280B Homework Exercises (Winter 2012)
The problems below either come from the lecture notes or from Folland. At the moment in the lecture notes please use the following translation table for this course:

a) Random variable = Measurable function on some measure space.

b) Probability measure is a measure where the total measure of the space is 1.

c) $\mathbb{E}[X]$ or $\mathbb{E}X$ is short hand for

$$\int_{\Omega} X(\omega) \, d\mu(\omega),$$

i.e. it is simply shorthand notation for the integral of $X$ relative to a measure, in this case the measure is assumed to $\mu$ and the underlying set is $\Omega.$
Exercise 1.1 (Measurability). Please review or read Lemma 9.3 of the lecture notes. I will likely use this lemma without mention throughout the course.

Exercise 1.2 (Differentiating past the integral). Please review (or read if it is new to you) Corollary 10.30, Corollary 10.31, and Proposition 10.33 in the lecture notes (Ver. 1). Also review Section 10.3 of the lecture notes (Ver. 1).

Exercise 1.3 (Folland 2.28 on p. 60.). Compute the following limits and justify your calculations:

1. \( \lim_{n \to \infty} \int_0^\infty \frac{\sin(\frac{x}{n})}{(1 + x^2)} \, dx. \)
2. \( \lim_{n \to \infty} \int_0^1 \frac{1 + nx^2}{1 + x^2} \, dx. \)
3. \( \lim_{n \to \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1 + x^2)} \, dx. \)
4. For all \( a \in \mathbb{R} \) compute,
\[
f(a) := \lim_{n \to \infty} \int_a^\infty n(1 + n^2 x^2)^{-1} \, dx.
\]

[Hints: for parts 1. and 2. you might use the binomial expansion to estimate the denominators.]

Exercise 1.4. Let \( \Omega := \{1, 2, 3, 4\} \) and \( \mathcal{M} := \{A, B\} \) where \( A := \{1, 2\} \) and \( B := \{2, 3\} \).

a) Show \( \sigma(\mathcal{M}) = 2^\Omega. \)

b) Find two distinct probability measures, \( \mu \) and \( \nu \) on \( 2^\Omega \) such that \( \mu(A) = \nu(A) \) and \( \mu(B) = \nu(B) \), i.e.
\[
\int_{\Omega} f \, d\mu = \int_{\Omega} f \, d\nu
\] (1.1)
holds for all \( f \in \mathcal{M}. \)

Moral: the assumption that \( \mathcal{M} \) is multiplicative can not be dropped from multiplicative system theorem.

Exercise 1.5. Suppose that \( u : \mathbb{R} \to \mathbb{R} \) is a continuously differentiable function such that \( \dot{u}(t) \geq 0 \) for all \( t \) and \( \lim_{t \to \pm \infty} u(t) = \pm \infty \). Show that
\[
\int_{\mathbb{R}} f(x) \, dx = \int_{\mathbb{R}} f(u(t)) \dot{u}(t) \, dt
\] (1.2)
for all measurable functions \( f : \mathbb{R} \to [0, \infty] \). In particular applying this result to \( u(t) = at + b \) where \( a > 0 \) implies,
\[
\int_{\mathbb{R}} f(x) \, dx = a \int_{\mathbb{R}} f(at + b) \, dt.
\]
Exercise 1.6. Let \((\Omega, \mathcal{B}, P)\) be a probability space and \(X,Y : \Omega \to \mathbb{R}\) be a pair of random variables such that
\[
\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)g(X)]
\]
for every pair of bounded measurable functions, \(f,g : \mathbb{R} \to \mathbb{R}\). Show \(P(X = Y) = 1\). **Hint:** Let \(\mathbb{H}\) denote the bounded Borel measurable functions, \(h : \mathbb{R}^2 \to \mathbb{R}\) such that
\[
\mathbb{E}[h(X,Y)] = \mathbb{E}[h(X,X)].
\]
Use the multiplicative systems Theorem ?? to show \(\mathbb{H}\) is the vector space of all bounded Borel measurable functions. Then take \(h(x,y) = 1_{\{x=y\}}\).

