therefore

\[
|q_m - q_n| \leq m^{-1} + n^{-1} + N^{-1} \text{ for all } m, n \geq M.
\]

It now follows that \( q = \{q_m\}_{m=1}^{\infty} \subset C \) and therefore \( q \) represents a point \( \bar{q} \in \mathbb{R} \). Using Remark A.4 and the triangle inequality,

\[
|\bar{a}(m) - \bar{q}| \leq |\bar{a}(m) - i(q_m)| + |i(q_m) - \bar{q}|
\]

\[
\leq i \left( m^{-1} \right) + i \left( q_m - \bar{q} \right) \to 0 \text{ as } m \to \infty
\]

and therefore \( \lim_{m \to \infty} \bar{a}(m) = \bar{q} \).

Definition A.7. A number \( M \in \mathbb{R} \) is an upper bound for a set \( A \subset \mathbb{R} \) if \( \lambda \leq M \) for all \( \lambda \in A \) and a number \( m \in \mathbb{R} \) is an lower bound for a set \( A \subset \mathbb{R} \) if \( \lambda \geq m \) for all \( \lambda \in A \). Upper and lower bounds need not exist. If \( A \) has an upper (lower) bound, \( A \) is said to be bounded from above (below).

Theorem A.8. To each non-empty set \( A \subset \mathbb{R} \) which is bounded from above (below) there is a unique least upper bound denoted by \( \sup A \in \mathbb{R} \) (respectively greatest lower bound denoted by \( \inf A \in \mathbb{R} \)).

Proof. Suppose \( A \) is bounded from above and for each \( n \in \mathbb{N} \), let \( m_n \in \mathbb{Z} \) be the smallest integer such that \( i \left( \frac{m_n}{2^n} \right) \) is an upper bound for \( A \). The sequence \( q_n := \frac{m_n}{2^n} \) is Cauchy because \( q_m \in [q_n - 2^{-n}, q_n] \cap \mathbb{Q} \) for all \( m \geq n \), i.e.

\[
|q_m - q_n| \leq 2^{-\min(m,n)} \to 0 \text{ as } m, n \to \infty.
\]

Passing to the limit, \( n \to \infty \), in the inequality \( i(q_n) \geq \lambda \), which is valid for all \( \lambda \in A \) implies

\[
\bar{q} = \lim_{n \to \infty} i(q_n) \geq \lambda \text{ for all } \lambda \in A.
\]

Thus \( \bar{q} \) is an upper bound for \( A \). If there were another upper bound \( M \in \mathbb{R} \) for \( A \) such that \( M < \bar{q} \), it would follow that \( M \leq i(q_n) < \bar{q} \) for some \( n \). But this is a contradiction because \( \{q_n\}_{n=1}^{\infty} \) is a decreasing sequence, \( i(q_n) \geq i(q_m) \) for all \( m \geq n \) and therefore \( i(q_n) \geq \bar{q} \) for all \( n \). Therefore \( \bar{q} \) is the unique least upper bound for \( A \). The existence of lower bounds is proved analogously.

Proposition A.9. If \( \{a_n\}_{n=1}^{\infty} \subset \mathbb{R} \) is an increasing (decreasing) sequence which is bounded from above (below), then \( \{a_n\}_{n=1}^{\infty} \) is convergent and

\[
\lim_{n \to \infty} a_n = \sup \{a_n : n \in \mathbb{N}\} \left( \lim_{n \to \infty} a_n = \inf \{a_n : n \in \mathbb{N}\} \right).
\]

If \( A \subset \mathbb{R} \) is a set bounded from above then there exists \( \{\lambda_n\} \subset A \) such that \( \lambda_n \uparrow M := \sup A \), as \( n \to \infty \), i.e. \( \lambda_n \) is increasing and \( \lim_{n \to \infty} \lambda_n = M \).

Proof. Let \( M := \sup \{a_n : n \in \mathbb{N}\} \), then for each \( N \in \mathbb{N} \) there must exist \( m \in \mathbb{N} \) such that \( M - i(N^{-1}) < a_m \leq M \). Since \( a_n \) is increasing, it follows that

\[
M - i(N^{-1}) < a_n \leq M \text{ for all } n \geq m.
\]
From this we conclude that \( \lim a_n \) exists and \( \lim a_n = M \). If \( M = \sup A \), for each \( n \in \mathbb{N} \) we may choose \( \lambda_n \in A \) such that
\[
M - i \left( n^{-1} \right) < \lambda_n \leq M.
\] (A.1)

By replacing \( \lambda_n \) by \( \max \{ \lambda_1, \ldots, \lambda_n \} \) if necessary we may assume that \( \lambda_n \) is increasing in \( n \). It now follows easily from Eq. (A.1) that \( \lim_{n \to \infty} \lambda_n = M \). ■

### A.1 The Decimal Representation of a Real Number

Let \( \alpha \in \mathbb{R} \) or \( \alpha \in \mathbb{Q} \), \( m, n \in \mathbb{Z} \) and \( S := \sum_{k=n}^{m} \alpha^k \). If \( \alpha = 1 \) then \( \sum_{k=n}^{m} \alpha^k = m - n + 1 \) while for \( \alpha \neq 1 \),
\[
\alpha S - S = \alpha^{m+1} - \alpha^{n}
\]
and solving for \( S \) gives the important geometric summation formula,
\[
\sum_{k=n}^{m} \alpha^k = \frac{\alpha^{m+1} - \alpha^{n}}{\alpha - 1} \quad \text{if } \alpha \neq 1.
\] (A.2)

Taking \( \alpha = 10^{-1} \) in Eq. (A.2) implies
\[
\sum_{k=n}^{m} 10^{-k} = \frac{10^{-(m+1)} - 10^{-n}}{10^{-1} - 1} = \frac{10^{-1} - 10^{-(m-n+1)}}{9}
\]
and in particular, for all \( M \geq n \),
\[
\lim_{m \to \infty} \sum_{k=n}^{m} 10^{-k} = \frac{1}{9 \cdot 10^{n-1}} \geq \sum_{k=n}^{M} 10^{-k}.
\]

Let \( \mathbb{D} \) denote those sequences \( \alpha \in \{0, 1, 2, \ldots, 9\}^\mathbb{Z} \) with the following properties:

1. there exists \( N \in \mathbb{N} \) such that \( \alpha_{-n} = 0 \) for all \( n \geq N \) and
2. \( \alpha_n \neq 0 \) for some \( n \in \mathbb{Z} \).

