Set Operations, Functions, and Counting

Let \( \mathbb{N} \) denote the positive integers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) be the non-negative integers and \( \mathbb{Z} = \mathbb{N}_0 \cup \{-\mathbb{N}_0\} \) – the positive and negative integers including 0, \( \mathbb{Q} \) the rational numbers, \( \mathbb{R} \) the real numbers, and \( \mathbb{C} \) the complex numbers. We will also use \( \mathbb{F} \) to stand for either of the fields \( \mathbb{R} \) or \( \mathbb{C} \).

5.1 Set Operations and Functions

**Notation 5.1** Given two sets \( X \) and \( Y \), let \( Y^X \) denote the collection of all functions \( f : X \to Y \). If \( X = \mathbb{N} \), we will say that \( f \in Y^\mathbb{N} \) is a sequence with values in \( Y \) and often write \( f_n \) for \( f(n) \) and express \( f \) as \( \{f_n\}_{n=1}^\infty \). If \( X = \{1, 2, \ldots, N\} \), we will write \( Y^N \) in place of \( Y^{\{1, 2, \ldots, N\}} \) and denote \( f \in Y^N \) by \( f = (f_1, f_2, \ldots, f_N) \) where \( f_n = f(n) \).

**Notation 5.2** More generally if \( \{X_\alpha : \alpha \in A\} \) is a collection of non-empty sets, let \( X_A = \prod_{\alpha \in A} X_\alpha \) and \( \pi_\alpha : X_A \to X_\alpha \) be the canonical projection map defined by \( \pi_\alpha(x) = x_\alpha \). If \( X_\alpha = X \) for some fixed space \( X \), then we will write \( \prod_{\alpha \in A} X_\alpha \) as \( X^A \) rather than \( X_A \).

Recall that an element \( x \in X_A \) is a “choice function,” i.e. an assignment \( x_\alpha := x(\alpha) \in X_\alpha \) for each \( \alpha \in A \). The axiom of choice states that \( X_A \neq \emptyset \) provided that \( X_\alpha \neq \emptyset \) for each \( \alpha \in A \).

**Notation 5.3** Given a set \( X \), let \( 2^X \) denote the power set of \( X \) – the collection of all subsets of \( X \) including the empty set.

The reason for writing the power set of \( X \) as \( 2^X \) is that if we think of 2 meaning \( \{0, 1\} \), then an element of \( a \in 2^X = \{0, 1\}^X \) is completely determined by the set

\[ A := \{x \in X : a(x) = 1\} \subset X. \]

In this way elements in \( \{0, 1\}^X \) are in one to one correspondence with subsets of \( X \).

For \( A \in 2^X \) let

\[ A^c := X \setminus A = \{x \in X : x \notin A\} \]

and more generally if \( A, B \subset X \) let

\[ B \setminus A := \{x \in B : x \notin A\} = A \cap B^c. \]

We also define the symmetric difference of \( A \) and \( B \) by

\[ A \Delta B := (B \setminus A) \cup (A \setminus B). \]

As usual if \( \{A_\alpha\}_{\alpha \in I} \) is an indexed collection of subsets of \( X \) we define the union and the intersection of this collection by

\[ \bigcup_{\alpha \in I} A_\alpha := \{x \in X : \exists \alpha \in I \; x \in A_\alpha\} \quad \text{and} \quad \bigcap_{\alpha \in I} A_\alpha := \{x \in X : x \in A_\alpha \forall \alpha \in I\}. \]

**Example 5.4.** Let \( A, B \), and \( C \) be subsets of \( X \). Then

\[ A \cap (B \cup C) = [A \cap B] \cup [A \cap C]. \]

Indeed, \( x \in A \cap (B \cup C) \iff x \in A \) and \( x \in B \cup C \iff x \in A \) and \( x \in B \) or \( x \in A \) and \( x \in C \iff x \in A \cap B \) or \( x \in A \cap C \iff x \in [A \cap B] \cup [A \cap C] \).

**Notation 5.5** We will also write \( \bigsqcup_{\alpha \in I} A_\alpha \) for \( \bigcup_{\alpha \in I} A_\alpha \) in the case that \( \{A_\alpha\}_{\alpha \in I} \) are pairwise disjoint, i.e. \( A_\alpha \cap A_\beta = \emptyset \) if \( \alpha \neq \beta \).

Notice that \( \cup \) is closely related to \( \exists \) and \( \cap \) is closely related to \( \forall \). For example let \( \{A_n\}_{n=1}^\infty \) be a sequence of subsets from \( X \) and define

\[ \{A_n \text{ i.o.}\} := \{x \in X : \# \{n : x \in A_n\} = \infty\} \quad \text{and} \quad \{A_n \text{ a.a.}\} := \{x \in X : x \in A_n \text{ for all } n \text{ sufficiently large}\}. \]

(One should read \( \{A_n \text{ i.o.}\} \) as \( A_n \) infinitely often and \( \{A_n \text{ a.a.}\} \) as \( A_n \) almost always.) Then \( x \in \{A_n \text{ i.o.}\} \) iff

\[ \forall N \in \mathbb{N} \exists n \geq N \exists x \in A_n \]

and this may be expressed as

\[ \{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{n \geq N} A_n. \]

