4.3 Problems

7. It wants us to show this equation has AT MOST two solutions. It’s really common in a situation like this to do a proof by contradiction. Assume it has at least 3 solutions, and call three of them \(x_1, x_2,\) and \(x_3\). Try it from there.

10: Like the hint in the problem suggests, do it by contradiction. Why must \(F\) be continuous? Since \(F'(x) = 0\) for \(x < 0\) what is \(F(x)\) equal to on this interval? Similarly, since \(F'(x) = 1\) for \(x ≥ 0\), what is \(F(x)\) equal to on \((0, ∞)\)? Then consider \(F(0) = \lim_{x→0} \frac{F(x)−F(0)}{x−0}\) from both sides.

11. Again, we see the AT MOST condition, so we think to do contradiction. Assume \(f(x) = 0\) has at least \(n + 1\) solutions. Try it from there.

15. It says \(g(x)f'(x) = f(x)g'(x)\) for all \(x\). This looks like a certain derivative rule. Figure out which rule, and move stuff around to make it look more like that rule.

Ben Chow’s answers

4.3.#7: Suppose there exist solutions \(x_1 < x_2 < x_3\) to \(x^n + ax + b = 0\), where \(n\) is an even natural number. Let \(f(x) = x^n + ax + b\). Then \(f'(x) = nx^{n−1} + a\), where \(n−1\) is an odd natural number. Observe that \(f'\) is strictly increasing (and actually a bijection from \(\mathbb{R}\) to \(\mathbb{R}\)). By Rolle’s Theorem there exist \(x_4 ∈ (x_1, x_2)\) and \(x_5 ∈ (x_2, x_3)\) such that \(f'(x_4) = f'(x_5) = 0\). However, \(f'(x) = 0\) has exactly one solution since \(f' : \mathbb{R} → \mathbb{R}\) is a bijection (we only need that \(f'(x) = 0\) has at most 1 solution which follows from \(f'\) being strictly increasing).

4.3.#10: \(F'(x) = 0\) for \(x < 0\) implies that \(F(x) = c\) for \(x < 0\), where \(c ∈ \mathbb{R}\) by Lemma 4.19. \(F'(x) = 1\) for \(x ≥ 0\) implies that \(F(x) = x − A\), where \(A ∈ \mathbb{R}\) by Proposition 4.20. Since \(F\) is differentiable, \(F\) is continuous by Proposition 4.5. Hence \(A = F(0) = \lim_{x→0} F(x) = 0\). Thus \(F(x) = x\) for \(x ≥ 0\). This implies that \(\frac{dF}{dx}(0) = \lim_{x→0} \frac{F(x)−F(0)}{x−0} = 0\), whereas \(\frac{dF}{dx}(0) = \lim_{x→0} \frac{F(x)−F(0)}{x−0} = 1\). Since \(\frac{dF}{dx}(0) ≠ \frac{dF}{dx}(0)\), \(F\) is not differentiable at \(x = 0\), which is a contradiction.

4.3.#11: Suppose that \(f(x) = 0\) has at least \(n + 1\) solutions. List them inincreasing order: \(x_1 < x_2 < \cdots < x_{n+1}\). Then, by Rolle’s theorem, there exist \(x'_1 \in (x_1, x_2), x'_2 \in (x_2, x_3), \ldots, x'_n \in (x_n, x_{n+1})\) such that \(0 = f(x'_1) = f(x'_2) = \cdots = f(x'_n)\). Hence \(f'(x) = 0\) has at least \(n\) solutions. We have proved the contrapositive of the statement.

4.3.#15: Writing the equation as \(g(x)f'(x) − f(x)g'(x) = 0\) and seeing this as the numerator of the quotient rule, we compute that

\[
\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) − f(x)g'(x)}{\left(g(x)\right)^2} = 0,
\]

which holds since \(g(x) ≠ 0\) for all \(x ∈ \mathbb{R}\). Hence \(\frac{f(x)}{g(x)} = c\), where \(c ∈ \mathbb{R}\).