Associated to each \( \alpha \in \mathbb{D} \) is the sequence \( a = a(\alpha) \) defined by
\[
a_n := \sum_{k=-\infty}^{n} \alpha_k 10^{-k}.
\]

Since for \( m > n \),
\[
|a_m - a_n| = \left| \sum_{k=n+1}^{m} \alpha_k 10^{-k} \right| \leq 9 \sum_{k=n+1}^{m} 10^{-k} \leq 9 \frac{1}{10} \frac{1}{9^{m-n}} = \frac{1}{10^n},
\]
it follows that
\[
|a_m - a_n| \leq \frac{1}{10^{\min(m,n)}} \to 0 \text{ as } m, n \to \infty.
\]

Therefore \( a = a(\alpha) \in \mathbb{C} \) and we may define a map \( D : \{ \pm 1 \} \times \mathbb{D} \to \mathbb{R} \) defined by \( D(\varepsilon, \alpha) = \varepsilon a(\alpha) \). As is customary we will denote \( D(\varepsilon, \alpha) = \varepsilon a(\alpha) \) as
\[
\varepsilon \cdot \alpha_m \ldots \alpha_0.\alpha_1\alpha_2 \ldots \alpha_n 
\]
where \( m \) is the largest integer in \( \mathbb{Z} \) such that \( \alpha_k = 0 \) for all \( k < m \). If \( m > 0 \) the expression in Eq. (A.3) should be interpreted as
\[
\varepsilon \cdot 0.0 \ldots 0\alpha_m \alpha_{m+1} \ldots.
\]

An element \( \alpha \in \mathbb{D} \) has a tail of all 9’s starting at \( N \in \mathbb{N} \) if \( \alpha_n = 9 \) and for all \( n \geq N \) and \( \alpha_{N-1} \neq 9 \). If \( \alpha \) has a tail of 9’s starting at \( N \in \mathbb{N} \), then for \( n > N \),
\[
a_n(\alpha) = \sum_{k=-\infty}^{N-1} \alpha_k 10^{-k} + 9 \sum_{k=N}^{n} 10^{-k}
\]
\[
= \sum_{k=-\infty}^{N-1} \alpha_k 10^{-k} + \frac{9}{10^{N-1}} \cdot \frac{1 - 10^{-(n-N)}}{9}
\]
\[
\to \sum_{k=-\infty}^{N-1} \alpha_k 10^{-k} + 10^{-(N-1)} \text{ as } n \to \infty.
\]

If \( \alpha’ \) is the digits in the decimal expansion of \( \sum_{k=-\infty}^{N-1} \alpha_k 10^{-k} + 10^{-(N-1)} \), then
\[
\alpha’ \in \mathbb{D}' := \{ \alpha \in \mathbb{D} : \alpha \text{ does not have a tail of all 9’s} \}
\]
and we have just shown that \( D(\varepsilon, \alpha) = D(\varepsilon, \alpha’) \). In particular this implies
\[
D(\{ \pm 1 \} \times \mathbb{D}') = D(\{ \pm 1 \} \times \mathbb{D}).
\] (A.4)

### Theorem A.10 (Decimal Representation)

The map
\[
D : \{ \pm 1 \} \times \mathbb{D}' \to \mathbb{R} \setminus \{0\}
\]
is a bijection.
Proof. Suppose $D(\varepsilon, \alpha) = D(\delta, \beta)$ for some $(\varepsilon, \alpha)$ and $(\delta, \beta)$ in $\{\pm 1\} \times \mathbb{D}$. Since $D(\varepsilon, \alpha) > 0$ if $\varepsilon = 1$ and $D(\varepsilon, \alpha) < 0$ if $\varepsilon = -1$ it follows that $\varepsilon = \delta$. Let $a = a(\alpha)$ and $b = a(\beta)$ be the sequences associated to $\alpha$ and $\beta$ respectively. Suppose that $\alpha \neq \beta$ and let $j \in \mathbb{Z}$ be the position where $\alpha$ and $\beta$ first disagree, i.e. $\alpha_n = \beta_n$ for all $n < j$ while $\alpha_j \neq \beta_j$. For sake of definiteness suppose $\alpha_j > \beta_j$. Then for $n > j$ we have

$$b_n - a_n = (\beta_j - \alpha_j) 10^{-j} + \sum_{k=j+1}^{n} (\beta_k - \alpha_k) 10^{-k} \geq 10^{-j} - 9 \sum_{k=j+1}^{n} 10^{-k} \geq 10^{-j} - 9 \frac{1}{10^j} = 0.$$ 