Similarly, \( x \in \{A_n \text{ a.a.}\} \) iff

\[ \forall N \in \mathbb{N} \exists n \geq N \exists x \in A_n \]
Definition 5.6. If $f : X \to Y$ is a function and $B \subset Y$, then
$$f^{-1}(B) := \{ x \in X : f(x) \in B \}.$$ 
If $A \subset X$ we also write,
$$f(A) := \{ f(x) : x \in A \} \subset Y.$$ 

Example 5.7. If $f : X \to Y$ is a function and $B \subset Y$, then $f^{-1}(B^c) = [f^{-1}(B)]^c$ or to be more precise,
$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B).$$ 
To prove this notice that
$$x \in f^{-1}(B^c) \iff f(x) \in B^c \iff f(x) \notin B \iff f^{-1}(B) \iff x \notin [f^{-1}(B)]^c.$$ 
On the other hand, if $A \subset X$ it is not necessarily true that $f(A^c) = [f(A)]^c$. For example, suppose that $f : \{1, 2\} \to \{1, 2\}$ is defined by $f(1) = f(2) = 1$ and $A = \{1\}$. Then $f(A) = f(A^c) = \{1\}$ where $[f(A)]^c = \{1\}^c = \{2\}$. 

Notation 5.8 If $f : X \to Y$ is a function and $\mathcal{E} \subset 2^Y$ let
$$f^{-1}\mathcal{E} := f^{-1}(\mathcal{E}) := \{ f^{-1}(E) \mid E \in \mathcal{E} \}.$$ 
If $\mathcal{G} \subset 2^X$, let
$$f_\mathcal{G} := \{ A \in 2^X \mid f^{-1}(A) \in \mathcal{G} \}.$$ 

Definition 5.9. Let $\mathcal{E} \subset 2^X$ be a collection of sets, $A \subset X$, $i_A : A \to X$ be the inclusion map ($i_A(x) = x$ for all $x \in A$) and
$$\mathcal{E}_A = i_A^{-1}(\mathcal{E}) = \{ A \cap E : E \in \mathcal{E} \}.$$ 

5.1.1 Exercises

Let $f : X \to Y$ be a function and $\{A_i\}_{i \in I}$ be an indexed family of subsets of $Y$, verify the following assertions.

Exercise 5.1. $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$. 

Exercise 5.2. Suppose that $B \subset Y$, show that $B \setminus (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (B \setminus A_i)$. 

Exercise 5.3. $f^{-1}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f^{-1}(A_i)$. 

Exercise 5.4. $f^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f^{-1}(A_i)$. 

Exercise 5.5. Find a counterexample which shows that $f(C \cap D) = f(C) \cap f(D)$ need not hold.

5.2 Cardinality

In this section, $X$ and $Y$ be sets.

Definition 5.10 (Cardinality). We say $\text{card}(X) \leq \text{card}(Y)$ if there exists an injective map, $f : X \to Y$ and $\text{card}(Y) \geq \text{card}(X)$ if there exists a surjective map $g : Y \to X$. We say $\text{card}(X) = \text{card}(Y)$ (also denoted as $X \sim Y$) if there exists bijections, $f : X \to Y$.

Proposition 5.11. If $X$ and $Y$ are sets, then $\text{card}(X) \leq \text{card}(Y)$ iff $\text{card}(Y) \geq \text{card}(X)$. 

Proof. If $f : X \to Y$ is an injective map, define $g : Y \to X$ by $g|_{f(X)} = f^{-1}$ and $g|_{Y \setminus f(X)} = x_0 \in X$ chosen arbitrarily. Then $g : Y \to X$ is surjective. 

If $g : Y \to X$ is a surjective map, then $Y_x := g^{-1}\{x\} \neq \emptyset$ for all $x \in X$ and so by the axiom of choice there exists $f \in \prod_{x \in X} Y_x$. Thus $f : X \to Y$ such that $f(x) \in Y_x$ for all $x$. As the $\{Y_x\}_{x \in X}$ are pairwise disjoint, it follows that $f$ is injective. 

Theorem 5.12 (Schröder–Bernstein Theorem). If $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(X)$, then $\text{card}(X) = \text{card}(Y)$. Stated more explicitly; if there exists injective maps $f : X \to Y$ and $g : Y \to X$, then there exists a bijection map, $h : X \to Y$. 

Proof. See the Appendix, Theorem B.10.

Exercise 5.6. If $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$, $Y = Y_1 \cup Y_2$ with $Y_1 \cap Y_2 = \emptyset$, and $X_i \sim Y_i$ for $i = 1, 2$, then $X \sim Y$. This exercise generalizes to an arbitrary number of factors.
5.3 Finite Sets

**Notation 5.13 (Integer Intervals)** For $n \in \mathbb{N}$ we let

$$J_n := \{1, 2, \ldots, n\} := \{k \in \mathbb{N} : k \leq n\}.$$

**Definition 5.14.** We say a non-empty set, $X$, is **finite** if $\text{card}(X) = \text{card}(J_n)$ for some $n \in \mathbb{N}$. We will also write $\#(X) = n$\footnote{You should read $\#(X) = n$, as $X$ is a set with $n$ elements.} to indicate that $\text{card}(X) = \text{card}(J_n)$. It is shown in Theorem 5.17 below that $\#(X)$ is well defined, i.e. it is not possible for $\text{card}(X) = \text{card}(J_n)$ and $\text{card}(X) = \text{card}(J_m)$ unless $m = n$.

