**Math 140B HW1 Answers**

**HW1, #1.** Let $h$ and $k$ be real functions defined in a neighborhood $N$ of $a \in \mathbb{R}$ satisfying \( \lim_{x \to a} h(x) = 0 \) and \( |k(x)| \leq C \) for all $x \in N$, where $C$ is a constant. Prove that

\[
\lim_{x \to a} h(x) k(x) = 0.
\]

**Answer:** Let $\varepsilon > 0$. Since \( \lim_{x \to 0} h(x) = 0 \) there exist \( \delta > 0 \) such that if \( 0 < |x - 0| < \delta \), then \( |h(x) - 0| < \frac{\varepsilon}{C} \). Hence, if \( 0 < |x - 0| < \delta \), then using \( |k(x)| \leq C \) we obtain

\[
|h(x)k(x) - 0| = |h(x)||k(x)| < \frac{\varepsilon}{C} \cdot C = \varepsilon.
\]

This proves \( \lim_{x \to 0} h(x)k(x) = 0 \).

**HW1, #2.** Let $\alpha > 0$ be a constant.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by \( f(x) = |x|^{\alpha} \sin x \). Prove that $f'(0)$ exists and equals 0.

**Answer:** For $x \neq 0$ we have

\[
\left| \frac{f(x) - f(0)}{x - 0} \right| = |x|^{\alpha} \left| \frac{\sin x}{x} \right| \leq |x|^{\alpha}.
\]

Since \( \lim_{x \to 0} |x|^{\alpha} = 0 \), by the squeeze theorem we conclude that \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0 \).

Note that #1 is a version of the squeeze theorem. As such, alternatively we could have argued as follows. For \( x \neq 0 \) we have \( \frac{f(x) - f(0)}{x - 0} = h(x)k(x) \), where \( h(x) = |x|^{\alpha} \) and \( k(x) = \frac{\sin x}{x} \). Since \( \lim_{x \to a} h(x) = 0 \) and \( |k(x)| \leq 1 \), by #1 we have that \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} h(x)k(x) = 0 \).

\( \square \)

(b) Define $g : \mathbb{R} \to \mathbb{R}$ by

\[
g(x) = \begin{cases} 
|x|^{1+\alpha} \cos \left( \frac{1}{x} \right) & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

Prove that \( g'(0) = 0 \).

**Answer:** For \( x \neq 0 \) we have

\[
\left| \frac{g(x) - g(0)}{x - 0} \right| = \frac{|x|^{1+\alpha}}{|x|} \left| \cos \left( \frac{1}{x} \right) \right| \leq |x|^{\alpha}.
\]

Again, by the squeeze theorem we conclude that \( g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = 0 \).

\( \square \)

**HW1, #3.** (Compare with #1 on p. 114.) Let $f$ and $g$ be real functions defined on $\mathbb{R}$ satisfying:

1. \( |f(x) - f(y)| \leq g(x - y) \) for all \( x, y \in \mathbb{R} \),

2. \( \lim_{x \to 0} \frac{g(x)}{x} = 0 \).

Prove that $f$ is constant.

**Answer:** Again an application of the squeeze theorem. We have

\[
\left| \frac{f(x) - f(y)}{x - y} \right| \leq \left| \frac{g(x - y)}{x - y} \right|,
\]

for all $x, y \in \mathbb{R}$.
so \( \lim_{x \to 0} \frac{g(x)}{x} = 0 \) implies \( f'(y) = \lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0 \). This implies by Theorem 5.11(b) that \( f \) is constant.

**HW1, #4.** (a) Let \( g : \mathbb{R} \to \mathbb{R} \) be a differentiable function satisfying \( g'(x) > 0 \) for all \( x \neq 0 \). Prove that \( g \) is one-to-one.