Therefore $b_n - a_n \geq 0$ for all $n$ and $\lim (b_n - a_n) = 0$ iff $\beta_j = \alpha_j + 1$ and $\beta_k = 9$ and $\alpha_k = 0$ for all $k > j$. In summary, $D(\varepsilon, \alpha) = D(\delta, \beta)$ with $\alpha \neq \beta$ implies either $\alpha$ or $\beta$ has an infinite tail of nines which shows that $D$ is injective when restricted to $\{\pm 1\} \times \mathbb{D}'$. To see that $D$ is surjective it suffices to show any $\bar{b} \in \mathbb{R}$ with $0 < \bar{b} < 1$ is in the range of $D$. For each $n \in \mathbb{N}$, let $a_n = \alpha_1 \ldots \alpha_n$ with $\alpha_i \in \{0, 1, 2, \ldots, 9\}$ such that

$$i(a_n) < \bar{b} \leq i(a_n) + i(10^{-n}). \quad \text{(A.5)}$$ 

Since $a_{n+1} = a_n + a_{n+1} 10^{-(n+1)}$ for some $a_{n+1} \in \{0, 1, 2, \ldots, 9\}$, we see that $a_{n+1} = \alpha_1 \ldots \alpha_n \alpha_{n+1}$, i.e. the first $n$ digits in the decimal expansion of $a_{n+1}$ are the same as in the decimal expansion of $a_n$. Hence this defines $\alpha_n$ uniquely for all $n \geq 1$. By setting $\alpha_n = 0$ when $n \leq 0$, we have constructed from $\bar{b}$ an element $\alpha \in \mathbb{D}$. Because of Eq. (A.5), $D(1, \alpha) = \bar{b}$. \hfill \blacksquare

Notation A.11 From now on we will identify $\mathbb{Q}$ with $i(\mathbb{Q}) \subset \mathbb{R}$ and elements in $\mathbb{R}$ with their decimal expansions.

To summarize, we have constructed a complete ordered field $\mathbb{R}$ “containing” $\mathbb{Q}$ as a dense subset. Moreover every element in $\mathbb{R}$ (modulo those of the form $m 10^{-n}$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$) has a unique decimal expansion.

Corollary A.12. The set $(0, 1) := \{a \in \mathbb{R} : 0 < a < 1\}$ is uncountable while $\mathbb{Q} \cap (0, 1)$ is countable.

Proof. By Theorem A.10, the set $\{0, 1, 2, \ldots, 8\}^\mathbb{N}$ can be mapped injectively into $(0, 1)$ and therefore it follows from Lemma 6.6 that $(0, 1)$ is uncountable. For each $m \in \mathbb{N}$, let $A_m := \{ \frac{n}{m} : n \in \mathbb{N} \text{ with } n < m \}$. Since $\mathbb{Q} \cap (0, 1) = \bigcup_{m=1}^{\infty} A_m$ and $#(A_m) < \infty$ for all $m$, another application of Lemma 6.6 shows $\mathbb{Q} \cap (0, 1)$ is countable. \hfill \blacksquare

A.2 Exercises

Exercise A.5. Show to every $a \in \mathbb{R}$ with $a \geq 0$ there exists a unique number $b \in \mathbb{R}$ such that $b \geq 0$ and $b^2 = a$. Of course we will call $b = \sqrt{a}$. Also show that $a \rightarrow \sqrt{a}$ is an increasing function on $[0, \infty)$. Hint: To construct $b = \sqrt{a}$ for $a > 0$, to each $n \in \mathbb{N}$ let $m_n \in \mathbb{N}_0$ be chosen so that

$$\frac{m_n^2}{n^2} < a \leq \frac{(m_n + 1)^2}{n^2} \text{ i.e. } i \left( \frac{m_n^2}{n^2} \right) < a \leq i \left( \frac{(m_n + 1)^2}{n^2} \right)$$

and let $q_n := \frac{m_n}{n}$. Then show $b = \left\{ q_n \right\}_{n=1}^{\infty} \in \mathbb{R}$ satisfies $b > 0$ and $b^2 = a$. 

Appendix: More Set Theoretic Properties (highly optional)

B.1 Appendix: Zorn’s Lemma and the Hausdorff Maximal Principle (optional)

Definition B.1. A partial order \( \leq \) on \( X \) is a relation with following properties:

1. If \( x \leq y \) and \( y \leq z \) then \( x \leq z \).
2. If \( x \leq y \) and \( y \leq x \) then \( x = y \).
3. \( x \leq x \) for all \( x \in X \).

Example B.2. Let \( Y \) be a set and \( X = 2^Y \). There are two natural partial orders on \( X \):

1. Ordered by inclusion, \( A \subseteq B \) is \( A \prec B \) and
2. Ordered by reverse inclusion, \( A \subseteq B \) if \( B \subseteq A \).

Definition B.3. Let \( (X, \leq) \) be a partially ordered set we say \( X \) is linearly or totally ordered if for all \( x,y \in X \) either \( x \leq y \) or \( y \leq x \). The real numbers \( \mathbb{R} \) with the usual order \( \leq \) is a typical example.

Definition B.4. Let \( (X, \leq) \) be a partially ordered set. We say \( x \in X \) is a maximal element if for all \( y \in X \) such that \( y \geq x \) implies \( y = x \), i.e. there is no element larger than \( x \). An upper bound for a subset \( E \) of \( X \) is an element \( x \in X \) such that \( x \geq y \) for all \( y \in E \).

Example B.5. Let
\[
X = \{ a = \{1\} \ b = \{1,2\} \ c = \{3\} \ d = \{2,4\} \ e = \{2\} \}
\]
ordered by set inclusion. Then \( b \) and \( d \) are maximal elements despite that fact that \( b \nsubseteq d \) and \( d \nsubseteq b \). We also have:

1. If \( E = \{a,c,e\} \), then \( E \) has no upper bound.
2. If \( E = \{a,e\} \), then \( b \) is an upper bound.
3. If \( E = \{e\} \), then \( b \) and \( d \) are upper bounds.

Theorem B.6. The following are equivalent.

