Definition. Let \( f : \mathbb{C} \to \mathbb{R} \). We say that \( \lim_{z \to \infty} f(z) = \infty \) if for any \( M \), there exists \( R \) such that if \( |z| \geq R \), then \( f(z) \geq M \).

Theorem 8.8. Let \( a_0, a_1, \ldots, a_n \) be complex numbers, where \( n \geq 1 \) and \( a_n \neq 0 \). Then the polynomial defined by
\[
P(z) = \sum_{k=0}^{n} a_k z^k \quad \text{for} \quad z \in \mathbb{C}
\]
has a zero. I.e., there exists \( z \in \mathbb{C} \) such that \( P(z) = 0 \).

Proof. \(|P(z)|\) acheives its minimum.

**Step 1.** Since \( a_n \neq 0 \), we may assume without loss of generality that \( a_n = 1 \) (by dividing \( P(z) \) by \( a_n \)). Let \( r = |z| \). Since
\[
P(z) = z^n + \sum_{k=0}^{n-1} a_k z^k ,
\]
we have
\[
|P(z)| \geq |z^n| - \sum_{k=0}^{n-1} |a_k z^k| = r^n - \sum_{k=0}^{n-1} |a_k| r^k = r^n \left(1 - \sum_{k=0}^{n-1} |a_k| r^{k-n}\right).
\]
Since each power \( k - n \) of \( r \) in the sum is negative, we have
\[
\lim_{r \to \infty} \sum_{k=0}^{n-1} |a_k| r^{k-n} = 0.
\]
This implies that
\[
\lim_{z \to \infty} |P(z)| = \infty
\]
in the sense of the definition at the top of the page.

Define \( M = 2 |P(0)| \). Since \( \lim_{z \to \infty} |P(z)| = \infty \), there exists \( R \) such that if \( |z| \geq R \), then \( f(z) \geq M \). Consider the set \( K = \{ z \in \mathbb{C} : |z| \leq R \} \). Since \( K \) is closed and bounded, \( K \) is compact. Hence the restriction of \( f \) to \( K \) acheives its minimum (since \( f \) is continuous). That is, there exists \( z_0 \in \mathbb{C} \) such that
\[
|P(z_0)| = \inf_{z \in K} |P(z)| .
\]
On the other hand,
\[
\inf_{z \in \mathbb{C} - K} |P(z)| \geq M = 2 |P(0)| \geq 2 \inf_{z \in K} |P(z)| = 2 |P(z_0)|
\]
since $0 \in K$. Hence

$$|P(z_0)| = \inf_{z \in \mathbb{C}} |P(z)|.$$  

This completes Step 1.

**Step 1.** If there does not exist a zero of $P(z)$, then we have a contradiction.

Suppose that there does not exist a zero of $P(z)$. Then $|P(z_0)| > 0$. So we can adjust the polynomial $P(z)$ by defining

$$Q(z) = \frac{P(z + z_0)}{P(z_0)}.$$  

We have $Q(0) = 1$ and $|Q(z)| \geq 1 = |Q(0)|$ for all $z \in \mathbb{C}$ (this true because $|P(z + z_0)| \geq |P(z_0)|$ for all $z$).

Since $Q$ is a degree $n$ polynomial, we have

$$Q(z) = \sum_{k=0}^{n} b_k z^k,$$

where $b_n \neq 0$. Note that $b_0 = Q(0) = 1$. Let $j \geq 1$ be the largest integer such that $b_1 = \cdots = b_{j-1} = 0$ and $b_j \neq 0$. We write

$$Q(z) = b_n z^n + \cdots + b_j z^j + 1.$$  

Since $-\frac{b_j}{|b_j|}$ is a unit vector, we may write it as

$$-\frac{b_j}{|b_j|} = e^{-i(j\theta)},$$  

where $i$ is the imaginary unit complex number. That is,

$$b_j e^{i(j\theta)} = -|b_j|.$$  

Consider points of the form $z = re^{i\theta}$, where $r > 0$. We have

$$Q(re^{i\theta}) = b_n r^n e^{i(n\theta)} + \cdots + b_j r^j e^{i(j\theta)} + 1.$$  

So (assuming that $-|b_j| r^j + 1 > 0$, i.e., $r < |b_j|^{-1/j}$)

$$|Q(re^{i\theta})| \leq \sum_{k=j+1}^{n} |b_k r^k e^{i(k\theta)}| + |b_j r^j e^{i(j\theta)} + 1|$$

$$= \sum_{k=j+1}^{n} |b_k| r^k - |b_j| r^j + 1$$

$$= \left( \sum_{k=j+1}^{n} |b_k| r^{k-j} - |b_j| \right) r^j + 1$$

where $|b_j| > 0$. Since

$$\lim_{r \to 0} \left( \sum_{k=j+1}^{n} |b_k| r^{k-j} - |b_j| \right) = -|b_j| < 0$$

and since the function $r \mapsto \sum_{k=j+1}^{n} |b_k| r^{k-j} - |b_j|$ is continuous, we have that for $r > 0$ sufficiently small and $\theta$ as above,

$$|Q(re^{i\theta})| < 1.$$  

This contradicts $|Q(z)| \geq 1$ for all $z \in \mathbb{C}$. □