Theorem 13.1 (General $2 \times 2$ inverse). Let $A$ be a non-zero $2 \times 2$ matrix written as
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
and set
\[
\delta = \det(A) := ad - bc.
\]
Then $A^{-1}$ exists iff $\det(A) \neq 0$ and in this case
\[
A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

Proof. If
\[
C := \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},
\]
then a simple computation shows;
\[
CA = AC = \delta I. \tag{13.1}
\]
So if $A^{-1}$ exists, then $A \neq 0$ and hence $C \neq 0$. Since
\[
\delta A^{-1} = A^{-1} (\delta I) = A^{-1} (AC) = C
\]
it follows that $\delta \neq 0$.

Conversely if $\delta \neq 0$ then it follows from Eq. (13.2) that $A^{-1} = \frac{1}{\delta} C$ as claimed.

Alternative Proof. Given $y \in \mathbb{R}^2$ we will solve the best we can the equation $Ax = y$. In components this becomes;

\[
ax_1 + bx_2 = y_1 \tag{13.3}
\]
\[
cx_1 + dx_2 = y_2. \tag{13.4}
\]
Performing
\[
d \begin{bmatrix} 13.3 \\ 13.4 \end{bmatrix} - b \begin{bmatrix} 13.3 \\ 13.4 \end{bmatrix} \implies (ad - bc) x_1 = dy_1 - by_2
\]
\[
-c \begin{bmatrix} 13.3 \\ 13.4 \end{bmatrix} + a \begin{bmatrix} 13.3 \\ 13.4 \end{bmatrix} \implies (ad - bc) x_2 = ay_2 - cy_1.
\]
So if $Ax = y$ then,
\[
\det(A)x = \begin{bmatrix} dy_1 - by_2 \\ -cy_1 + ay_2 \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} y_1
\]
\[
= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} y.
\]

Case 1. If $\det(A) = 0$ then the vector $y$ would have to satisfy
\[
\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} y = 0
\]
and this means $A$ is not onto, i.e. $Ax = y$ does not have a solution for all $y$ and hence $A$ is not invertible.

Case 2. If $\det(A) \neq 0$ we have found
\[
x = A^{-1}y = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} y.
\]

It is now straightforward to verify that
\[
\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det(A)I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]
so that $A^{-1}A = I = AA^{-1}$.

Proposition 13.2 (Properties of the inverse). Suppose $A$ and $B$ are invertible $n \times n$ matrices. Then;

1. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.
2. $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
3. $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof. 1. We need only check that $(A^{-1}) A = I = A (A^{-1})$ which is certainly true.

2. We need only check that $(AB) (B^{-1}A^{-1}) = I = (B^{-1}A^{-1}) (AB)$ which is again true.
3. Again it suffices to show $A^T (A^{-1})^T = I = (A^{-1})^T A^T$. This is seen as follows

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$  

\[ \square \]

**Proposition 13.3.** Suppose that $A$ and $B$ are $n \times n$ matrices such that $AB = I$, then $BA = I$, i.e. $B = A^{-1}$.  

**Proof.** First proof. If $y \in \mathbb{R}^n$ and $x = By$, then $Ax = ABy = Iy = y$ which shows that $A$ is onto which implies that $A$ has $n - \text{ pivots which implies } \text{Nul} (A) = \{0\}.$ If $x \in \mathbb{R}^n$ we have $ABAx = IAx = Ax$, so that $A(BAx - x) = 0$. As $\text{Nul} (A) = \{0\}$ we may conclude that $BAx - x = 0$ for all $x \in \mathbb{R}^n$, i.e. $BA = I$. 

Second proof. If $Bx = 0$ then $0 = A0 = ABx = x$ which implies $\text{Nul} (B) = 0$. Therefore $B$ has $n - \text{ pivots which implies } B$ is onto as well. Thus we may define $C$ so that $Bc_i = e_i$ for all $i$ and for this $C$ we will have $BC = I$. However,

$$C = IC = (AB)C = ABC = AI = A$$

and therefore $BA = I$ as desired.  

\[ \square \]

**Theorem 13.4.** Suppose that $A$ is a $n \times n$ matrix. Then $A^{-1}$ exists iff the equation $Ax = y$ has a unique solution $x$ for all $y \in \mathbb{R}^n$ iff $rref(A) = I$. Moreover $A^{-1} = [b_1|\ldots|b_n] =: B$ where $Ab_i = e_i$ for $i = 1, 2, \ldots, n$.  

**Proof.** This has already been proved but here is the proof again. If $A^{-1}$ exists and $Ax = y$, then 

$$x = Ix = (A^{-1}A)x = A^{-1}(Ax) = A^{-1}y$$

and if $y \in \mathbb{R}^n$ and $x := A^{-1}y$ we find

$$Ax = A(A^{-1}y) = (AA^{-1})y = Iy = y.$$ 

Conversely suppose $Ax = y$ has a unique solution for all $y \in \mathbb{R}^n$ and let $B$ be as above. Then

$$AB = [Ab_1|\ldots|Ab_n] = [e_1|\ldots|e_n] = I.$$ 

So to finish the proof we must show $BA = I$ as well. Now apply Proposition 13.3.  

\[ \square \]

