Math 142A. Instructor: Chow

Homework #2. Due 4:00 pm on Monday, July 13, 2015.

Homeworks are due at 4 pm each Monday. You may either put it on the
desk at the front of the classroom SOLIS 104 or in the drop box in the
basement of APM.

**Problem 1:** Do Exercise 2.2.1 on p. 37. **ANS:**
(a) False. \((-1)^n\)
(b) False. \(\{\frac{1}{n}\}\)
(c) False. Diverges to infinity.
(d) False. Could limit to \(\sqrt{2}\) for example.
(e) False. \(\{\frac{1}{n}\}\) in (0,2) converges to 0.

**Problem 2:** By an \(\varepsilon-N\) argument very similar to the proof of Lemma
2.21, prove that the set \([b,\infty)\) is closed for any \(b \in \mathbb{R}\).

**ANS:** Suppose there exists \(b \in \mathbb{R}\) such that \([b,\infty)\) is not closed. Then
there exists a sequence \(\{d_n\}\) in \([b,\infty)\) such that \(d_n \to d\) and \(d \notin [b,\infty)\),
that is, \(d < b\). Let \(\varepsilon = b - d\), which is positive. By the definition of
convergence, there exists \(N \in \mathbb{N}\) such that \(|d_n - d| < \varepsilon = b - d\). Hence
\(d_N \leq d + |d_N - d| < d + \varepsilon = b\). This contradicts \(d_N \geq b\) (from \(d_N \in [b,\infty)\)).
We conclude that for all \(b \in \mathbb{R}\), the set \([b,\infty)\) is closed.

**Problem 3:** Do Exercise 2.2.2 on p. 37. **Hint:** One approach is to use
the fact that \(x \in (-\infty,0]\) if and only if \(-x \in [0,\infty)\).

**ANS:** Let \(\{a_n\}\) be a sequence in \((-\infty,0]\) that converges to a number \(a\).
Then \(\{-a_n\}\) is a sequence in \([0,\infty)\) which converges to \(-a\). By Lemma 2.21,
\(-a \in [0,\infty)\), that is, \(a \in (-\infty,0]\).

**Problem 4:** (i) Prove that if \(A\) is a closed subset of \(\mathbb{R}\) and if \(B\) is a
closed subset of \(\mathbb{R}\), then \(A \cap B\) is closed subset of \(\mathbb{R}\).

**ANS:** Let \(\{a_n\}\) be a sequence in \(A \cap B\) that converges to a number \(a\).
Since \(\{a_n\}\) is a sequence in \(A \cap B\) converging to \(a\) and since \(A\) is closed, we
have \(a \in A\). Similarly, \(a \in B\). Hence \(x \in A \cap B\).

(ii) Use this to show that any interval \([a,b]\) is closed.

**ANS:** \([a,b] = [a,\infty) \cap (-\infty, b]\), so it is the intersection of two closed sets,
which is closed.

**Problem 5:** Do Exercise 2.2.3 on p. 37.

**ANS:** Let \(x \in \mathbb{R}\).
Case 1. \(x \in \mathbb{Q}\). Consider the sequence \(\left\{ x + \frac{\sqrt{2}}{n} \right\} \).
Case 2. \(x \notin \mathbb{Q}\). Consider the sequence \(\left\{ x + \frac{1}{n} \right\} \).
Problem 6: Let \( \{a_n\} \) be a sequence and let \( a \in \mathbb{R} \). Prove that if there exists a sequence \( \{\varepsilon_n\} \) such that \( \varepsilon_n \to 0 \) and \( |a_n - a| < \varepsilon_n \) for all \( n \geq 1 \), then \( a_n \to a \).

ANS: Let \( \varepsilon > 0 \). Since \( \varepsilon_n \to 0 \), there exist \( N \in \mathbb{N} \) such that if \( n \geq N \), then \( \varepsilon_n < \varepsilon \). Hence, if \( n \geq N \), then

\[
|a_n - a| < \varepsilon_n < \varepsilon.
\]

Problem 7: Do Exercise 2.3.3 on p. 42.

