Math 142A. Instructor: Chow
Homework #1. Due 4:00 pm on Monday, July 6, 2015.

1.1 #2, 11, 13, 14, 15. 1.2 #1, 2, 3, 4(a). 1.3 #7, 11, 14, 17, 18.
2.1 #1, 2, 6, 8, 11.

1.1.1. (a) False. \( S = [0,1) \).
(b) True. If \( x \in S \), then \( x > 0 \), so that \( x \geq 0 \). (and \( S \) is nonempty)
Hence 0 is a lower bound for \( S \). Hence 0 \( \leq \inf S \).
(c) True. \( \sup S \) is an upper bound for \( S \) (the least). Hence, \( \sup S \geq x \)
for all \( x \in S \). Since \( B \subset S \), we conclude that \( \sup S \geq x \) for all \( x \in B \).
That is, \( \sup S \) is an upper bound for \( B \). So \( \sup S \geq \sup B \).

1.1.11. (a) Let \( x \in \mathbb{Q} \) and let \( y \in \mathbb{R} \setminus \mathbb{Q} \). Suppose \( x + y \in \mathbb{Q} \). Then
\( y = (x + y) + (-x) \in \mathbb{Q} \), which is a contradiction. Hence \( x + y \in \mathbb{R} \setminus \mathbb{Q} \).
(b) Let \( x \in \mathbb{Q} \) and let \( y \in (\mathbb{R} \setminus \mathbb{Q}) \) with \( x \neq 0 \) and \( y \neq 0 \). Suppose \( xy \in \mathbb{Q} \).
Then \( y = \frac{1}{x} \cdot xy \in \mathbb{Q} \), which is a contradiction. Hence \( xy \in \mathbb{R} \setminus \mathbb{Q} \).

1.1.13. Since \( S \) is nonempty, there exists \( x \in S \). Hence \( \inf S \leq x \leq \sup S \).

1.1.14. Let \( x \in S \). Then \( \inf S \leq x \leq \sup S \). Since \( \inf S = \sup S \), this
implies \( x = \inf S = \sup S \). Since \( S \) is nonempty, \( S = \{ \inf S \} \).

1.1.15. Suppose \( S \) has a maximum. Let \( c \) be its maximum. Then \( c \in S \)
and \( c \) is an upper bound for \( S \).

1.2.1. (a) False. \( \mathbb{Z} \cap (0,1) \) is empty.
(b) False. (positive real numbers) \( \cap (-1,0) \) is empty.
(c) True. Let \( a < b \) be real numbers. By Theorem 1.9, there exists \( c \in \mathbb{Q} \)
with \( a < c < b \). Suppose \( c \in \mathbb{N} \) (otherwise we are done). Then by Theorem
1.9 there exists \( d \in \mathbb{Q} \) with \( d \in (c,b) \cap (c,c + 1) \). Since \( d \in (c,c + 1) \), \( d \) is
not an integer.

1.2.2. Let \( S \) be a nonempty set of integers that is bounded below. Define
\( T = \{ n \in \mathbb{Z} : -n \in S \} \). Then \( T \) is bounded above and nonempty. (Reason:
If \( c \) is a lower bound for \( S \), then \( -c \) is an upper bound for \( T \).) By Proposition
1.7, \( T \) has a maximum, which we call \( M \). So \( M \in T \) and \( M \) is an upper
bound for \( T \). Then \( -M \) is the minimum of \( S \). (Reason: \( -M \in S \) and \( -M \)
is a lower bound for \( S \).)

1.2.3. (\( \Rightarrow \)) Suppose that \( S \) has a minimum. Let \( m \) be the minimum of
\( S \). Then \( m \) is a lower bound and \( m \in S \). If \( c \) is a lower bound for \( S \), then
\( m \in S \) implies \( c \leq m \). By this and \( m \) being a lower bound, \( m = \inf S \). Hence
\( \inf S \in S \).

(\( \Leftarrow \)) Suppose \( \inf S \in S \). Then \( \inf S \) is a lower bound for \( S \) and \( \inf S \in S \),
so that \( \inf S \) is the minimum of \( S \).

1.2.4(a). \( S = \{ 1/n : n \in \mathbb{N} \} \). Maximum = 1. No minimum. \( \inf S = 0 \).
1 \in S \text{ and } 1 \geq 1/n \text{ for all } n \in \mathbb{N}. \text{ So } 1 \text{ is the maximum of } S.

