The problems below either come from the lecture notes or from Folland.
HomeWork #1 Due Friday April 6, 2012

Exercise 1.1. Suppose that $X$ is a separable Banach space. Show there exists $\varphi_n \in X^*$ such that
\[ \|x\| = \sup_n |\varphi_n(x)| \quad \text{for all} \quad x \in X. \tag{1.1} \]

Use this to conclude that Borel $\sigma$–algebra of $X$ ($\mathcal{B}_X := \sigma(\text{open balls})$) and the $\sigma$–algebra generated by $\varphi \in X^*$ are the same, i.e. $\sigma(X^*) = \mathcal{B}_X$. So if $(\Omega, \mathcal{F}, \mu)$ is a measure space and $X$ is separable, a function $u : \Omega \to X$ is weakly integrable iff $u : \Omega \to X$ is $\mathcal{F}/\mathcal{B}_X$–measurable and
\[ \int_{\Omega} \|u(\omega)\| \, d\mu(\omega) < \infty. \]

Exercise 1.2. Suppose that $X$ and $Y$ are Banach spaces, $T \in L(X,U)$, $(\Omega, \mathcal{B}, \mu)$ is a measure space, and $u : \Omega \to X$ is an integrable function in the sense that:
1. For all $\lambda \in X^*$, $\lambda \circ u \in L^1(\mu)$ and
2. there exists a unique element $x \in X$ (denoted by $\int_{\Omega} u(\omega) \, d\mu(\omega)$) such that $f(\int_{\Omega} u \, d\mu) = \int_{\Omega} [f \circ u] \, d\mu$ for all $f \in X^*$.
\[ \int_{\Omega} T[u(\omega)] \, d\mu(\omega) = T \int_{\Omega} u(\omega) \, d\mu(\omega). \tag{1.2} \]

[For situations where the hypothesis of this exercise hold see Theorems ?? and ??.]

Exercise 1.3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and write $L^p$ for $L^p(\Omega, \mathcal{F}, \mu)$. Show;
1. if $f \in L^p \cap L^\infty$ for some $p < \infty$, then $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$.
2. If we further assume $\mu(X) < \infty$, show $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$ for all measurable functions $f : X \to \mathbb{C}$. In particular, $f \in L^\infty$ iff $\lim_{q \to \infty} \|f\|_q < \infty$.

Hints: Use Corollary ?? on interpolation of $L^p$–norms (see Eq. (1.3) below) to show $\lim \sup_{q \to \infty} \|f\|_q < \|f\|_\infty$. To show $\lim \inf_{q \to \infty} \|f\|_q \geq \|f\|_\infty$, let $M < \|f\|_\infty$ and make use of Chebyshev’s inequality. Here we define
\[ \lim \sup_{q \to \infty} \|f\|_q = \lim_{M \to \infty} \sup_{q \geq M} \|f\|_q \quad \text{and} \quad \lim \inf_{q \to \infty} \|f\|_q = \lim_{M \to \infty} \inf_{q \geq M} \|f\|_q. \]

Exercise 1.4. Use the inequality
\[ s_1 \ldots s_n \leq \sum_{i=1}^n \frac{s_i^{p_i}}{p_i} \quad \text{for} \quad s_i \geq 0 \quad \text{and} \quad \sum_{i=1}^n \frac{1}{p_i} = 1. \]
as in the notes to give another proof of the statement that

$$\left\| \prod_{i=1}^{n} f_i \right\| \leq \prod_{i=1}^{n} \|f_i\|_{p_i}, \quad \text{where} \quad \sum_{i=1}^{n} p_i^{-1} = r^{-1}.$$  

**Exercise 1.5 (Part of Folland 6.3 on p. 186).** Prove Eq. (??) in Corollary ???. In detail suppose that 0 < $p_0 < p_1 \leq \infty$, $\lambda \in (0, 1)$ and $p_\lambda \in (p_0, p_1)$ be defined by

$$\frac{1}{p_\lambda} = \frac{1 - \lambda}{p_0} + \frac{\lambda}{p_1}$$

as in Eq. (1.4). Show

$$\|f\|_{p_\lambda} \leq \max (\lambda, (1 - \lambda)) \left(\|f\|_{p_0} + \|f\|_{p_1}\right).$$