Exercise 1.7 (Density of \(\mathcal{A} - \) simple functions). Let \((\Omega, \mathcal{B}, P)\) be a probability space and assume that \(\mathcal{A}\) is a sub-algebra of \(\mathcal{B}\) such that \(\mathcal{B} = \sigma(\mathcal{A})\). Let \(\mathbb{H}\) denote the bounded measurable functions \(f : \Omega \to \mathbb{R}\) such that for every \(\varepsilon > 0\) there exists an \(\mathcal{A} - \) simple function \(\varphi : \Omega \to \mathbb{R}\) such that \(\mathbb{E}|f - \varphi| < \varepsilon\). Show \(\mathbb{H}\) consists of all bounded measurable functions, \(f : \Omega \to \mathbb{R}\). **Hint:** let \(\mathbb{M}\) denote the collection of \(\mathcal{A} - \) simple functions.

Exercise 1.8 (Density of \(\mathcal{A}\) in \(\mathcal{B} = \sigma(\mathcal{A}) - \) NOT to be collected). Keeping the same notation as in Exercise 1.7 but now take \(f = 1_B\) for some \(B \in \mathcal{B}\) and given \(\varepsilon > 0\), write \(\varphi = \sum_{i=0}^n \lambda_i 1_{A_i}\) where \(\lambda_0 = 0\), \(\{\lambda_i\}_{i=1}^n\) is an enumeration of \(\varphi(\Omega) \setminus \{0\}\), and \(A_i = \{\varphi = \lambda_i\}\). Show 1.

\[
\mathbb{E}|1_B - \varphi| = P(A_0 \cap B) + \sum_{i=1}^n |1 - \lambda_i| P(B \cap A_i) + |\lambda_i| P(A_i \setminus B) \geq P(A_0 \cap B) + \sum_{i=1}^n \min\{P(B \cap A_i), P(A_i \setminus B)\}. \tag{1.3}
\]

2. Now let \(\psi = \sum_{i=0}^n \alpha_i 1_{A_i}\) with
\[
\alpha_i = \begin{cases} 1 & \text{if } P(A_i \setminus B) \leq P(B \cap A_i) \\ 0 & \text{if } P(A_i \setminus B) > P(B \cap A_i) \end{cases}
\]
Then show that
\[
\mathbb{E}|1_B - \psi| = P(A_0 \cap B) + \sum_{i=1}^n \min\{P(B \cap A_i), P(A_i \setminus B)\} \leq \mathbb{E}|1_B - \varphi|.
\]

Observe that \(\psi = 1_D\) where \(D = \bigcup_{i=0}^n A_i \in \mathcal{A}\) and so you have shown; for every \(\varepsilon > 0\) there exists a \(D \in \mathcal{A}\) such that
\[
P(B\Delta D) = \mathbb{E}|1_B - 1_D| < \varepsilon.
\]
[This problem is the same as one of the propositions in the notes!]

Exercise 1.9. Suppose that \(\{(X_i, \mathcal{B}_i)\}_{i=1}^n\) are measurable spaces and for each \(i\), \(\mathcal{M}_i\) is a multiplicative system of real bounded measurable functions on \(X_i\) such that \(\sigma(\mathcal{M}_i) = \mathcal{B}_i\) and there exist \(\chi_n \in \mathcal{M}_i\) such that \(\chi_n \to 1\) boundedly as \(n \to \infty\). Given \(f_i : X_i \to \mathbb{R}\) let \(f_1 \otimes \cdots \otimes f_n : X_1 \times \cdots \times X_n \to \mathbb{R}\) be defined by
\[
(f_1 \otimes \cdots \otimes f_n) (x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n).
\]
Show
\[
\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n := \{f_1 \otimes \cdots \otimes f_n : f_i \in \mathcal{M}_i \text{ for } 1 \leq i \leq n\}
\]
is a multiplicative system of bounded measurable functions on
\[
(X := X_1 \times \cdots \times X_n, \mathcal{B} := \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n)
\]
such that \(\sigma(\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n) = \mathcal{B}\). (It is enough to write your solution in the special case where \(n = 2\).)

\[1\] Recall from Definition ?? than \(f\) is an \(\mathcal{A} - \) simple function if \(f\) is a simple function such that \(f^{-1}(\{y\}) \in \mathcal{A}\) for all \(y \in \mathbb{R}\).
HomeWork # 2 Due Friday January 20, 2012