1. **The axiom of choice:** to each collection \( \{X_a\}_{a \in A} \) of non-empty sets there exists a “choice function,” \( x : A \to \prod_{a \in A} X_a \) such that \( x(a) \in X_a \) for all \( a \in A \), i.e. \( \prod_{a \in A} X_a \neq \emptyset \).

2. **The Hausdorff Maximal Principle:** Every partially ordered set has a maximal (relative to the inclusion order) linearly ordered subset.

3. **Zorn’s Lemma:** If \( X \) is partially ordered set such that every linearly ordered subset of \( X \) has an upper bound, then \( X \) has a maximal element.

**Proof.** (2 \( \Rightarrow \) 3) Let \( X \) be a partially ordered subset as in 3 and let \( \mathcal{F} = \{ E \subseteq X : E \text{ is linearly ordered} \} \) which we equip with the inclusion partial ordering. By 2, there exist a maximal element \( E \in \mathcal{F} \). By assumption, the linearly ordered set \( E \) has an upper bound \( x \in X \). The element \( x \) is maximal, for if \( y \in Y \) and \( y \geq x \), then \( E \cup \{y\} \) is still an linearly ordered set containing \( E \). So by maximality of \( E \), \( E = E \cup \{y\} \), i.e. \( y \in E \) and therefore \( y \leq x \) showing which combined with \( x \geq y \) implies that \( y = x \).

(3 \( \Rightarrow \) 1) Let \( \{X_a\}_{a \in A} \) be a collection of non-empty sets, we must show \( \prod_{a \in A} X_a \) is not empty. Let \( \mathcal{G} \) denote the collection of functions \( g : D(g) \to \prod_{a \in A} X_a \) such that \( D(g) \) is a subset of \( A \) and for all \( \alpha \in D(g) \), \( g(\alpha) \in X_\alpha \). Notice that \( \mathcal{G} \) is not empty, for we may let \( \alpha_0 \in A \) and \( x_0 \in X_{\alpha_0} \) and then set \( D(g) = \{\alpha_0\} \) and \( g(\alpha_0) = x_0 \) to construct an element of \( \mathcal{G} \). We now put a partial order on \( \mathcal{G} \) as follows. We say that \( f \leq g \) for \( f,g \in \mathcal{G} \) provided that \( D(f) \subseteq D(g) \) and \( f = g \mid_{D(f)} \). If \( \Phi \subseteq \mathcal{G} \) is a linearly ordered set, let \( D(h) = \cup_{g \in \Phi} D(g) \) and for \( \alpha \in D(g) \) let \( h(\alpha) = g(\alpha) \). Then \( h \in \mathcal{G} \) is an upper bound for \( \Phi \). So by Zorn’s

---

1. If \( X \) is a countable set we may prove Zorn’s Lemma by induction. Let \( \{x_n\}_{n=1}^\infty \) be an enumeration of \( X \), and define \( E_n \subseteq X \) inductively as follows. For \( n = 1 \) let \( E_1 = \{x_1\} \), and if \( E_n \) have been chosen, let \( E_{n+1} = E_n \cup \{x_{n+1}\} \) if \( x_{n+1} \) is an upper bound for \( E_n \) otherwise let \( E_{n+1} = E_n \). The set \( E = \cup_{n=1}^\infty E_n \) is a linearly ordered (you check) subset of \( X \) and hence by assumption \( E \) has an upper bound, \( x \in X \). I claim that his element is maximal, for if there exists \( y = x_m \in X \) such that \( y \geq x \), then \( x_m \) would be an upper bound for \( E_{m-1} \) and therefore \( y = x_m \in E \subseteq E \). That is to say if \( y \geq x \), then \( y \in E \) and hence \( y \leq x \), so \( y = x \). (Hence we may view Zorn’s lemma as a “jazzy” up version of induction.)

2. Similarly one may show that 3 \( \Rightarrow \) 2. Let \( \mathcal{F} = \{ E \subseteq X : E \text{ is linearly ordered} \} \) and order \( \mathcal{F} \) by inclusion. If \( M \subset \mathcal{F} \) is linearly ordered, let \( E = \cup M = \bigcup_{A \in M} A \). If \( x,y \in E \) then \( x \in A \) and \( y \in B \) for some \( A,B \in M \). Now \( M \) is linearly ordered by set inclusion so \( A \subseteq B \), or \( B \subseteq A \). If \( x,y \in A \) or \( x,y \in B \), since \( A \) and \( B \) are linearly ordered we must have either \( x \leq y \) or \( y \leq x \). That is to say \( E \) is linearly ordered. Hence by 3, there exists a maximal element \( E \in \mathcal{F} \) which is the assertion in 2.
Lemma there exists a maximal element \( h \in \mathcal{G} \). To finish the proof we need only show that \( D(h) = A \). If this were not the case, then let \( \alpha_0 \in A \setminus D(h) \) and \( x_0 \in X_{\alpha_0} \). We may now define \( D(h) = D(h) \cup \{ \alpha_0 \} \) and

\[
\tilde{h}(\alpha) = \begin{cases} h(\alpha) & \text{if } \alpha \in D(h) \\ x_0 & \text{if } \alpha = \alpha_0. \end{cases}
\]

Then \( h \leq \tilde{h} \) while \( h \neq \tilde{h} \) violating the fact that \( h \) was a maximal element.

(1 \( \Rightarrow \) 2) Let \( (X, \leq) \) be a partially ordered set. Let \( \mathcal{F} \) be the collection of linearly ordered subsets of \( X \) which we order by set inclusion. Given \( x_0 \in X \), \( \{x_0\} \in \mathcal{F} \) is linearly ordered set so that \( \mathcal{F} \neq \emptyset \). Fix an element \( P_0 \in \mathcal{F} \). If \( P_0 \) is not maximal there exists \( P_1 \in \mathcal{F} \) such that \( P_0 \subseteq P_1 \). In particular we may choose \( x \notin P_0 \) such that \( P_0 \cup \{x\} \in \mathcal{F} \). The idea now is to keep repeating this process of adding points \( x \in X \) until we construct a maximal element \( P \) of \( \mathcal{F} \). We now have to take care of some details. We may assume with out loss of generality that \( \mathcal{F} \) is not maximal there exists \( \mathcal{F} \neq \emptyset \) is a non-empty set.