**Lemma 5.15.** Suppose $n \in \mathbb{N}$ and $k \in J_{n+1}$, then $\text{card}(J_{n+1} \setminus \{k\}) = \text{card}(J_n)$.

**Proof.** Let $f : J_n \rightarrow J_{n+1} \setminus \{k\}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x < k \\ x+1 & \text{if } x \geq k \end{cases}$$

Then $f$ is the desired bijection.

Alternatively. If $n = 1$, then $J_2 = \{1, 2\}$ and either $J_2 \setminus \{k\} = J_1$ or $J_2 \setminus \{k\} = \{2\}$, either way $\text{card}(J_2 \setminus \{k\}) = \text{card}(J_1)$. Now suppose that result holds for a given $n \in \mathbb{N}$ and $k \in J_{n+2}$. If $k = \{n+2\}$ we have $J_{n+2} \setminus \{k\} = J_{n+1}$ so card $(J_{n+2} \setminus \{k\}) = \text{card}(J_{n+1})$, while if $k \in J_{n+1} \subset J_{n+2}$, then $J_{n+2} \setminus \{k\} = (J_{n+1} \setminus \{k\}) \cup \{n+2\} \sim J_n \cup \{n+2\} \sim J_n \cup \{n+1\} = J_{n+1}$. $\blacksquare$

**Lemma 5.16.** If $m, n \in \mathbb{N}$ with $n > m$, then every map, $f : J_n \rightarrow J_m$, is not injective.

**Proof.** If $f : J_n \rightarrow J_m$ were injective, then $f|_{J_{m+1}} : J_{m+1} \rightarrow J_m$ would be injective as well. Therefore it suffices to show there is no injective map, $f : J_{m+1} \rightarrow J_m$ for all $m \in \mathbb{N}$. We prove this last assertion by induction on $m$. The case $m = 1$ is trivial as $J_1 = \{1\}$ so the only function, $f : J_2 \rightarrow J_1$ is the function, $f(1) = 1 = f(2)$ which is not injective.

Now suppose $m \geq 1$ and there were an injective map, $f : J_{m+2} \rightarrow J_{m+1}$. Letting $k := f(m+2)$ we would have, $f|_{J_{m+1}} : J_{m+1} \rightarrow J_{m+1} \setminus \{k\} \sim J_m$, which would produce and injective map from $J_{m+1}$ to $J_m$. However this contradicts the induction hypothesis and thus completes the proof. $\blacksquare$

**Theorem 5.17.** If $m, n \in \mathbb{N}$, then $\text{card}(J_m) \leq \text{card}(J_n)$ iff $m \leq n$. Moreover, $\text{card}(J_n) = \text{card}(J_m)$ iff $m = n$ and hence $\text{card}(J_m) < \text{card}(J_n)$ iff $m < n$.

**Proof.** As $J_m \subset J_n$ if $m \leq n$ and $J_m = J_n$ if $m = n$, it is only the forward implications that have any real content. If $\text{card}(J_m) \leq \text{card}(J_n)$, there exists an injective map, $g : J_m \rightarrow J_n$. According to Lemma 5.16 this can only happen if $m \leq n$. If $\text{card}(J_n) = \text{card}(J_m)$, then $\text{card}(J_n) \leq \text{card}(J_m)$ and $\text{card}(J_m) \leq \text{card}(J_n)$ and hence $m \leq n$ and $n \leq m$, i.e. $m = n$. $\blacksquare$

**Proposition 5.18.** If $X$ is a finite set with $\#(X) = n$ and $S$ is a non-empty subset of $X$, then $S$ is a finite set and $\#(S) \leq n$. Moreover if $\#(S) = n$, then $S = X$.

**Proof.** It suffices to assume that $X = J_n$ and $S \subset J_n$. We now give two proofs of the result.

**Proof 1.** Let $S_1 = S$ and $f(1) := \min S \geq 1$. If $S_2 := S_1 \setminus \{f(1)\}$ is not empty, let $f(2) := \min S_2 \geq 2$. We then continue this construction inductively. So if $f(k) = \min S_k \geq k$ has been constructed, then we define $S_{k+1} := S_k \setminus \{f(k)\}$. If $S_{k+1} \neq \emptyset$ we define $f(k+1) := \min S_{k+1} \geq k + 1$. As $f(k) \geq k$ for all $k$ that $f$ is defined, this process has to stop after at most $n$ steps. Say it stops at $k$ so that $S_k+1 = \emptyset$. Then $f : J_k \rightarrow S$ is a bijection and therefore $S$ is finite and $\#(S) = k \leq n$. Moreover, the only way that $k = n$ is if $f(k) = k$ at each step of the construction so that $f : J_n \rightarrow S$ is the identity map in this case, i.e. $S = J_n$.

