1.10. Homework #10 (Due in section 3/10/2011).

- Look at (do not hand in) from chapter I (p. 50-53): I.5.P9
- Hand in from chapter I (p. 50-53): I.5.E.4, I.5.E.5 parts a and b only, I.5.P8
- Hand in the following exercises as well.

Exercises 1-4 refer to the following Markov matrix:

\[
P = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 1/4 & 3/4 & 0
\end{bmatrix}
\]

(1.4)

We will let \( \{X_n\}_{n=0}^{\infty} \) denote the Markov chain associated to \( P \).

**Exercise 1.** Make a jump diagram for this matrix and identify the recurrent and transient classes. Also find the invariant distributions for the chain restricted to each of the recurrent classes.

**Exercise 2.** Find all of the invariant distributions for \( P \).

**Exercise 3.** Compute the hitting probabilities, \( h_5 = P_5(X_n \text{ hits } \{3, 4\}) \) and \( h_6 = P_6(X_n \text{ hits } \{3, 4\}) \).

**Exercise 4.** Find \( \lim_{n \to \infty} P_6(X_n = j) \) for \( j = 1, 2, 3, 4, 5, 6 \).

**Exercise 5.** Suppose that \( \{T_1, T_2\} \) are independent random variables with \( T_i \overset{d}{=} E(\lambda_i) \) with \( \lambda_i > 0 \) for \( i = 1, 2 \). Show

\[
P(T_1 + T_2 \in (w, w + dw)) = 1_{w \geq 0} \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[ e^{-\lambda_1 w} - e^{-\lambda_2 w} \right] dw,
\]

i.e. show

\[
\mathbb{E}[f(T_1 + T_2)] = \int_0^\infty f(w) \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[ e^{-\lambda_1 w} - e^{-\lambda_2 w} \right] dw
\]

for all bounded or non-negative functions \( f \). If \( \lambda_1 = \lambda_2 = \lambda \) the above formula should be interpreted as

\[
\mathbb{E}[f(T_1 + T_2)] = \int_0^\infty f(w) \lambda^2 w e^{-\lambda w} \, dw.
\]

(See Exercise 7 for an extension of this last formula.)

**Exercise 6.** For \( n \in \mathbb{N} \) and \( t > 0 \) show

\[
V_n(t) := \int_{0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq t} ds_1 \cdots ds_n = \frac{t^n}{n!}
\]

**Hints:** first observe that \( V_1(t) = t \). Now show

\[
V_n(t) = \int_0^t V_{n-1}(s) \, ds
\]

and complete the proof by induction.

**Exercise 7.** Suppose that \( \{T_i\}_{i=1}^n \) are i.i.d. exponential random times with parameter \( \lambda \) and let \( W_n = T_1 + \cdots + T_n \).

Shown

\[
P(W_n \in (w, w + dw)) = \frac{\lambda^n w^{n-1}}{(n-1)!} e^{-\lambda w} dw,
\]

i.e. show

\[
\mathbb{E}[f(W_n)] = \int_0^\infty f(w) \frac{\lambda^n w^{n-1}}{(n-1)!} e^{-\lambda w} dw \text{ for all } f \geq 0.
\]

**Hints:** 1) write out \( \mathbb{E}[f(W_n)] \) as an \( n \)-fold iterated integral over \( t_1, \ldots, t_n \geq 0 \). Then make the change of variables \( s_i = t_1 + \cdots + t_i \) for \( i = 1, 2, \ldots, n \) (you can do this one by one) and observe the new integral is now over \( 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n < \infty \). 2) The integrals involving \( s_1, \ldots, s_{n-1} \) may now be computed using Exercise 6.