**Corollary 13.5.** If $A$ is an $n \times n$ matrix is invertible, then

$$\text{rref}([A|I]) = [I|A^{-1}].$$

**Proof.** Recall that $A^{-1}e_i = b_i$ where $Ab_i = e_i$ and hence

$$\text{rref} ([A|e_i]) = [I|b_i] = [I|A^{-1}e_i].$$

Therefore

$$\text{rref} ([A|e_1|e_2|\ldots|e_n]) = [I|A^{-1}e_1|A^{-1}e_2|\ldots|A^{-1}e_n] = [I|A^{-1}].$$

\[ \square \]

**Theorem 13.6 (Invertible Matrix Theorem 9, p. 129).** Let $A$ be a square matrix then the following are equivalent;

1. $A^{-1}$ exists.
2. a) $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one to one, i.e. $Ax = y$ has at most one solution.
   b) $\text{Nul} (A) = \{0\}$
   c) $Ax = 0$ has only the trivial solution $x = 0$.
   d) the columns of $A$ are linearly independent.
   e) $\text{rref}(A)$ has a pivot in every column.
   f) $\text{rref}(A) = I$
3. a) $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is onto.
   b) $Ax = y$ is consistent for all $y \in \mathbb{R}^n$.
   c) $\text{Ran} (A) = \text{col} (A) = \mathbb{R}^n$.
   d) $\text{rref}(A)$ has a pivot in every row.
   e) $\text{rref}(A) = I$
4. There is a matrix $C$ such that $CA = I$.
5. There is a matrix $D$ such that $AD = I$.
6. $A^T$ is invertible.

13.1 Vector Spaces

Let us recall the following basic properties about $\mathbb{R}^n$.

**Theorem 13.7 (See top of p. 32).** For all $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$;

1. **Associativity of addition:** $x + (y + z) = (x + y) + z$.
2. **Commutativity of addition:** $y + z = z + y$.
3. **Identity element of addition:** $0 + y = y$ for all $y \in \mathbb{R}^n$.
4. **Inverse elements of addition:** $-y + y = 0$ for all $y \in \mathbb{R}^n$. 

...
5. **Distributivity of scalar multiplication with respect to vector addition:** \( a(y + z) = ay + az \).

6. **Distributivity of scalar multiplication with respect to field addition** \( (a + b)y = ay + by \).

7. **Compatibility of scalar multiplication with the multiplication on \( \mathbb{R} \)**: \( a(by) = (ab)y \).

8. **Identity element of scalar multiplication** \( 1y = y \) for all \( y \in \mathbb{R}^n \).

We now turn these properties into a definition.

**Definition 13.8 (See Definition on p. 217).** A **vector space** is a non-empty set \( V \) of objects, called vectors, equipped with an addition operation “+” and scalar (\( = \mathbb{R} \) or maybe \( \mathbb{C} \)) multiplication “·” satisfying all of the properties above: i.e. For all \( u, v, w \in V \) and \( a, b \in \mathbb{R} \):

1. **Associativity of addition:** \( u + (v + w) = (u + v) + w \).
2. **Commutativity of addition:** \( v + w = w + v \).
3. **Identity element of addition:** There is an element \( 0 \in V \) such that \( 0 + v = v \) for all \( v \).
4. **Inverse elements of addition:** \( -v + v = 0 \) for all \( v \in V \). (In fact \( -v = (-1) \cdot v \).)
5. **Distributivity of scalar multiplication with vector addition:** \( a \cdot (v + w) = a \cdot v + a \cdot w \).
6. **Distributivity of scalar multiplication with respect to field addition** \( (a + b) \cdot v = a \cdot v + b \cdot v \).
7. **Compatibility of scalar multiplication with the multiplication on \( \mathbb{R} \)**: \( a \cdot (b \cdot v) = (ab) \cdot v \).
8. **Identity element of scalar multiplication** \( 1 \cdot v = v \) for all \( v \in V \).

**Example 13.9.** The infinite blackboard with geometric addition and scalar multiplication.

**Example 13.10 (The Main Umbrella Example).** Let \( D \) be a non-empty set and let \( V \) denote all functions, \( f : D \to \mathbb{R} \). For \( f, g \in V \) and \( \lambda \in \mathbb{R} \) we define \( f + g \) and \( \lambda \cdot f \) by

\[
(f + g)(t) = f(t) + g(t) \quad \text{(addition in } \mathbb{R})
\]

\[
(\lambda \cdot f)(t) = \lambda f(t) \quad \text{(multiplication in } \mathbb{R}).
\]

It can now be checked that \( V \) is a vector space so that functions have now become vectors! Essentially all other examples of vector spaces we give will be related to an example of this form.

**Example 13.11.** \( \mathbb{R}^3 = \{x : \{1, 2, 3\} \to \mathbb{R}\} \) and more generally \( \mathbb{R}^n = \{x : \{1, 2, \ldots, n\} \to \mathbb{R}\} \).

**Example 13.12.** The vector space of \( 2 \times 2 \) matrices;

\[ M_{2 \times 2} = \{ A : A \text{ is a } 2 \times 2 \text{ - matrix }\} = \{ A : \{(1,1), (1,2), (2,1), (2,2)\} \to \mathbb{R}\} \).

This can be generalized.

**Definition 13.13 (Subspace).** Let \( V \) be a vector space. A non-empty subset, \( H \subset V \), is a **subspace** of \( V \) if \( H \) is closed under addition and scalar multiplication.