**Proof 2.** We prove this by induction on $n$. When $n = 1$ the only non-empty subset of $S$ of $J_1$ is $J_1$ itself. Thus $\#(S) = 1$ and $S = J_1$. Now suppose that the result hold for some $n \in \mathbb{N}$ and let $S \subset J_{n+1}$. If $n+1 \notin S$, then $S \subset J_n$ and by the induction hypothesis we know $\#(S) = k \leq n < n + 1$. So now suppose that $n+1 \in S$ and let $S' := S \setminus \{n+1\} \subset J_n$. Then by the induction hypothesis, $S'$ is a finite set and $\#(S') = k \leq n$, i.e. there exists a bijection, $f' : J_k \rightarrow S'$ and $S' = J_n$ is $k = n$. Therefore $f : J_{n+1} \rightarrow S$ given by $f = f'$ on $J_k$ and $f(k+1) = n+1$ is a bijections from $J_{n+1}$ to $S$. Therefore $\#(S) = k+1 \leq n+1$ with equality iff $S' = J_n$ which happens iff $S = J_{n+1}$. $\blacksquare$

**Proposition 5.19.** If $f : J_n \rightarrow J_n$ is a map, then the following are equivalent,

1. $f$ is injective,
2. $f$ is surjective,
3. $f$ is bijective.

**Proof.** If $n = 1$, the only map $f : J_1 \rightarrow J_1$ is $f(1) = 1$. So in this case there is nothing to prove. So now suppose the proposition holds for level $n$ and $f : J_{n+1} \rightarrow J_{n+1}$ is a given map.
If \( J_{n+1} \rightarrow J_{n+1} \) is an injective map and \( f(J_{n+1}) \) is a proper subset of \( J_{n+1} \), then \( \text{card}(J_{n+1}) < \text{card}(f(J_{n+1})) = \text{card}(J_{n+1}) \) which is absurd. Thus \( f \) is injective implies \( f \) is surjective.

Conversely suppose that \( J_{n+1} \rightarrow J_{n+1} \) is surjective. Let \( g : J_{n+1} \rightarrow J_{n+1} \) be a right inverse, i.e. \( f \circ g = \text{id} \), which is necessarily injective, see the proof of Proposition [5.18]. By the previous paragraph we know that \( g \) is necessarily surjective and therefore \( f = g^{-1} \) is a bijection.

**Theorem 5.20.** A subset \( S \subseteq \mathbb{N} \) is finite iff \( S \) is bounded. Moreover if \( \#(S) = n \in \mathbb{N} \) then the sup \( \{S \geq n \} \) with equality iff \( S = J_n \).

**Proof.** If \( S \) is bounded then \( S \subseteq J_n \) for some \( n \in \mathbb{N} \) and hence \( S \) is a finite set by Proposition [5.18]. Also observe that if \( \#(S) = n = \sup(S) \), then \( S \subseteq J_n \) and \( \#(S) = n = \#(J_n) \). Thus it follows from Proposition [5.18] that \( S = J_n \).

Conversely suppose that \( S \subseteq \mathbb{N} \) is a finite set and let \( n = \#(S) \). We will now complete the proof by induction. If \( n = 1 \) we have \( S \sim J_1 \) and therefore \( S = \{k\} \) for some \( k \in \mathbb{N} \). In particular sup \( S = k \geq 1 \) with equality iff \( S = J_1 \).

Suppose the truth of the statement for some \( n \in \mathbb{N} \) and let \( S \subseteq \mathbb{N} \) be a set with \( \#(S) = n + 1 \). If we choose a point, \( k \in S \), we have by Lemma [5.15] that \( \#(S \setminus \{k\}) = n \). Hence by the induction hypothesis, \( \sup(S \setminus \{k\}) \geq n \) with equality iff \( S \setminus \{k\} = J_n \). If \( \sup(S \setminus \{k\}) > n \) then \( \sup(S) \geq \sup(S \setminus \{k\}) \geq n + 1 \) as desired. If \( \sup(S \setminus \{k\}) = n \) then \( S \setminus \{k\} = J_n \) therefore \( S \ni k > n \). Hence it follows that \( \sup(S) = k \geq n + 1 \).

**Corollary 5.21.** Suppose \( S \) is a non-empty subset of \( \mathbb{N} \). Then \( S \) is an unbounded subset of \( \mathbb{N} \) iff \( \text{card}(J_n) \leq \text{card}(S) \) for all \( n \in \mathbb{N} \).

**Proof.** If \( S \) is bounded we know \( \text{card}(S) = \text{card}(J_k) \) for some \( k \in \mathbb{N} \) which would violate the hypothesis that \( \text{card}(J_n) \leq \text{card}(S) \) for all \( n \in \mathbb{N} \). Conversely if \( \text{card}(S) \leq \text{card}(J_n) \) for some \( n \in \mathbb{N} \), then there exists and injective map, \( f : S \rightarrow J_n \). Therefore \( \text{card}(S) = \text{card}(f(S)) = \text{card}(J_k) \) for some \( k \leq n \). So \( S \) is finite and hence bounded in \( \mathbb{N} \) by Theorem [5.20].