**Hint:** Use the inequality

$$st \leq \frac{s^a}{a} + \frac{t^b}{b},$$

where $a, b \geq 1$ with $a^{-1} + b^{-1} = 1$ are chosen appropriately.(see Lemma ?? for Eq. (??)) applied to the right side of the interpolation inequality;

$$\|f\|_{p_\lambda} \leq \|f\|^{1-\lambda}_{p_0} \|f\|^{\lambda}_{p_1}. \quad (1.3)$$

**Proposition 1.1.** Suppose that $0 < p_0 < p_1 \leq \infty$, $\lambda \in (0, 1)$ and $p_\lambda \in (p_0, p_1)$ be defined by

$$\frac{1}{p_\lambda} = \frac{1 - \lambda}{p_0} + \frac{\lambda}{p_1} \quad (1.4)$$

with the interpretation that $\lambda/p_1 = 0$ if $p_1 = \infty$. Then $L^{p_\lambda} \subseteq L^{p_0} + L^{p_1}$, i.e. every function $f \in L^{p_\lambda}$ may be written as $f = g + h$ with $g \in L^{p_0}$ and $h \in L^{p_1}$. For $1 \leq p_0 < p_1 \leq \infty$ and $f \in L^{p_0} + L^{p_1}$ let

$$\|f\| := \inf \{\|g\|_{p_0} + \|h\|_{p_1} : f = g + h\}.$$  

Then $(L^{p_0} + L^{p_1}, \|\cdot\|)$ is a Banach space and the inclusion map from $L^{p_\lambda}$ to $L^{p_0} + L^{p_1}$ is bounded; in fact $\|f\| \leq 2 \|f\|_{p_\lambda}$ for all $f \in L^{p_\lambda}$.

**Proof.** Let $M > 0$, then the local singularities of $f$ are contained in the set $E := \{|f| > M\}$ and the behavior of $f$ at “infinity” is solely determined by $f$ on $E'$, hence let $g = f1_{E}$ and $h = f1_{E'}$, so that $f = g + h$. By our earlier discussion we expect that $g \in L^{p_0}$ and $h \in L^{p_1}$ and this is the case since

$$\|g\|^p_{p_0} = \int |f|^p_{p_0} 1_{|f| > M} = M^{p_0} \int \left| \frac{f}{M} \right|^p_{p_0} 1_{|f| > M} \leq M^{p_0} \int \left| \frac{f}{M} \right|^p_{p_\lambda} 1_{|f| > M} \leq M^{p_0 - p_\lambda} \|f\|^{p_\lambda}_{p_\lambda} < \infty$$

and

$$\|h\|^p_{p_1} = \|f1_{|f| \leq M}\|^p_{p_1} = \int |f|^p_{p_1} 1_{|f| \leq M} = M^{p_1} \int \left| \frac{f}{M} \right|^p_{p_1} 1_{|f| \leq M} \leq M^{p_1} \int \left| \frac{f}{M} \right|^p_{p_\lambda} 1_{|f| \leq M} \leq M^{p_1 - p_\lambda} \|f\|^{p_\lambda}_{p_\lambda} < \infty.$$  

Moreover this shows

$$\lambda = \frac{p_0}{p_\lambda} \cdot \frac{p_1 - p_\lambda}{p_1 - p_0},$$

1 A little algebra shows that $\lambda$ may be computed in terms of $p_0$, $p_\lambda$ and $p_1$ by
\[ \|f\| \leq M^{1-p_\lambda/p_0} \|f\|_{p_\lambda}^{p_\lambda/p_0} + M^{1-p_\lambda/p_1} \|f\|_{p_\lambda}^{p_\lambda/p_1}. \]

Taking \( M = \alpha \|f\|_\lambda \) with \( \alpha > 0 \) implies

\[ \|f\| \leq \left( \alpha^{1-p_\lambda/p_0} + \alpha^{1-p_\lambda/p_1} \right) \|f\|_\lambda \]

and then taking \( \alpha = 1 \) shows \( \|f\| \leq 2 \|f\|_\lambda \). The proof that \((L^{p_0} + L^{p_1}, \|\cdot\|)\) is a Banach space is left as Exercise 1.6 to the reader.

**Exercise 1.6.** Show \((L^{p_0} + L^{p_1}, \|\cdot\|)\) is a Banach space. **Hint:** you may find using Theorem ?? (on the sum – criteria for completeness) is helpful here.