Exercise 2.2 (Look at but do not hand in). Folland 3.28 on p. 107.
Exercise 2.3 (Look at but do not hand in). Folland 3.29 on p. 107.
Exercise 2.5. Folland 3.33 on p. 108.
Exercise 2.9 (Look at but do not hand in.). Folland 3.40 on p. 108.
Exercise 3.1 (Global Integration by Parts Formula). Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are locally absolutely continuous functions\(^1\) such that $f'g$, $fg'$, and $fg$ are all Lebesgue integrable functions on $\mathbb{R}$. Prove the following integration by parts formula;

$$\int_{\mathbb{R}} f' (x) \cdot g (x) \, dx = - \int_{\mathbb{R}} f (x) \cdot g' (x) \, dx. \quad (3.1)$$

Similarly show that; if $f, g : [0, \infty) \to [0, \infty)$ are locally absolutely continuous functions such that $f'g$, $fg'$, and $fg$ are all Lebesgue integrable functions on $[0, \infty)$, then

$$\int_{0}^{\infty} f' (x) \cdot g (x) \, dx = - f (0) g (0) - \int_{0}^{\infty} f (x) \cdot g' (x) \, dx. \quad (3.2)$$

**Outline:**
1. First use the theory developed to see that Eq. (3.1) holds if $f (x) = 0$ for $|x| \geq N$ for some $N < \infty$.
2. Let $\psi : \mathbb{R} \to [0, 1]$ be a continuously differentiable function such that $\psi (x) = 1$ if $|x| \leq 1$ and $\psi (x) = 0$ if $|x| \geq 2$\(^2\). For any $\varepsilon > 0$ let $\psi_\varepsilon (x) = \psi (\varepsilon x)$ Write out the identity in Eq. (3.1) with $f (x)$ being replaced by $f (x) \psi_\varepsilon (x)$.
3. Now use the dominated convergence theorem to pass to the limit as $\varepsilon \downarrow 0$ in the identity you found in step 2.
4. A similar outline works to prove Eq. (3.2).

Exercise 3.2. Let $(X, d)$ be a metric space. Suppose that $\{x_n\}_{n=1}^{\infty} \subset X$ is a sequence and set $\varepsilon_n := d(x_n, x_{n+1})$. Show that for $m > n$ that

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} \varepsilon_k \leq \sum_{k=n}^{\infty} \varepsilon_k.$$ 

Conclude from this that if

$$\sum_{k=1}^{\infty} \varepsilon_k = \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$$

then $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Moreover, show that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence and $x = \lim_{n \to \infty} x_n$, then

$$d(x, x_n) \leq \sum_{k=n}^{\infty} \varepsilon_k.$$ 

Exercise 3.3. Show that $(X, d)$ is a complete metric space iff every sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ is a convergent sequence in $X$. You may find it useful to prove the following statements in the course of the proof.

\(^1\) This means that $f$ and $g$ restricted to any bounded interval in $\mathbb{R}$ are absolutely continuous on that interval.

\(^2\) You may assume the existence of such a $\psi$, we will deal with this later.
1. If \( \{x_n\} \) is Cauchy sequence, then there is a subsequence \( y_j := x_{n_j} \) such that \( \sum_{j=1}^{\infty} d(y_{j+1}, y_j) < \infty \).

2. If \( \{x_n\}_{n=1}^{\infty} \) is Cauchy and there exists a subsequence \( y_j := x_{n_j} \) of \( \{x_n\} \) such that \( x = \lim_{j \to \infty} y_j \) exists, then \( \lim_{n \to \infty} x_n \) also exists and is equal to \( x \).

**Exercise 3.4.** Suppose that \( f : [0, \infty) \to [0, \infty) \) is a \( C^2 \) – function such that \( f(0) = 0, f' > 0 \) and \( f'' \leq 0 \) and \( (X, \rho) \) is a metric space. Show that \( d(x, y) = f(\rho(x, y)) \) is a metric on \( X \). In particular show that

\[
d(x, y) := \frac{\rho(x, y)}{1 + \rho(x, y)}
\]

is a metric on \( X \). (Hint: use calculus to verify that \( f(a + b) \leq f(a) + f(b) \) for all \( a, b \in [0, \infty) \).)