For \( P \in \mathcal{F} \), let \( \mathcal{F}^* = \{x \in X : P \cup \{x\} \in \mathcal{F}\} \). As the above argument shows, \( \mathcal{F}^* \neq \emptyset \) for all \( P \in \mathcal{F} \). Using the axiom of choice, there exists \( f \in \prod_{P \in \mathcal{F}} \mathcal{F}^* \). We now define \( g : \mathcal{F} \to \mathcal{F} \) by

\[
g(P) = \begin{cases} P & \text{if } P \text{ is maximal} \\ P \cup \{f(x)\} & \text{if } P \text{ is not maximal}. \end{cases} \quad (B.1)
\]

The proof is completed by Lemma B.7 below which shows that \( g \) must have a fixed point \( P \in \mathcal{F} \). This fixed point is maximal by construction of \( g \).

**Lemma B.7.** The function \( g : \mathcal{F} \to \mathcal{F} \) defined in Eq. (B.1) has a fixed point.\(^3\)

**Proof.** The idea of the proof is as follows. Let \( P_1 \in \mathcal{F} \) be chosen arbitrarily. Notice that \( \Phi = \{g^n(P_0)\}_{n=0}^\infty \in \mathcal{F} \) is a linearly ordered set and it is therefore easily verified that \( P_1 = \bigcup_{n=0}^\infty g^n(P_0) \in \mathcal{F} \). Similarly we may repeat the process to construct \( P_2 = \bigcup_{n=0}^\infty g^n(P_1) \in \mathcal{F} \) and \( P_3 = \bigcup_{n=0}^\infty g^n(P_2) \in \mathcal{F} \), etc. etc. Then take \( P_\infty = \bigcup_{n=0}^\infty P_n \) and start again with \( P_0 \) replaced by \( P_\infty \). Then keep going this way until eventually the sets stop increasing in size, in which case we have found our fixed point. The problem with this strategy is that we may never win. (This is very reminiscent of constructing measurable sets and the way out is to use measure theoretic like arguments.)

Let us now start the formal proof. Again let \( P_0 \in \mathcal{F} \) and let \( \mathcal{F}_1 = \{P \in \mathcal{F} : P_0 \subset P \} \). Notice that \( \mathcal{F}_1 \) has the following properties:

1. \( P_0 \in \mathcal{F}_1 \).
2. If \( \Phi \subset \mathcal{F}_1 \) is a totally ordered (by set inclusion) subset then \( \cup \Phi \in \mathcal{F}_1 \).
3. If \( P \in \mathcal{F}_1 \) then \( g(P) \in \mathcal{F}_1 \).

Let us call a general subset \( \mathcal{F}' \subset \mathcal{F} \) satisfying these three conditions a tower and let

\[
\mathcal{F}_0 = \cap \{\mathcal{F}' : \mathcal{F}' \text{ is a tower}\}.
\]

Standard arguments show that \( \mathcal{F}_0 \) is still a tower and clearly is the smallest tower containing \( P_0 \). (Morally speaking \( \mathcal{F}_0 \) consists of all of the sets we were trying to constructed in the “idea section” of the proof.) We now claim that \( \mathcal{F}_0 \) is a linearly ordered subset of \( \mathcal{F} \). To prove this let \( \Gamma \subset \mathcal{F}_0 \) be the linearly ordered set

\[
\Gamma = \{C \in \mathcal{F}_0 : \text{for all } A \in \mathcal{F}_0 \text{ either } A \subset C \text{ or } C \subset A \}.
\]

Shortly we will show that \( \Gamma \subset \mathcal{F}_0 \) is a tower and hence that \( \mathcal{F}_0 = \Gamma \). That is to say \( \mathcal{F}_0 \) is linearly ordered. Assuming this for the moment let us finish the proof.

Let \( P = \cup \mathcal{F}_0 \) which is in \( \mathcal{F}_0 \) by property 2 and is clearly the largest element in \( \mathcal{F}_0 \). By 3, it now follows that \( P \cap g(P) \in \mathcal{F}_0 \) and by maximality of \( P \), we have \( g(P) = P \), the desired fixed point. So to finish the proof, we must show that \( \Gamma \) is a tower. First off it is clear that \( P_0 \cap \Gamma \) so in particular \( \Gamma \) is not empty. For each \( C \in \Gamma \) let

\[
\Phi_C := \{A \in \mathcal{F}_0 : \text{either } A \subset C \text{ or } g(C) \subset A \}.
\]

We will begin by showing that \( \Phi_C \subset \mathcal{F}_0 \) is a tower and therefore that \( \Phi_C = \mathcal{F}_0 \). 1. \( P_0 \in \Phi_C \) since \( P_0 \subset C \) for all \( C \in \Gamma \subset \mathcal{F}_0 \). 2. If \( \Phi \subset \Phi_C \subset \mathcal{F}_0 \) is totally ordered by set inclusion, then \( A_\Phi := \cup \Phi \in \mathcal{F}_0 \). We must show \( A_\Phi \in \Phi_C \), that is that \( A_\Phi \subset C \) or \( C \subset A_\Phi \). Now if \( A \subset C \) for all \( A \in \Phi \), then \( A_\Phi \subset C \) and hence \( A_\Phi \in \Phi_C \). On the other hand if there is some \( A \in \Phi \) such that \( g(C) \subset A \) then clearly \( g(C) \subset A_\Phi \) and again \( A_\Phi \in \Phi_C \). Given \( A \in \Phi_C \) we must show \( g(A) \in \Phi_C \), i.e. that

\[
g(A) \subset C \text{ or } g(C) \subset g(A). \quad (B.2)
\]