**Exercise 5.7.** Suppose that \( m, n \in \mathbb{N} \), show \( J_{m+n} = J_m \cup (m+J_n) \) and \((m+J_n) \cap J_m = \emptyset \). Use this to conclude if \( X \) is a disjoint union of two non-empty finite sets, \( X_1 \) and \( X_2 \), then \( \#(X) = \#(X_1) + \#(X_2) \).

**Exercise 5.8.** Suppose that \( m, n \in \mathbb{N} \), show \( J_m \times J_n \approx J_{mn} \). Use this to conclude if \( X \) and \( Y \) are two non-empty sets, then \( \#(X \times Y) = \#(X) \cdot \#(Y) \).

### 5.4 Countable and Uncountable Sets

**Definition 5.22 (Countability).** A set \( X \) is said to be **countable** if \( X = \emptyset \) or if there exists a surjective map, \( f : \mathbb{N} \rightarrow X \). Otherwise \( X \) is said to be **uncountable**.

**Remark 5.23.** From Proposition [5.11] it follows that \( X \) is **countable** iff there exists an injective map, \( g : X \rightarrow \mathbb{N} \). This may be succinctly stated as; \( X \) is **countable** iff \( \text{card}(X) \leq \text{card}(\mathbb{N}) \). From a practical point of view as set \( X \) is countable iff the elements of \( X \) may be arranged into a linear list,

\[
X = \{x_1, x_2, x_3, \ldots \}.
\]

**Example 5.24.** The integers, \( \mathbb{Z} \), are countable. In fact \( \mathbb{N} \sim \mathbb{Z} \), for example define \( f : \mathbb{N} \rightarrow \mathbb{Z} \) by

\[
(f(1), f(2), f(3), f(4), f(5), f(6), f(7), \ldots) = (0, 1, -1, 2, -2, 3, -3, \ldots).
\]

**Lemma 5.25.** If \( S \subseteq \mathbb{N} \) is an unbounded set, then \( \text{card}(S) = \text{card}(\mathbb{N}) \).

**Proof.** The main idea is that any subset, \( S \subseteq \mathbb{N} \), may be given as an finite or infinite list written in increasing order, i.e.

\[
S = \{n_1, n_2, n_3, \ldots \} \quad \text{with} \quad n_1 < n_2 < n_3 < \ldots.
\]

If the list is finite, say \( S = \{n_1, \ldots, n_k\} \), then \( n_k \) is an upper bound for \( S \). So \( S \) will be unbounded iff only if the list is infinite in which case \( f : \mathbb{N} \rightarrow S \) defined by \( f(k) = n_k \) defines a bijection.

**Formal proof.** Define \( f : \mathbb{N} \rightarrow S \) via, let

\[
S_1 := S \quad \text{and} \quad f(1) := \min S_1, \\
S_2 := S_1 \setminus \{f(1)\} \quad \text{and} \quad f(2) := \min S_2, \\
S_3 := S_2 \setminus \{f(2)\} \quad \text{and} \quad f(3) := \min S_3 \\
\vdots
\]

In more detail, let \( T \) denote those \( n \in \mathbb{N} \) such that there exists \( f : J_n \rightarrow S \) and \( \{S_k \subset S\} \}_{k=1}^n \) satisfying, \( S_1 = f, \) \( f(k) = \min S_k \) and \( S_{k+1} = S_k \setminus \{f(k)\} \) for \( 1 \leq k < n \). If \( n \in T \), we may define \( S_n+1 := S_n \setminus \{f(n)\} \) and \( f(n+1) := \min S_{n+1} \) in order to show \( n+1 \in T \). Thus \( T = \mathbb{N} \) and we have constructed an injective map, \( f : \mathbb{N} \rightarrow S \). Moreover \( \cap_{k \in \mathbb{N}} S_k \subseteq \mathbb{N} \setminus J_n \) for all \( n \) and therefore \( \cap_{k \in \mathbb{N}} S_k = \emptyset \). Thus it follows that \( f \) is a bijection.

- End of Lecture 14, 10/31/2012.

The following theorem summarizes most of what we need to know about counting and countability.

**Theorem 5.26.** The following properties hold:

1. \( \mathbb{N} \times \mathbb{N} \) is countable and in fact \( \mathbb{N} \times \mathbb{N} \sim \mathbb{N} \), i.e. there exists a bijective map, \( h \), from \( \mathbb{N} \) to \( \mathbb{N} \times \mathbb{N} \).
2. If $X$ and $Y$ are countable, then $X \times Y$ is countable.