**Exercise 3.5.** Let \( \{(X_n, d_n)\}_{n=1}^{\infty} \) be a sequence of metric spaces, \( X := \prod_{n=1}^{\infty} X_n \), and for \( x = (x(n))_{n=1}^{\infty} \) and \( y = (y(n))_{n=1}^{\infty} \) in \( X \) let

\[
d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x(n), y(n))}{1 + d_n(x(n), y(n))}.
\]

Show:

1. \( (X, d) \) is a metric space,
2. a sequence \( \{x_k\}_{k=1}^{\infty} \subset X \) converges to \( x \in X \) iff \( x_k(n) \to x(n) \in X_n \) as \( k \to \infty \) for each \( n \in \mathbb{N} \) and
3. \( X \) is complete if \( X_n \) is complete for all \( n \).

**Exercise 3.6 (Look at but do not hand in).** Let \( (X, \|\cdot\|) \) be a normed space over \( \mathbb{F} (\mathbb{R} \text{ or } \mathbb{C}) \). Show the map

\[
(\lambda, x, y) \in \mathbb{F} \times X \times X \to x + \lambda y \in X
\]

is continuous relative to the norm on \( \mathbb{F} \times X \times X \) defined by

\[
\|(\lambda, x, y)\|_{\mathbb{F} \times X \times X} := |\lambda| + \|x\| + \|y\|.
\]

**Exercise 3.7.** Let \( X = \mathbb{N} \) and for \( p, q \in [1, \infty) \) let \( \|\cdot\|_p \) denote the \( \ell^p(\mathbb{N}) \) – norm. Show \( \|\cdot\|_p \) and \( \|\cdot\|_q \) are inequivalent norms for \( p \neq q \) by showing

\[
\sup_{f \neq 0} \frac{\|f\|_p}{\|f\|_q} = \infty \text{ if } p < q.
\]

**Exercise 3.8 (Look at but do not hand in.).** Let \( d : C(\mathbb{R}) \times C(\mathbb{R}) \to [0, \infty) \) be defined by

\[
d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n},
\]

where \( \|f\|_n := \sup\{|f(x)| : |x| \leq n\} = \max\{|f(x)| : |x| \leq n\} \).

1. Show that \( d \) is a metric on \( C(\mathbb{R}) \).
2. Show that a sequence \( \{f_n\}_{n=1}^{\infty} \subset C(\mathbb{R}) \) converges to \( f \in C(\mathbb{R}) \) as \( n \to \infty \) iff \( f_n \) converges to \( f \) uniformly on bounded subsets of \( \mathbb{R} \).
3. Show that \((C(\mathbb{R}), d)\) is a complete metric space.

[The solution to this exercise is similar to the solution to Exercise 3.5]

**Exercise 3.9.** Let \( X = C([0, 1], \mathbb{R}) \) and for \( f \in X \), let

\[
\|f\|_1 := \int_0^1 |f(t)| \, dt.
\]

Show that \((X, \|\cdot\|_1)\) is normed space and show by example that this space is not complete.
Exercise 3.10. By making the change of variables, \( u = \ln x \), prove the following facts:

\[
\begin{align*}
\int_0^{1/2} x^{-a} |\ln x|^b \, dx < \infty & \iff a < 1 \text{ or } a = 1 \text{ and } b < -1 \\
\int_2^{\infty} x^{-a} |\ln x|^b \, dx < \infty & \iff a > 1 \text{ or } a = 1 \text{ and } b < -1 \\
\int_0^1 x^{-a} |\ln x|^b \, dx < \infty & \iff a < 1 \text{ and } b > -1 \\
\int_1^{\infty} x^{-a} |\ln x|^b \, dx < \infty & \iff a > 1 \text{ and } b > -1.
\end{align*}
\]

Suppose \( 0 < p_0 < p_1 \leq \infty \) and \( m \) is Lebesgue measure on \((0, \infty)\). Use the above results to manufacture a function \( f \) on \((0, \infty)\) such that \( f \in L^p((0, \infty), m) \) iff (a) \( p \in (p_0, p_1) \), (b) \( p \in [p_0, p_1] \) and (c) \( p = p_0 \).