There are three cases to consider: either \( A \subset C \), \( A = C \), or \( g(C) \subset A \). In the case \( A = C \), \( g(C) = g(A) \subset g(A) \) and if \( g(C) \subset A \) then \( g(C) \subset A \subset g(A) \) and Eq. (B.2) holds in either of these cases. So assume that \( A \subset C \). Since \( C \in \Gamma \), either \( g(A) \subset C \) (in which case we are done) or \( C \subset g(A) \). Hence we may assume that

\[
A \subset C \subset g(A).
\]

Now if \( C \) were a proper subset of \( g(A) \) it would then follow that \( g(A) \setminus A \) would consist of at least two points which contradicts the definition of \( g \). Hence we
must have \( g(A) = C \subseteq C \) and again Eq. \([B.2]\) holds, so \( \Phi_C \) is a tower. It is now easy to show \( \Gamma \) is a tower. It is again clear that \( P_0 \in \Gamma \) and Property 2. may be checked for \( \Gamma \) in the same way as it was done for \( \Phi_C \) above. For Property 3., if \( C \in \Gamma \) we may use \( \Phi_C = F_0 \) to conclude for all \( A \in F_0 \), either \( A \subseteq C \subseteq g(C) \) or \( g(C) \subseteq A \), i.e. \( g(C) \in \Gamma \). Thus \( \Gamma \) is a tower and we are done.  

B.2 Cardinality

In mathematics, the essence of counting a set and finding a result \( n \), is that it establishes a one to one correspondence (or bijection) of the set with the set of numbers \( \{1, 2, \ldots, n\} \). A fundamental fact, which can be proved by mathematical induction, is that no bijection can exist between \( \{1, 2, \ldots, n\} \) and \( \{1, 2, \ldots, m\} \) unless \( n = m \); this fact (together with the fact that two bijections can be composed to give another bijection) ensures that counting the same set in different ways can never result in different numbers (unless an error is made). This is the fundamental mathematical theorem that gives counting its purpose; however you count a (finite) set, the answer is the same. In a broader context, the theorem is an example of a theorem in the mathematical field of (finite) combinatorics—hence (finite) combinatorics is sometimes referred to as "the mathematics of counting."

Many sets that arise in mathematics do not allow a bijection to be established with \( \{1, 2, \ldots, n\} \) for any natural number \( n \); these are called infinite sets, while those sets for which such a bijection does exist (for some \( n \)) are called finite sets. Infinite sets cannot be counted in the usual sense; for one thing, the mathematical theorems which underlie this usual sense for finite sets are false for infinite sets. Furthermore, different definitions of the concepts in terms of which these theorems are stated, while equivalent for finite sets, are inequivalent in the context of infinite sets.

The notion of counting may be extended to them in the sense of establishing (the existence of) a bijection with some well understood set. For instance, if a set can be brought into bijection with the set of all natural numbers, then it is called "countably infinite." This kind of counting differs in a fundamental way from counting of finite sets, in that adding new elements to a set does not necessarily increase its size, because the possibility of a bijection with the original set is not excluded. For instance, the set of all integers (including negative numbers) can be brought into bijection with the set of natural numbers, and even seemingly much larger sets like that of all finite sequences of rational numbers are still (only) countably infinite. Nevertheless there are sets, such as the set of real numbers, that can be shown to be "too large" to admit a bijection with the natural numbers, and these sets are called "uncountable." Sets for which there exists a bijection between them are said to have the same cardinality, and in the most general sense counting a set can be taken to mean determining its cardinality. Beyond the cardinalities given by each of the natural numbers, there is an infinite hierarchy of infinite cardinalities, although only very few such cardinalities occur in ordinary mathematics (that is, outside set theory that explicitly studies possible cardinalities).

Counting, mostly of finite sets, has various applications in mathematics. One important principle is that if two sets \( X \) and \( Y \) have the same finite number of elements, and a function \( f : X \to Y \) is known to be injective, then it is also surjective, and vice versa. A related fact is known as the pigeonhole principle, which states that if two sets \( X \) and \( Y \) have finite numbers of elements \( n \) and \( m \) with \( n > m \), then any map \( f : X \to Y \) is not injective (so there exist two distinct elements of \( X \) that \( f \) sends to the same element of \( Y \)); this follows from the former principle, since if \( f \) were injective, then so would its restriction to a strict subset \( S \) of \( X \) with \( m \) elements, which restriction would then be surjective, contradicting the fact that for \( x \) in \( X \) outside \( S \), \( f(x) \) cannot be in the image of the restriction. Similar counting arguments can prove the existence of certain objects without explicitly providing an example. In the case of infinite sets this can even apply in situations where it is impossible to give an example; for instance there must exists real numbers that are not computable numbers, because the latter set is only countably infinite, but by definition a non-computable number cannot be precisely specified.

The domain of enumerative combinatorics deals with computing the number of elements of finite sets, without actually counting them; the latter usually being impossible because infinite families of finite sets are considered at once, such as the set of permutations of \( \{1, 2, \ldots, n\} \) for any natural number \( n \).

B.3 Formalities of Counting

**Definition B.8.** We say \( \text{card} (X) \leq \text{card} (Y) \) if there exists an injective map, \( f : X \to Y \) and \( \text{card} (Y) \geq \text{card} (X) \) if there exists a surjective map \( g : Y \to X \). We say \( \text{card} (X) = \text{card} (Y) \) if there exists bijections, \( f : X \to Y \).