**Answer:** Let \( x_1, x_2 \in \mathbb{R} \) with \( x_1 < x_2 \).

**Case 1.** \( 0 \notin (x_1, x_2) \). By Theorem 5.10, there exists \( c \in (x_1, x_2) \) such that

\[
 g(x_2) - g(x_1) = (x_2 - x_1) g'(c) .
\]

Since \( c \neq 0 \), we have \( g'(c) > 0 \), which implies \( g(x_2) - g(x_1) > 0 \).

**Case 2.** \( 0 \in (x_1, x_2) \). By applying Theorem 5.10 on \([x_1, 0] \) and \([0, x_2] \), we conclude that there exist \( c_1 \in (x_1, 0) \) and \( c_2 \in (0, x_2) \) such that

\[
 g(0) - g(x_1) = (0 - x_1) g'(c_1) > 0 ,
\]

\[
 g(x_2) - g(0) = (x_2 - 0) g'(c_2) > 0 .
\]

Thus

\[
 g(x_2) - g(x_1) = (g(x_2) - g(0)) + (g(0) - g(x_1)) > 0 .
\]

Hence, \( g \) is one-to-one (in fact, it is strictly increasing). \( \square \)

(b) **(Compare with #3 on p. 114.)** Let \( n \) be a positive integer and let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function satisfying \( |f'(x)| \leq Mx^{2n} \), where \( M \) is a nonnegative constant. Prove for \( \varepsilon \in \mathbb{R} \) with \( |\varepsilon| \) sufficiently small (how small depends on \( n \) and \( M \)) that the function

\[
 f_{\varepsilon}(x) = x^{2n+1} + \varepsilon f(x)
\]

is one-to-one on \( \mathbb{R} \).

**Answer:** Since

\[
 f'_\varepsilon(x) = (2n + 1)x^{2n} + \varepsilon f'(x)
\]

and \( |f'(x)| \leq Mx^{2n} \) for all \( x \in \mathbb{R} \) and where \( M \geq 0 \), we have

\[
 f'_\varepsilon(x) \geq (2n + 1)x^{2n} - \varepsilon Mx^{2n} .
\]

Choose \( \varepsilon \in (0, \frac{2n+1}{M}) \) if \( M > 0 \) and choose any \( \varepsilon > 0 \) if \( M = 0 \). Then \( f'_\varepsilon(x) \geq ax^{2n} \), where \( a = 2n + 1 - \varepsilon M > 0 \). Hence \( f'_\varepsilon(x) > 0 \) for \( x \neq 0 \). We may now apply part (a) to conclude that \( f_{\varepsilon} \) is one-to-one. \( \square \)

**HW1, #5.** **(This is part of #8 on pp. 114–115.)** Suppose \( f \) is differentiable and \( f' \) is continuous on \([a, b] \) and let \( \varepsilon > 0 \). Prove that there exists \( \delta > 0 \) such that

\[
 \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon
\]

whenever \( 0 < |t - x| < \delta \), \( a \leq x \leq b \), \( a \leq t \leq b \).

**Answer:** Let \( \varepsilon > 0 \). Since \( f' \) is continuous on \([a, b] \) and since \([a, b] \) is compact, by Theorem 4.19 \( f' \) is uniformly continuous. Then there exists \( \delta > 0 \) such that

\[
 |f'(y) - f'(x)| < \varepsilon \quad \text{whenever } |y - x| < \delta .
\]
Let \( a \leq x, t \leq b \). Then there exists \( c \) between \( x \) and \( t \) such that
\[
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(c) - f'(x)|.
\]
Hence, if \( 0 < |t - x| < \delta \), then \( |c - x| < \delta \), which implies \( |f'(c) - f'(x)| < \varepsilon \). \( \square \)

**HW1, #6.** Suppose that \( b \in \mathbb{R} \), \( f \) is a twice-differentiable real function on \((b, \infty)\), and define
\[
M_k = \sup_{x \in (b, \infty)} |f^{(k)}(x)| < \infty \quad \text{for } k = 0, 1, 2.
\]
Prove that
\[
M_1^2 \leq 4M_0M_2.
\]
**Hint:** Let \( a \in (b, \infty) \) and \( x = a + h \), where \( h > 0 \). Then there exists \( c \in (a, a + h) \) such that
\[
f(a + h) = f(a) + f'(a)h + \frac{f''(c)}{2}h^2.
\]
Show that this implies
\[
|f'(a)| \leq \frac{2M_0}{h} + \frac{M_2}{2}h.
\]
This inequality is true for any \( h > 0 \). What choice of \( h \) yields the smallest value for the right-hand side?

