Test #1 Review Day 10/13/2010

Definition 9.1. If $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ then

1. Linear systems like

   \begin{align*}
   a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
   a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
   & \vdots \\
   a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
   \end{align*}

   are equivalent to the matrix equation $A\mathbf{x} = \mathbf{b}$ where

   \[ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = m \times n \text{- coefficient matrix,} \]

   \[ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \]

2. $A\mathbf{x}$ is a linear combination of the columns of $A$ –

   \[ A\mathbf{x} = \sum_{i=1}^{n} x_ia_i. \]

   and $T(\mathbf{x}) := A\mathbf{x}$ defines a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$, i.e. $T$ preserves vector addition and scalar multiplication.

3. $\text{span} \{a_1, \ldots, a_n\} = \{ b \in \mathbb{R}^m : A\mathbf{x} = \mathbf{b} \text{ has a solution} \}

   = \{ b \in \mathbb{R}^m : A\mathbf{x} = \mathbf{b} \text{ is consistent} \}

   = \text{Ran}(A) = \{ \mathbf{b} = A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \}.

4. $\text{Nul}(A) := \{ \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0} \}

   = \{ \text{all solutions to the homogeneous equation: } A\mathbf{x} = 0 \}

5. Theorem: If $A\mathbf{p} = \mathbf{b}$ then the general solution to $A\mathbf{x} = \mathbf{b}$ is of the form

   $\mathbf{x} = \mathbf{p} + \mathbf{v}_h$ where $\mathbf{v}_h$ is a generic solution to $A\mathbf{v}_h = \mathbf{0}$, i.e. $\mathbf{x} = \mathbf{p} + \mathbf{v}_h$ with $\mathbf{v}_h \in \text{Nul}(A)$.

6. You should know the definition of linear independence (dependence), i.e. $\Gamma := \{ a_1, \ldots, a_n \} \subset \mathbb{R}^m$ are linearly independent iff the only solution to $\sum_{i=1}^{n} x_ia_i = 0$ is $\mathbf{x} = 0$. Equivalently put,

   \[ \Gamma := \{ a_1, \ldots, a_n \} \subset \mathbb{R}^m \text{ are L.I. } \iff \text{Nul}(A) = \{ \mathbf{0} \} \]

   \[ \iff A\mathbf{x} = 0 \text{ has only the trivial solution.} \]

7. You should know to perform row reduction in order to put a matrix into it reduced row echelon form.

8. You should be able to write down the general solution to the equation $A\mathbf{x} = \mathbf{b}$ and find equation that $\mathbf{b}$ must satisfy so that $A\mathbf{x} = \mathbf{b}$ is consistent.

9. You should be able to find the eigenvalues of $2 \times 2$ matrices, i.e. those $\lambda$ such that $\text{Nul}(A - \lambda I) \neq 0$.

10. Theorem: Let $A$ be a $m \times n$ matrix and $U := \text{ref}(A)$. Then;
a) $Ax = b$ has a solution iff $\text{rref}([A|b])$ does not have a pivot in the last column.

b) $Ax = b$ has a solution for every $b \in \mathbb{R}^m$ iff $\text{span}\{a_1, \ldots, a_n\} = \text{Ran}(A) = \mathbb{R}^m$ iff $U := \text{rref}(A)$ has a pivot in every row, i.e. $U$ does not contain a row of zeros.

c) If $m > n$ (i.e. there are more equations than unknowns), then $Ax = b$ will be inconsistent for some $b \in \mathbb{R}^m$. This is because there can be at most $n$-pivots and since $m > n$ there must be a row of zeros in $\text{rref}(A)$.

d) $Ax = b$ has at most one solution iff $\text{Nul}(A) = 0$ iff $Ax = 0$ has only the trivial solution iff $\{a_1, \ldots, a_n\}$ are linearly independent, iff $\text{rref}(A)$ has no free variables — i.e. there is a pivot in every column.

e) If $m < n$ (i.e. there are fewer equations than unknowns) then $Ax = 0$ will always have a non-trivial solution or equivalently put the columns of $A$ are necessarily linearly dependent. This is because there can be at most $m$ pivots and so at least one column does not have pivot and there is at least one free variable.

9.2 How to study:

Look over your homework problems, do the practice exam, and do more problems from the book. Here is one more to look at which reviews many of the notions we have had so far.

Example 9.2. Compute $T\left(\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right)$ given that

$$
T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.
$$

We first solve for $x_1$ and $x_2$ such that

$$
\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.
$$

Since

$$
\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}
$$

it follows that we must take $x_1 = 3$ and $x_2 = -3$. Therefore,

$$
\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
$$

and since $T$ is linear we find:

$$
T\left(\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right) = 3T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) - 3T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)
$$

$$
= 3\begin{bmatrix} 6 \\ 8 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}.
$$

This all worked because

$$
\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \in \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}
$$

and the coefficients $x_1 = 3$ and $x_2 = -3$ were uniquely determined because

$$
\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}
$$

are linearly independent.

Alternatively method. By inspection we have

$$
\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.
$$

Using the linearity of $T$ then implies,

$$
T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - T\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 6 \\ 8 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}
$$

and therefore

$$
T\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right) = 3T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}.
$$