**Proposition B.9.** We have \( \text{card} (X) \leq \text{card} (Y) \) iff \( \text{card} (Y) \geq \text{card} (X) \).

**Proof.** If \( f : X \to Y \) is an injective map, define \( g : Y \to X \) by \( g(f(x)) = f^{-1} \) and \( g(Y \setminus f(X)) = x_0 \in X \) chosen arbitrarily. Then \( g : Y \to X \) is surjective.

If \( g : Y \to X \) is a surjective map, then \( Y_x := g^{-1}(\{x\}) \neq \emptyset \) for all \( x \in X \) and so by the axiom of choice there exists \( f \in \prod_{x \in X} Y_x \). Thus \( f : X \to Y \) such that \( f(x) \in Y_x \) for all \( x \). As the \( Y_x \) are pairwise disjoint, it follows that \( f \) is injective.

**Theorem B.10 (Schröder-Bernstein Theorem).** If \( \text{card} (X) \leq \text{card} (Y) \) and \( \text{card} (Y) \leq \text{card} (X) \), then \( \text{card} (X) = \text{card} (Y) \). Stated more explicitly; if there exists injective maps \( f : X \to Y \) and \( g : Y \to X \), then there exists a bijective map, \( h : X \to Y \).
We continue this process of inverse iterates as long as it is possible, i.e. we can construct $y_{n+1}$ if $x_n \in g(Y)$ and $x_{n+1} \in f(X)$ . There are now three possibilities:

1. ancestor $(x)$ has infinite length so the process never gets stuck in which case we say $x \in X_\infty$, read as start in $X$ and end never get stuck.

2. ancestor $(x)$ is finite and the last term in the sequence is in $X$, in which case we say $x \in X_X$ (read as start in $X$ and end in (get stuck in) $X$).

3. ancestor $(x)$ is finite and the last term in the sequence is in $Y$, in which case we say $x \in X_Y$ (read as start in $X$ and end in (get stuck in) $Y$).

In this way we partition $X$ into three disjoint sets, $X_\infty, X_X$, and $X_Y$. Similarly we may partition $Y$ into $Y_\infty, Y_X$, and $Y_Y$. Let us now observe that,

1. $f(X_\infty) = Y_\infty$. Indeed if $x \in X_\infty$ then ancestor $(f(x)) = (x, ancestor (x))$ is an infinite sequence, i.e. $f(x) \in Y_\infty$. Moreover if $y \in Y_\infty$, then ancestor $(y) = (y, ancestor (x))$ where $f(x) = y$ so that $x \in X_\infty$ and $y \in f(X_\infty)$. Thus we have shown $f : X_\infty \rightarrow Y_\infty$ is a bijection, i.e. $\text{card}(X_\infty) = \text{card}(Y_\infty)$.

2. $f(X_X) = Y_X$. Indeed if $x \in X_X$ then again ancestor $(f(x)) = (x, ancestor (x))$ which ends in $X$ so that $f(x) \in Y_X$. Moreover if $y \in Y_X$, then ancestor $(y) = (y, ancestor (x))$ where $f(x) = y$ so that ancestor $(x)$ ends in $X$, i.e. $y \in f(X_X)$ . Thus we have shown $f : X_X \rightarrow Y_X$ is a bijection, i.e. $\text{card}(X_X) = \text{card}(Y_X)$.

3. By the same argument as in item 2. it follow that $g : Y_Y \rightarrow X_Y$ is a bijection, i.e. $\text{card}(Y_Y) = \text{card}(X_Y)$.

The last three statement implies $\text{card}(X) = \text{card}(Y)$. We may in fact define a bijection, $h : X \rightarrow Y$, by

$$h(x) = \begin{cases} f(x) & \text{if } x \in X_\infty \cup X_X \\ g|_{Y_X}(x) & \text{if } x \in X_Y \end{cases}$$

Definition B.11. We say $\text{card}(X) < \text{card}(Y)$ if $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(X) \neq \text{card}(Y)$ , i.e. $\text{card}(X) < \text{card}(Y)$ if there exists an injective map, $f : X \rightarrow Y$, but not bijective map exists. Similarly we say $\text{card}(Y) > \text{card}(X)$ if $\text{card}(Y) \geq \text{card}(X)$ and $\text{card}(Y) \neq \text{card}(X)$, i.e. $\text{card}(Y) > \text{card}(X)$ if there exists a surjective map $g : Y \rightarrow X$ but no bijective map exists.

Proposition B.12. For any non-empty set $X$, $\text{card}(X) < \text{card}(2^X)$.

Proof. Define $f : X \rightarrow 2^X$ by $f(x) = \{x\}$. Then $f$ is an injective map and hence $\text{card}(X) \leq \text{card}(2^X)$. Now suppose that $g : X \rightarrow 2^X$ is any map. Let $X_0 = \{x \in X : x \notin g(x)\} \subset X$. I claim that $X_0 \notin g(X)$.

Indeed suppose there exists $x_0 \in X$ such that $g(x_0) = X_0$. If $x_0 \notin X_0$, then $x_0 \notin g(x_0) = X_0$ which is impossible. Similarly if $x_0 \notin X_0 = g(x_0)$ then $x_0 \in X_0$ and again we have reached a contradiction. Thus we must conclude that $X_0 \notin g(X)$. Thus there are no surjective maps, $g : X \rightarrow 2^X$ so that $\text{card}(X) \neq \text{card}(2^X)$.

Proposition B.13. If $\text{card}(X) < \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(Z)$, then $\text{card}(X) < \text{card}(Z)$.

Proof. If there exists an injective map, $f : Z \rightarrow X$ then composing this with and injective map, $g : X \rightarrow Y$ gives an injective map, $g \circ f : Z \rightarrow X$ and there for $\text{card}(Z) \leq \text{card}(X)$. But this would imply that $\text{card}(X) = \text{card}(Z)$.