3. If $\{X_n\}_{n \in \mathbb{N}}$ are countable sets then $X := \bigcup_{n=1}^{\infty} X_n$ is a countable set.

4. If $X$ is countable, then either there exists $n \in \mathbb{N}$ such that $X \sim J_n$ or $X \sim \mathbb{N}$.

5. If $S \subset \mathbb{N}$ and $S \sim J_n$ for some $n \in \mathbb{N}$ then $S$ is bounded.

6. If $X$ is a set and $\text{card} \, J_n \leq \text{card} \, X$ for all $n \in \mathbb{N}$ then $\text{card} \, \mathbb{N} \leq \text{card} \, X$.

7. If $A \subset X$ is a subset of a countable set $X$ then $A$ is countable.

**Proof.** We take each item in turn.

1. Put the elements of $\mathbb{N} \times \mathbb{N}$ into an array of the form

   $$(1,1) \ (1,2) \ (1,3) \ldots$$

   $$(2,1) \ (2,2) \ (2,3) \ldots$$

   $$(3,1) \ (3,2) \ (3,3) \ldots$$

   and then “count” these elements by counting the sets $\{(i,j) : i+j = k\}$ one at a time. For example let $h(1) = (1,1), h(2) = (2,1), h(3) = (1,2), h(4) = (3,1), h(5) = (2,2), h(6) = (1,3)$ and so on. In other words we put $\mathbb{N} \times \mathbb{N}$ into the following list form,

   $$\mathbb{N} \times \mathbb{N} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,1), \ldots, (1,4), \ldots \}.$$

2. If $f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$ are surjective functions, then the function $(f \times g) : \mathbb{N} \to X \times Y$ is surjective where $(f \times g)(m,n) := (f(m),g(n))$ for all $(m,n) \in \mathbb{N} \times \mathbb{N}$.

3. By assumption there exists surjective maps, $f_n : \mathbb{N} \to J_n$ for each $n \in \mathbb{N}$.

4. Let $h(n) := (a(n), b(n))$ be the bijection constructed for item 1. Then $f : \mathbb{N} \to X$ defined by $f(n) := f_{a(n)}(b(n))$ is a surjective map.

5. To see this let $f : \mathbb{N} \to X$ be a surjective map and let $g(x) := \min f^{-1}\{\{x\}\}$ for all $x \in X$. Then $g : X \to \mathbb{N}$ is an injective map. Let $S := g(X)$, then $g : X \to S \subset \mathbb{N}$ is a bijection. So it remains to show $S \sim \mathbb{N}$ or $S \sim J_n$ for some $n \in \mathbb{N}$. If $S$ is unbounded, then $S \sim \mathbb{N}$ as we have already seen. So it suffices to consider the case where $S$ is bounded. If $S$ is bounded by 1 then $S = \{1\} = J_1$ and we are done. Now assume the result is true if $S$ is bounded by $n \in \mathbb{N}$ and now suppose that $S$ is bounded by $n+1$. If $n+1 \notin S$, then $S$ is bounded by $n$ and so by induction, $S \sim J_k$ for some $k \leq n < n+1$. If $n+1 \in S$, then from above, $S \setminus \{n+1\} \sim J_n$ for some $k \leq n$, i.e. there exists a bijection, $f : J_k \to S \setminus \{n+1\}$. We then extend $f$ to $f_{k+1}$ by setting $f(k+1) := n+1$ which shows $J_{k+1} \sim S$.

6. We again prove this by induction on $n$. If $n = 1$, then $S = \{m\}$ for some $m \in \mathbb{N}$ which is bounded. So suppose for some $n \in \mathbb{N}$, every subset $S \subset \mathbb{N}$ with $S \sim J_n$ is bounded. Now suppose that $S \subset \mathbb{N}$ with $S \sim J_{n+1}$. Then

   $f(J_n) \sim J_n$ and hence $f(J_n)$ is bounded in $\mathbb{N}$. Then $\max f(J_n) \vee \{f(n)+1\}$ is an upper bound for $S$. This completes the inductive argument.

6. For each $n \in \mathbb{N}$ there exists and injection, $f_n : J_n \to X$. By replacing $X$ by $X_0 := \bigcup_{n \in \mathbb{N}} f_n(J_n)$ we may assume that $X = \bigcup_{n \in \mathbb{N}} f_n(J_n)$. Thus there exists a surjective map, $f : \mathbb{N} \to X$ by item 3. Let $g : X \to \mathbb{N}$ be defined by $g(x) := \min f^{-1}\{\{x\}\}$ for all $x \in X$ and let $S := g(X)$. To finish the proof we need only show that $S$ is unbounded. If $S$ were bounded, then we would find $k \in \mathbb{N}$ such that $J_k \sim S \sim X$. However this is impossible since $\text{card} \, J_n \leq \text{card} \, X = \text{card} \, J_k$ would imply $n \leq k$ even though $n$ can be chosen arbitrarily in $\mathbb{N}$.

7. If $g : X \to \mathbb{N}$ is an injective map then $g|_A : A \to \mathbb{N}$ is an injective map and therefore $A$ is countable.

**Lemma 5.27.** If $X$ is a countable set which contains $Y \subset X$ with $Y \sim \mathbb{N}$, then $X \sim \mathbb{N}$.