ANS: We shall assume that \( \{a_n\} \) is monotone increasing. The monotone decreasing is analogous.

Since \( \{a_n\} \) is monotone increasing, \( \{a_n\} \) converges if and only if \( \{a_n\} \) is bounded.

Claim. \( \{a_n\} \) is bounded if and only if \( \{a_n^2\} \) is bounded.

Proof of the claim. (\( \Rightarrow \)) Suppose that \( \{a_n\} \) is bounded. Then there exists \( M \geq 0 \) such that \( |a_n| \leq M \) for all \( n \geq 1 \). Hence \( a_n^2 \leq M^2 \) for all \( n \geq 1 \). Therefore \( \{a_n^2\} \) is bounded.

(\( \Leftarrow \)) Suppose that \( \{a_n^2\} \) is bounded. Then there exists \( M \geq 0 \) such that \( a_n^2 \leq M \) for all \( n \geq 1 \). Then \( |a_n| \leq \sqrt{M} \) for all \( n \geq 1 \). Therefore \( \{a_n\} \) is bounded.

Case (i). \( \sup \{a_n|n \in \mathbb{N}\} \leq 0 \). Then \( a_n \leq 0 \) for all \( n \geq 1 \). Thus \( \{a_n\} \) being monotone increasing implies that \( \{a_n^2\} \) is monotone decreasing. Hence \( \{a_n^2\} \) converges if and only if \( \{a_n^2\} \) is bounded.

Case (ii). \( \sup \{a_n|n \in \mathbb{N}\} > 0 \). Then there exists \( N \in \mathbb{N} \) such that \( a_n > 0 \) for \( n \geq N \). Thus \( \{a_n\} \) being monotone increasing implies that \( \{a_n^2\}_{n \geq N} \) is monotone increasing. Hence \( \{a_n^2\} \) converges if and only if \( \{a_n^2\} \) is bounded.

Problem 8: Do Exercise 2.3.4 on p. 42.

ANS: Let \( \varepsilon = \frac{1 - |a|}{2} \). Then there exists \( N \in \mathbb{N} \) such that if \( n \geq N \), then \( |a_n - a| < \varepsilon \). Hence

\[
|a_n| \leq |a| + \varepsilon = |a| + \frac{1 - |a|}{2} = \frac{1 + |a|}{2} < 1
\]

for \( n \geq N \).

By Proposition 2.28, \( \left(\frac{1 + |a|}{2}\right)^n \to 0 \). From the comparison test (from calculus), \( (a_n)^n \to 0 \). (Alternatively, one can make an \( \varepsilon-N \) argument.)
Problem 9: Do Exercise 2.3.8 on p. 42.
ANS: \(s_n\) is monotone increasing since \(r \geq 0\) and \(b_n\) is nonnegative. On the other hand, since \(b_n\) is bounded, there exists \(M \geq 0\) such that \(b_n \leq M\). Hence
\[
s_n = \sum_{k=1}^{n} b_k r^k \leq \sum_{k=1}^{n} M r^k \leq M \frac{1}{1-r} < \infty
\]
since \(0 \leq r < 1\). That is, \(s_n\) is bounded above and hence bounded. By Theorem 2.25, \(s_n\) converges.

Problem 10: Do Exercise 2.3.9 on p. 42.
ANS: Since \(I_{n+1} \subseteq I_n\), we have \(a_{n+1} \geq a_n\) and \(b_{n+1} \leq b_n\). Thus \(\{a_n\}\) is monotone increasing and \(\{b_n\}\) is monotone decreasing.

By the monotone convergence theorem, \(\{a_n\}\) converges to \(a = \sup \{a_n | n \in \mathbb{N}\}\) and \(\{b_n\}\) converges to \(b = \inf \{b_n | n \in \mathbb{N}\}\). Fix \(n \in \mathbb{N}\). Then for all \(m \geq n\) we have \(a_n \leq a_m \leq b_m\). Hence \(a_n \leq \inf \{b_n | n \in \mathbb{N}\} = b\). Now, since \(a_n \leq b\) for all \(n \in \mathbb{N}\), we conclude that \(a = \sup \{a_n | n \in \mathbb{N}\} \leq b\). Since, for any \(n \in \mathbb{N}\) we have \(a_n \leq a \leq b \leq b_n\), we have \([a, b] \subseteq [a_n, b_n] = I_n\).