**Answer:** It suffices to prove the theorem for \( b = 0 \). Since \( M_2 > 0 \), we have \( M_0 > 0 \). Let \( a \in (0, \infty) \) and let \( x = a + h \), where \( h > 0 \). By Taylor’s Theorem 5.15, there exists \( c \in (a, a + h) \) such that
\[
f(a + h) = f(a) + f'(a)h + \frac{f''(c)}{2}h^2.
\]
That is,
\[
f'(a) = \frac{f(a + h) - f(a)}{h} - \frac{f''(c)}{2}h.
\]
Taking the supremum of the absolute value of this yields
\[
|f'(a)| \leq \frac{2M_0}{h} + \frac{M_2}{2}h
\]
for any \( h > 0 \). Choosing the minimizing value, which is given by \( \frac{h}{2} = \left( \frac{M_0}{M_2} \right)^{1/2} \), we obtain
\[
|f'(a)| \leq 2 \left( \frac{M_0M_2}{2} \right)^{1/2}.
\]
We conclude that
\[
M_1^2 = \sup_{a \in (0, \infty)} |f'(a)|^2 \leq 4M_0M_2. \quad \square
\]

**HW1, #7.** (Problem #23 on p. 117) The function \( f \) defined by \( f(x) = \frac{x^3 + 1}{3} \) has three fixed points \( \alpha, \beta, \gamma \), where
\[
-2 < \alpha < -1, \quad 0 < \beta < 1, \quad 1 < \gamma < 2.
\]
For arbitrarily chosen \( x_1 \), define the sequence \( \{x_n\} \) by setting \( x_{n+1} = f(x_n) \).
(a) If \( x_1 < \alpha \), prove that \( x_n \to -\infty \) as \( n \to \infty \).
(b) If \( \alpha < x_1 < \gamma \), prove that \( x_n \to \beta \) as \( n \to \infty \).
(c) If \( \gamma < x_1 \), prove that \( x_n \to +\infty \) as \( n \to \infty \).

Thus \( \beta \) can be located by this method, but \( \alpha \) and \( \gamma \) cannot.

**Answer:** (Pedantic.)

\[
\text{Graph of } y = \frac{x^3 + 1 - 3x}{3}.
\]

Define
\[
g(x) = f(x) - x = \frac{x^3 + 1 - 3x}{3}.
\]

Then
\[
g(x_n) = f(x_n) - x_n = x_{n+1} - x_n.
\]

Furthermore, the three zeros of \( g \) are \( \alpha \), \( \beta \), \( \gamma \). We have
\[
g(x) < 0 \text{ for } x < \alpha,
g(x) > 0 \text{ for } \alpha < x < \beta,
g(x) < 0 \text{ for } \beta < x < \gamma,
g(x) > 0 \text{ for } x > \gamma.
\]

(a) Suppose \( n \) is such that \( x_n < \alpha \). Then \( g(x_n) < 0 \), which says that \( x_{n+1} < x_n \). One can easily show by induction that if \( x_1 < \alpha \), then the sequence \( \{x_n\} \) is strictly decreasing. We have for \( n \geq 1 \),
\[
\frac{x_{n+1} - x_{n+2}}{x_n - x_{n+1}} = \frac{f(x_n) - f(x_{n+1})}{x_n - x_{n+1}} = f'(c)
\]
for some \( c \in (x_{n+1}, x_n) \) by the mean value theorem. Now \( f'(c) = c^2 > x_1^2 > 1 \) since \( c < x_1 < \alpha < -1 \). From this it is easy to deduce that \( x_n \to -\infty \).