**Proof.** By assumption there is an injective map, $g : X \to \mathbb{N}$ and a bijective map, $f : \mathbb{N} \to Y$. It then follows that $g \circ f : \mathbb{N} \to \mathbb{N}$ is injective from which it follows that $g(X)$ is unbounded. Indeed, $(g \circ f)(J_n) \subset g(X)$ for all $n$ implies $\text{card} \, (J_n) \leq \text{card} \, (g(X))$ for all $n$ which implies $g(X)$ is unbounded by Corollary 5.21. Therefore $X \sim g(X) \sim \mathbb{N}$ by Lemma 5.25.

**Corollary 5.28.** We have $\text{card} \, (\mathbb{Q}) = \text{card} \, (\mathbb{N})$ and in fact, for any $a < b$ in $\mathbb{R}$, $\text{card} \, (\mathbb{Q} \cap (a,b)) = \text{card} \, (\mathbb{N})$.

**Proof.** First off $\mathbb{Q}$ is a countable since $\mathbb{Q}$ may be expressed as a countable union of countable sets;

$$\mathbb{Q} = \bigcup_{m \in \mathbb{N}} \left\{ \frac{n}{m} : n \in \mathbb{Z} \right\}.$$  

From this it follows that $\mathbb{Q} \cap (a,b)$ is countable for all $a < b$ in $\mathbb{R}$. As these sets are not finite, they must have the cardinality of $\mathbb{N}$.

**Theorem 5.29 (Uncountability results).** If $X$ is an infinite set and $Y$ is a set with at least two elements, then $Y^X$ is uncountable. In particular $2^X$ is uncountable for any infinite set $X$.

**Proof.** Let us begin by showing $\mathbb{N}^N = \{0,1\}^N$ is uncountable. For sake of contradiction suppose $f : \mathbb{N} \to \{0,1\}^N$ is a surjection and write $f(n)$ as $(f_1(n), f_2(n), f_3(n), \ldots)$. Now define $a \in \{0,1\}^N$ by $a_n := 1 - f_n(n)$. By construction $f_n(n) \neq a_n$ for all $n$ and so $a \notin f(N)$. This contradicts the assumption that $f$ is surjective and shows $\mathbb{N}^N$ is uncountable. For the general
case, since $Y_0^X \subset Y^X$ for any subset $Y_0 \subset Y$, if $Y_0^X$ is uncountable then so is $Y^X$. In this way we may assume $Y_0$ to be a two point set which may as well be $Y_0 = \{0,1\}$. Moreover, since $X$ is an infinite set, we may find an injective map $x: \mathbb{N} \to X$ and use this to set up an injection, $i: 2^\mathbb{N} \to 2^X$ by setting $i((A)) = \{x_n: n \in \mathbb{N}\} \subset X$ for all $A \subset \mathbb{N}$. If $2^\mathbb{N}$ were countable we could find a surjective map $f: 2^X \to \mathbb{N}$ in which case $f \circ i: 2^\mathbb{N} \to \mathbb{N}$ would be surjective as well. However this is impossible since we have already seen that $2^\mathbb{N}$ is uncountable.

**Corollary 5.30.** The set $(0,1) := \{a \in \mathbb{R} : 0 < a < 1\}$ is uncountable while $\mathbb{Q} \cap (0,1)$ is countable. More generally, for any $a < b$ in $\mathbb{R}$, $\text{card}(\mathbb{Q} \cap (a,b)) = \text{card}(\mathbb{N})$ while $\text{card}(\mathbb{Q}^c \cap (a,b)) > \text{card}(\mathbb{N})$.

**Proof.** From Section 3.4 the set $\{0,1,2\ldots,8\}^\mathbb{N}$ can be mapped injectively into $(0,1)$ and therefore it follows from Theorem 5.29 that $(0,1)$ is uncountable. For each $m \in \mathbb{N}$, let $A_m := \{\frac{m}{n} : n \in \mathbb{N}$ with $n < m\}$. Since $\mathbb{Q} \cap (0,1) = \cup_{m=1}^\infty X_m$ and $\#(X_m) < \infty$ for all $m$, another application of Theorem 5.26 shows $\mathbb{Q} \cap (0,1)$ is countable.

The fact that these results hold for any other finite interval follows from the fact that $f : (0,1) \to (a,b)$ defined by $f(t) := a + t(b-a)$ is a bijection.

**Definition 5.31.** We say a non-empty set $X$ is **infinite** if $X$ is not a finite set.

**Example 5.32.** Any unbounded subset, $S \subset \mathbb{N}$, is an infinite set according to Theorem 5.20.

**Theorem 5.33.** Let $X$ be a non-empty set. The following are equivalent:

1. $X$ is an infinite set,
2. $\text{card}(J_n) \leq \text{card}(X)$ for all $n \in \mathbb{N}$,
3. $\text{card}(\mathbb{N}) \leq \text{card}(X)$,
4. $\text{card}(X \setminus \{x\}) = \text{card}(X)$ for some (or all) $x \in X$.