Problem 11: Do Exercise 2.4.1 on p. 46.
ANS:
(a) True. Holds for same bound \(M\).
(b) True. For example, \(a_{n+1} \geq a_n\) for all \(n \geq 1\) implies that \(a_{n+1} \geq a_k\) for all \(k \geq 1\) since \(n_{k+1} > n_k\) for all \(k \geq 1\).
(c) True. Converges to the same limit.
(d) False. Take \((-1)^n\), which doesn’t converge. The subsequence \(\{-1\}^{2n}\) converges.

Problem 12: Do Exercise 2.4.2 on p. 46.
ANS:
(a) True. Because it’s bounded. But the limit might not be in \((0, 1)\).
(b) False. Take \(\{\frac{1}{n}\}\).
(c) False. Take \(\{n\}\).
(d) True. Because \([0, \infty)\) is closed.
(e) False. Take \(\{n\}\).

Problem 13: Do Exercise 2.4.7 on p. 47.
ANS: Without loss of generality, assume that \(\{a_n\}\) is monotone increasing. Suppose that some subsequence \(\{a_{n_k}\}\) is bounded. Then there exists \(M \in \mathbb{R}\) such that \(a_{n_k} \leq M\) for all \(k \geq 1\). Let \(n \in \mathbb{N}\). Then there exists \(k \in \mathbb{N}\) such that \(n_k \geq n\). This implies \(a_n \leq a_{n_k} \leq M\). Hence \(\{a_n\}\) is bounded above and hence bounded.
Problem 14: Do Exercise 2.4.8 on p. 47.
ANS: If \( \{a_n\} \) has a convergent subsequence \( \{a_{n_k}\} \), then since it is monotone, we have \( \{a_{n_k}\} \) is bounded. Hence \( \{a_n\} \) is bounded, and since it is monotone, \( \{a_n\} \) converges.

Problem 15: Do Exercise 2.4.10 on p. 47.
ANS: Let \( \varepsilon > 0 \) be as stated. Let \( N \in \mathbb{N} \). Then there exists \( k \in \mathbb{N} \) such that \( n_k \geq N \). Since \( |a_{n_k} - a| \geq \varepsilon \) by hypothesis, there exists \( n \geq N \) such that \( |a_n - a| \geq \varepsilon \). Hence, there does not exist \( N \in \mathbb{N} \) such that \( |a_n - a| < \varepsilon \) for all \( n \geq N \).

Problem 16: Do Exercise 2.5.1 on p. 52.
ANS:
(a) False. \((0, 1)\).
(b) False. \(\mathbb{R}\) is closed.
(c) False. \(\mathbb{R}\) is closed.
(d) False. \((0, 1)\).
(e) False. \((0, 1)\) is a subset of \([0, 1]\).

Problem 17: Do Exercise 2.5.2 on p. 52.
ANS:
(a) False. \(\sqrt{n} \rightarrow 0\).
(b) False. It isn’t even closed.
(c) False. \(-\frac{1}{n} \rightarrow 0\) and 0 is not negative.

Problem 18: Do Exercise 2.5.3 on p. 52.
ANS:
(a) \(b - \frac{1}{n} \rightarrow b \notin [a, b]\). So every subsequence also converges to \(b\).
(b) The cover \([a, b - \frac{1}{n}]\) does not have a finite subcover.
(c) \(b - \frac{1}{n} \rightarrow b \notin [a, b]\).

Problem 19: Do Exercise 2.5.7 on p. 52.
ANS: Compact is the same as closed and bounded. This implies the results.

Problem 20: Do Exercise 2.5.10 on p. 52.
ANS: False. Let \(I_1 = [0, 0] = \{0\}\) and let \(I_n = [\frac{1}{n}, 1]\) for \(n \geq 2\). Then \(\{I_n\}\) covers \([0, 1]\) but has no finite subcover.