(b) Similarly to (a), if \( x_1 > \gamma \), then \( \{x_n\} \) is strictly increasing and for \( n \geq 1 \),
\[
\frac{x_{n+1} - x_{n+2}}{x_n - x_{n+1}} = f'(c) = c^2 > x_1^2 > 1,
\]
where \( c > x_1 \). This implies \( x_n \to \infty \).

(b) Suppose \( \alpha < x_1 < \gamma \). If \( \alpha = \beta \), then \( x_n \equiv \beta \).

(i) Suppose \( \alpha < x_1 < \beta \). Now suppose that \( n \) is such that \( \alpha < x_n < \beta \). Then \( x_{n+1} - x_n = g(x_n) > 0 \), so that \( x_{n+1} > x_n \). Moreover,
\[
f(x_n) < f(\beta) = \beta
\]
since $f$ is strictly increasing and $x_n < \beta$. Since \{x_n\} is a bounded increasing sequence in $(\alpha, \beta)$, it converges to some number $x_\infty \in (\alpha, \beta]$. Moreover, by the continuity of $f$, we have

$$f(x_\infty) = f(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = x_\infty.$$  

Since $\beta$ is the only fixed point of $f$ in $(\alpha, \beta]$, we conclude that $x_\infty = \beta$.  

(ii) The case where $\beta < x_1 < \gamma$ is similar. \(\Box\)

**HW1, #8. (#26 on pp. 119.)** Suppose $f$ is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number $A$ such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

**Hint:** Fix $x_0 \in [a, b]$, let

$$M_0 = \sup_{x \in [a, x_0]} |f(x)|, \quad M_1 = \sup_{x \in [a, x_0]} |f'(x)|.$$  

Show that if $x \in [a, x_0]$, then

$$|f(x)| \leq M_1 (x_0 - a) \leq A (x_0 - a) M_0.$$  

Then show that $M_0 = 0$ if $A (x_0 - a) < 1$. Conclude that $f(x) = 0$ for all $x \in [a, x_1]$. Finally, show by continuing in this way that we obtain $f(x) = 0$ for all $x \in [a, b]$.

**Answer:** Let $x_0 \in [a, b]$, let $M_0 = \sup_{y \in [a, x_0]} |f(y)|$, and let $M_1 = \sup_{y \in [a, x_0]} |f'(y)|$. Let $x \in [a, x_0]$. Then

$$|f(x)| = \left| f(x) - f(a) \right| \leq \sup_{a \leq y \leq b} |f'(y)||x - a| \leq M_1 (x - a) \leq M_1 (x_0 - a) \leq A M_0 (x_0 - a),$$

where the last inequality is because $|f'(y)| \leq A |f(y)|$ for $y \in [a, b]$.

Let $M = \sup_{y \in [a, b]} |f(y)|$. Let $x_0 = \min \left\{ a + \frac{1}{2M}, b \right\}$, so that $x_0 \leq b$ and $x_0 - a \leq \frac{1}{2M}$. Since $M_0 \leq M$, we have for all $x \in [a, x_0]$,

$$|f(x)| \leq A M_0 (x_0 - a) \leq \frac{A M_0}{2M} \leq \frac{A}{2} = \frac{1}{2} \sup_{y \in [a, x_0]} |f(y)|.$$  

This implies $\sup_{y \in [a, x_0]} |f(y)| = 0$, so that $f(x) = 0$ for all $x \in [a, x_0]$.

Assume $x_0 \neq b$ (otherwise we are done). By induction define $x_{n+1} = \min \left\{ x_n + \frac{1}{2M}, b \right\}$. By the same argument as above applied to the interval $[x_n, x_{n+1}]$, we have that if $f(x) = 0$ for all $x \in [a, x_n]$, then $f(x) = 0$ for all $x \in [a, x_{n+1}]$. We conclude that $f(x) = 0$ for all $x \in [a, x_n]$ for all $n \geq 1$. The result follows since there exists $n$ such that $x_n = b$. \(\Box\)