**Proof.** 1. $\implies$ 2. Suppose that $X$ is an infinite set. We show by induction that $\text{card}(J_n) \leq \text{card}(X)$ for all $n \in \mathbb{N}$. Since $X$ is not empty, there exists $x \in X$ and we may define $f: J_1 \to X$ by $f(1) = x$ in order to learn $\text{card}(J_1) \leq \text{card}(X)$. Suppose we have shown $\text{card}(J_n) \leq \text{card}(X)$ for some $n \in \mathbb{N}$, i.e. there exists and injective map $f: J_n \to X$. If $f(J_n) = X$ it would follow that $\text{card}(X) = \text{card}(J_n)$ and would violated the assumption that $X$ is not a finite set. Thus there exists $x \in X \setminus f(J_n)$ and we may define $f': J_{n+1} \to X$ by $f'(J_n) = f$ and $f'(n+1) = x$. Then $f': J_{n+1} \to X$ is injective and hence $\text{card}(J_{n+1}) \leq \text{card}(X)$.

2. $\iff$ 3. This is the content of Theorem 5.18.

3. $\implies$ 4. Let $x_1 \in X$ and $f : \mathbb{N} \to X$ be an injective map such that $f(1) = x_1$. We now define a bijections, $\psi : X \to X \setminus \{x_1\}$ by

$$
\psi(x) = \begin{cases} x & \text{if } x \neq f(0) \\
 f(i+1) & \text{if } x = f(i) \in f(0) 
\end{cases}
$$

4. $\implies$ 1. We will prove the contrapositive. If $X$ is a finite and $x \in X$, we have seen that card$(X \setminus \{x\}) < \text{card}(X)$, namely $\#(X \setminus \{x\}) = \#(X) - 1$.

The next two theorems summarizes the properties of cardinalities that have been proven above.

**Theorem 5.34 (Cardinality/Counting Summary I).** Given a non-empty set $X$, then one and only one of the following statements holds;

1. There exists a unique $n \in \mathbb{N}$ such that $\text{card}(X) = \text{card}(J_n)$.
2. $X$ is uncountable.
3. $X$ is countable.

If $X$ satisfies case 1, we say $X$ is a **finite set.** If $X$ satisfies case 2 we say $X$ is a **countably infinite set** and if $X$ satisfies case 3, we say $X$ is an **uncountably infinite set.**

**Theorem 5.35 (Cardinality/Counting Summary II).** Let $X$ and $Y$ be sets and $S$ be a subset of $\mathbb{N}$.

1. If $S \subset \mathbb{N}$ is an unbounded set, then $\text{card}(S) = \text{card}(\mathbb{N})$.
2. If $S \subset \mathbb{N}$ is a bounded set then $\text{card}(S) = \text{card}(J_n)$ for some $n \in \mathbb{N}$.
3. If $\{X_k\}_{k=1}^\infty$ are subsets of $X$ such that $\text{card}(X_k) \leq \text{card}(\mathbb{N})$, then $\text{card}(\cup_{k=1}^\infty X_k) \leq \text{card}(\mathbb{N})$.
4. If $X$ and $Y$ are sets such that $\text{card}(X) \leq \text{card}(\mathbb{N})$ and $\text{card}(Y) \leq \text{card}(\mathbb{N})$, then $\text{card}(X \times Y) \leq \text{card}(\mathbb{N})$.
5. $\text{card}(\mathbb{Q}) = \text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})$.
6. For any $a < b$ in $\mathbb{R}$, $\text{card}(\mathbb{Q} \cap (a,b)) = \text{card}(\mathbb{N})$ while $\text{card}(\mathbb{Q}^c \cap (a,b)) > \text{card}(\mathbb{N})$.

5.4.1 Exercises

**Exercise 5.9.** Show that $\mathbb{Q}^n$ is countable for all $n \in \mathbb{N}$.

**Exercise 5.10.** Let $\mathbb{Q}[t]$ be the set of polynomial functions, $p$, such that $p$ has rational coefficients. That is $p \in \mathbb{Q}[t]$ iff there exists $n \in \mathbb{N}_0$ and $a_k \in \mathbb{Q}$ for $0 \leq k \leq n$ such that

$$
p(t) = \sum_{k=0}^n a_k t^k \text{ for all } t \in \mathbb{R}.
$$

Show $\mathbb{Q}[t]$ is a countable set.
Definition 5.36 (Algebraic Numbers). A real number, $x \in \mathbb{R}$, is called algebraic number, if there is a non-zero polynomial $p \in \mathbb{Q}[t]$ such that $p(x) = 0$. [That is to say, $x \in \mathbb{R}$ is algebraic if it is the root of a non-zero polynomial with coefficients from $\mathbb{Q}$.]

Note that for all $q \in \mathbb{Q}$, $p(t) := t - q$ satisfies $p(q) = 0$. Hence all rational numbers are algebraic. But there are many more algebraic numbers, for example $y^{1/n}$ is algebraic for all $y \geq 0$ and $n \in \mathbb{N}$.

Exercise 5.11. Show that the set of algebraic numbers is countable. [Hint: any polynomial of degree $n$ has at most $n$ real roots.] In particular, “most” irrational numbers are not algebraic numbers, i.e. there is still any uncountable number of non-algebraic numbers.