**Exercise 1.7.** Folland 6.9 on p. 186.

**Exercise 1.8.** Folland 6.10 on p. 186. Use the strong form of the Dominated Convergence Theorem ??.

**Exercise 1.9 (Fatou’s Lemma).** If \( f_n \geq 0 \) and \( f_n \to f \) in measure, then \( \int f \leq \lim \inf_{n \to \infty} \int f_n \).

**Exercise 1.10.** Folland 6.27 on p. 196. **Hint:** see Theorem 6.20 in Folland (or Theorem ?? or the lecture notes.)

1.1 Optional exercises to ponder but not hand in.

**Exercise 1.11 (Folland 6.5 on p. 186. Do not hand in this problem.).** Suppose \( 0 < p < q \leq \infty \). Then \( L^p \not\subset L^q \) iff \( X \) contains sets of arbitrarily small positive measure. Also \( L^q \not\subset L^p \) iff \( X \) contains sets of arbitrarily large finite measure.

**Exercise 1.12.** Show that Egoroff’s Theorem remains valid when the assumption \( \mu(X) < \infty \) is replaced by the assumption that \( |f_n| \leq g \in L^1 \) for all \( n \). **Hint:** make use of Theorem ?? applied to \( f_n|_{X_k} \) where \( X_k := \{|g| \geq k^{-1}\} \).
HomeWork #2  Due Friday April 13, 2012

2.1 Baire Category Theorem Exercises

Exercise 2.1. Let \((X, \tau)\) be a topological space and \(E, G\) be subsets of \(X\). Prove;

1. \(E\) is nowhere dense iff \(E^c\) has dense interior.
2. \(G \subseteq X\) is dense iff \(G \cap W \neq \emptyset\) for all \(\emptyset \neq W \subseteq_o X\).

Exercise 2.2. Recall that \(R \subseteq X\) is a residual set if \(R^c\) is meager, i.e. \(R^c\) is the countable union of nowhere dense sets. Show \(R\) is residual iff \(R = \bigcap_{n=1}^{\infty} A_n\) for some \(\{A_n\}_{n=1}^{\infty}\) such that each \(A_n\) has dense interior, i.e. \(A_n^o = X\).

Exercise 2.3. Suppose that \((X, \tau_X)\) and \((Y, \tau_Y)\) are two topological spaces and \(\varphi : X \rightarrow Y\) is a homeomorphism, i.e. \(\varphi\) is continuous, invertible, and \(\varphi^{-1}\) is continuous. Show \(\varphi (A^o) = [\varphi (A)]^o\) and \(\varphi (\overline{A}) = \overline{\varphi (A)}\) for all \(A \subseteq X\).

Exercise 2.4. Let \((X, \|\cdot\|)\) be a normed space and \(E \subseteq X\) be a subspace.

1. If \(E\) is closed and proper subspace of \(X\) then \(E\) is nowhere dense.
2. If \(E\) is a proper finite dimensional subspace of \(X\) then \(E\) is nowhere dense.

Exercise 2.5. Now suppose that \((X, \|\cdot\|)\) is an infinite dimensional Banach space. Show that \(X\) can not have a countable algebraic basis. More explicitly, there is no countable subset \(S \subseteq X\) such that every element \(x \in X\) may be written as a finite linear combination of elements from \(S\). Hint: make use of Exercise 2.4 and the Baire category theorem.

2.2 Open Mapping and Closed Operator Exercises

Exercise 2.6. Let \(T : X \rightarrow Y\) be a linear map between normed vector spaces, show \(T\) is closed (i.e. has a closed graph) iff for all convergent sequences \(\{x_n\}_{n=1}^{\infty} \subseteq X\) such that \(\{Tx_n\}_{n=1}^{\infty} \subseteq Y\) is also convergent, we have \(\lim_{n \to \infty} Tx_n = T (\lim_{n \to \infty} x_n)\) \footnote{Compare this with the statement that \(T\) is continuous iff for every convergent sequences \(\{x_n\}_{n=1}^{\infty} \subseteq X\) we have \(\{Tx_n\}_{n=1}^{\infty} \subseteq Y\) is necessarily convergent and \(\lim_{n \to \infty} Tx_n = T (\lim_{n \to \infty} x_n)\).}

Exercise 2.7. Let \(X = \ell^1 (\mathbb{N})\) equipped with the \(\ell^1 (\mathbb{N})\) – norm \((\|f\| := \sum_{n=1}^{\infty} |f(n)|)\) and let

\[
Y = \left\{ f \in X : \sum_{n=1}^{\infty} n |f(n)| < \infty \right\}.
\]

We view \(Y\) as a normed space with the same \(\ell^1 (\mathbb{N})\) – norm as \(X\). Futher let \(T : Y \rightarrow X\) be the linear transformation defined by \((Tf)(n) = nf(n)\) for all \(f \in Y\) and \(n \in \mathbb{N}\). Show:
1. $Y$ is a proper dense subspace of $X$ and in particular $Y$ is not complete.
2. $T : Y \to X$ is a closed operator and is not bounded.
3. $T : Y \to X$ is algebraically invertible, $S := T^{-1} : X \to Y$ is bounded and surjective but is not an open mapping.