Definition B.14. Let $A_n := \{1,2,\ldots,n\}$ for all $n \in \mathbb{N}$ and write $n$ for $\text{card}(A_n)$.

Proposition B.15. We have $\text{card}(A_m) < \text{card}(A_n)$ for all $m < n$. Moreover if $\emptyset \neq X \subsetneq A_n$ then $\text{card}(X) = \text{card}(A_k)$ for some $k < n$.

Proof. If $f : A_1 \rightarrow A_2$, then either $f(1) = 1$ or $f(1) = 2$. In either case $f$ is injective but not bijective so that $\text{card}(A_2) < \text{card}(A_1)$. Let $S_n$ be the statement that $\text{card}(A_k) < \text{card}(A_l)$ for all $1 \leq k < l \leq n$ and for any proper subset $X \subset A_n$ we have $\text{card}(X) = \text{card}(A_m)$ for some $m < n$. Then we have just shown that $S_2$ is true. So suppose that $S_n$ is now true. As $f : A_k \rightarrow A_l$ defined by $f(m) = m$ for all $m \in A_k$ is a injection when $k < l$ we always have $\text{card}(A_k) \leq \text{card}(A_l)$. Now suppose that $\text{card}(A_k) = \text{card}(A_{n+1})$ for some $k < n$. Then there exists a bijection, $f : A_{n+1} \rightarrow A_k$. In this case $f(A_n)$ is a proper subset of $A_k$ and therefore $\text{card}(f(A_n)) < \text{card}(A_k)$ but on the other hand $\text{card}(f(A_n)) = \text{card}(A_n) \geq \text{card}(A_k)$ which is a contradiction. So no such bijection can exists and we have shown $\text{card}(A_k) < \text{card}(A_{n+1})$ for all $k < n$. Finally suppose that $X \subset A_{n+1}$ is proper subset. If $X \subset A_n$ then $\text{card}(X) = \text{card}(A_k)$ for some $k < n$ by the induction hypothesis. On the other hand if $n+1 \in X$, let $X^\prime := X \setminus \{n+1\} \subsetneq A_n$. Therefore by the induction hypothesis $\text{card}(X^\prime) = \text{card}(A_k)$ for some $k < n$. It is then clear that $\text{card}(X) = \text{card}(A_{k+1})$ where $k+1 < n$, indeed we map $X := X^\prime \cup \{n+1\} \rightarrow A_k \cup \{k+1\} = A_{k+1}$.

Example B.16. $\text{card}(A_n \setminus \{k\}) = n - 1$ for $k \in A_n$. Indeed, let $f : A_{n-1} \rightarrow A_n \setminus \{k\}$ be defined by
$$f(x) = \begin{cases} x & \text{if } x < k \\ x + 1 & \text{if } x \geq k \end{cases}$$

Then $f$ is the desired bijection. More generally if $X \subset Y$ and $\text{card}(X) = m < n = \text{card}(Y)$, then $\text{card}(Y \setminus X) = n - m$ and if $X$ and $Y$ are finite disjoint sets then $\text{card}(X \cup Y) = \text{card}(X) + \text{card}(Y)$. Similarly, $\text{card}(X \times Y) = \text{card}(X) \cdot \text{card}(Y)$.

**Proposition B.17.** If $f : A_n \to A_n$ is a map, then the following are equivalent,

1. $f$ is injective,
2. $f$ is surjective,
3. $f$ is bijective.

Moreover $\text{card}(\text{Bijec}(A_n)) = n!$.

**Proof.** If $n = 1$, the only map $f : A_1 \to A_1$ is $f(1) = 1$. So in this case there is nothing to prove. So now suppose the proposition holds for level $n$ and $f : A_{n+1} \to A_{n+1}$ is a given map.

If $f : A_{n+1} \to A_{n+1}$ is an injective map and $\text{card}(A_{n+1}) < \text{card}(f(A_{n+1})) = \text{card}(A_{n+1})$ which is absurd. Thus $f$ is injective implies $f$ is surjective.

Conversely suppose that $f : A_{n+1} \to A_{n+1}$ is surjective. Let $g : A_{n+1} \to A_{n+1}$ be a right inverse, i.e. $f \circ g = \text{id}$, which is necessarily injective, see the proof of Proposition B.9 By the previous paragraph we know that $g$ is necessarily surjective and therefore $f = g^{-1}$ is a bijection.

It now only remains to prove $\text{card}(\text{Bijec}(A_n)) = n!$ which we again do by induction. For $n = 1$ the result is clear. So suppose it holds at level $n$. If $f : A_{n+1} \to A_{n+1}$ is a bijection. Given $1 \leq k \leq n + 1$ let

$$\text{Bij}_k(A_{n+1}) := \{ f \in \text{Bij}(A_{n+1}) : f(n+1) = k \}.$$ 

For $f \in \text{Bij}_k(A_{n+1})$, we have $f : A_n \to A_{n+1} \setminus \{k\} \cong A_n$ is a bijection. Thus $\text{Bij}_k(A_{n+1}) \cong \text{Bij}(A_n)$ and

$$\text{Bij}(A_{n+1}) = \sum_{k=1}^{n+1} \text{Bij}_k(A_{n+1})$$

we have

$$\text{card}(\text{Bij}(A_{n+1})) = \sum_{k=1}^{n+1} \text{card}(\text{Bij}_k(A_{n+1}))$$

$$= \sum_{k=1}^{n+1} \text{card}(\text{Bij}(A_n)) = \sum_{k=1}^{n+1} n!$$

$$= (n + 1) n! = (n + 1)!.$$
References


Index

Cauchy, 13 19

Summable, 33