0 \leq 1/n \text{ for all } n \in \mathbb{N}. \text{ So } 0 \text{ is a lower bound for } S. \text{ Suppose } c > 0. \text{ Then by the Archimedean Property, there exists } n \in \mathbb{N} \text{ such that } 1/n < c. \text{ So } c \text{ is not a lower bound for } S. \text{ Hence } 0 \text{ is the greatest lower bound for } S, \text{ i.e., } 0 = \inf S. \text{ Since } 0 \notin S, \text{ the minimum of } S \text{ does not exist.}

1.2.6. \text{ Let } a \in \mathbb{R} \text{ and } S = \{x \in \mathbb{Q} : x < a\}. \text{ If } x \in S, \text{ then } x \leq a. \text{ Hence } a \text{ is an upper bound for } S. \text{ Suppose } b < a. \text{ Then by Theorem 1.9, there exists } c \in \mathbb{Q} \text{ with } b < c < a. \text{ Hence } b \text{ is not an upper bound for } S. \text{ We conclude that } a = \sup S.

1.3.7. \text{ The triangle inequality says that for any } x, y \in \mathbb{R} \text{ we have } |x + y| \leq |x| + |y|. \text{ Since } a = (a + b) + (-b), \text{ by the triangle inequality,}

|a| = |(a + b) + (-b)| \leq |a + b| + |-b| = |a + b| + |b|.

So |a| - |b| \leq |a + b|. \text{ Interchanging } a \text{ and } b \text{ yields } |b| - |a| \leq |b + a| = |a + b|.

Since |b| - |a| = -(|a| - |b|), \text{ we conclude that } ||a| - |b|| \leq |a + b|.

Replacing } b \text{ by } -b, \text{ we get}

||a| - |b|| = ||a| - |b|| \leq |a - b|.

1.3.11. \text{ Binomial formula: If } a, b \geq 0, \text{ then}

(a + b)^n = a^n + na^{n-1}b + \sum_{k=2}^{n} \binom{n}{k} a^{n-k}b^k

\geq a^n + na^{n-1}b.

In particular, taking } a = 1, \text{ we get

(1 + b)^n \geq 1 + nb.

1.3.14. \text{ This follows from } 0 \leq \frac{1}{2} (a - b)^2 = \frac{1}{2} (a^2 + b^2) - ab.

1.3.17. \text{ Let } a, b, c \geq 0.

(a) \text{ This follows from } 0 \leq (a - b)^2 + (a - c)^2 + (b - c)^2.

(b) \text{ Use } \#15. \text{ } a + b \geq 2(ab)^{1/2} \text{ and same for the others. Multiply all the inequalities together.}

(c) \frac{1}{2} (a^2b^2 + a^2c^2) \geq a^2bc \text{ and same for the others. Add all the inequalities together.}

1.3.18*. \text{ (a) Since } p(x) > 0 \text{ if } x > 0, p(x) < 0 \text{ if } x < 0, \text{ and } p(x) = 0 \text{ if } x = 0, \text{ it suffices to prove the following.}

(i) \text{ If } 0 < x < y, \text{ then } x^3 < y^3.

(ii) \text{ If } x < y < 0, \text{ then } x^3 < y^3.
Proofs: Suppose $0 < x < y$. Then $x^3 = xxx < xxy < yyy = y^3$.

The same inequalities hold when $x < y < 0$ (one has to be careful when checking this).

2.1.1. (a) False. Take the sequence $\{(-1)^n\}$.
(b) False. Take any sequence $\{a_n\}$ which diverges. Then $\{-a_n\}$ also diverges, but their sum $\{0\}$ converges.
(c) True. Because $b_n = (a_n + b_n) + (-a_n)$ and Theorem 2.10 (and Lemma 2.11 for $\{a_n\}$).

2.1.2. (a) Observe that $\frac{1}{\sqrt{n}} \geq 0$. Let $\varepsilon > 0$. By the Archimedean Property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon^2$. Then $\frac{1}{\sqrt{N^2}} < \varepsilon$. Hence, if $n \geq N$, then $0 \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon$.

(b) Observe that $\frac{1}{n+5} \geq 0$. Let $\varepsilon > 0$. By the Archimedean Property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Hence, if $n \geq N$, then $0 \leq \frac{1}{n+5} \leq \frac{1}{N+5} \leq \frac{1}{N} < \varepsilon$.

2.1.6. Let $\varepsilon = \frac{\varepsilon}{2}$. Since $a > 0$, we have $\varepsilon > 0$. Since $a_n \to a$, there exists $N \in \mathbb{N}$ such that $a_n > a - \varepsilon = \varepsilon > 0$ for $n \geq N$ (we used $a = 2\varepsilon$).

2.1.8. This follows from the sum property (Theorem 2.10) applied to the sequences $\{c_n\}$ and $\{e\}$.

2.1.11. (a) We need this for all $\varepsilon > 0$.
(b) This actually is equivalent to $a_n = a$ for all $n \in \mathbb{N}$.
(c) This says that there exists $N \in \mathbb{N}$ such that $a_n = a$ for all $n \geq N$. 

3