Exercise 2.8. Let $X$ be a vector space equipped with two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$ such that $\|\cdot\|_1 \leq \|\cdot\|_2$ and $X$ is complete relative to both norms. Show there is a constant $C < \infty$ such that $\|\cdot\|_2 \leq C \|\cdot\|_1$.

Exercise 2.9 (No slowest decay rate). Show that it is impossible to find a “magic sequence,” $\{a_n\}_{n \in \mathbb{N}} \subset (0, \infty)$, with the following property: if $\{\lambda_n\}_{n \in \mathbb{N}}$ is a sequence in $\mathbb{C}$, then $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ iff $\sup_n a_n^{-1} |\lambda_n| < \infty$. (Poetically speaking, there is no “slowest rate” of decay for the summands of absolutely convergent series.)

Outline: For sake of contradiction suppose such a magic sequence $\{a_n\}_{n \in \mathbb{N}} \subset (0, \infty)$ were to exists.

1. For $f \in \ell^\infty(\mathbb{N})$, let $(Tf)(n) := a_n f(n)$ for $n \in \mathbb{N}$. Verify that $Tf \in \ell^1(\mathbb{N})$ and $T : \ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})$ is a bounded linear operator.
2. Show $T : \ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})$ must be an invertible operator and that $T^{-1} : \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ is necessarily bounded, i.e. $T : \ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})$ is a homeomorphism.
3. Arrive at a contradiction by showing either that $T^{-1}$ is not bounded or by using the fact that, $D$, the set of finitely supported sequences, is dense in $\ell^1(\mathbb{N})$ but not in $\ell^\infty(\mathbb{N})$.

Exercise 2.10 (Do not hand in). Let $X = C([0,1])$ and $Y = C^1([0,1]) \subset X$ with both $X$ and $Y$ being equipped with the uniform norm. Let $T : Y \to X$ be the linear map, $Tf = f'$. Here $C^1([0,1])$ denotes those functions, $f \in C^1((0,1)) \cap C([0,1])$ such that $f'(1) := \lim_{x \uparrow 1} f'(x)$ and $f'(0) := \lim_{x \downarrow 0} f'(x)$ exist.

1. $Y$ is a proper dense subspace of $X$ and in particular $Y$ is not complete.
2. $T : Y \to X$ is a closed operator which is not bounded.

2.3 Uniform Boundedness Principle Exercises

Exercise 2.11. Suppose $T : X \to Y$ is a linear map between two Banach spaces such that $f \circ T \in X^*$ for all $f \in Y^*$. Show $T$ is bounded.

Exercise 2.12. Suppose $T_n : X \to Y$ for $n \in \mathbb{N}$ is a sequence of bounded linear operators between two Banach spaces such that $\lim_{n \to \infty} T_n x$ exists for all $x \in X$. Show $Tx := \lim_{n \to \infty} T_n x$ defines a bounded linear operator from $X$ to $Y$.

Exercise 2.13. Let $X$ be a Banach space, $\{T_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ be two sequences of bounded operators on $X$ such that $T_n \to T$ and $S_n \to S$ strongly, and suppose $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\lim_{n \to \infty} \|x_n - x\| = 0$. Show:

1. $T_n S_n \to TS$ strongly as $n \to \infty$ and that
2. $\lim_{n \to \infty} \|T_n x_n - T x\| = 0$.

Exercise 2.14. Let $X, Y$ and $Z$ be Banach spaces and $B : X \times Y \to Z$ be a bilinear map such that $B(x, \cdot) \in L(Y, Z)$ and $B(\cdot, y) \in L(X, Z)$ for all $x \in X$ and $y \in Y$. Show there is a constant $M < \infty$ such that $\|B(x, y)\| \leq M \|x\| \|y\|$ for all $(x, y) \in X \times Y$ and conclude from this that $B : X \times Y \to Z$ is continuous.