Chapter 4

Ricci Solitons and Special Solutions

In the study of the singularities which exist or form for solutions of partial differential equations including those which arise in geometry, a fundamental notion is that of rescaling and applying monotonicity formulas to obtain self-similar solutions which model the solutions near the singularities. Such techniques have been successfully applied to the study of minimal surfaces, harmonic maps, Yang-Mills connections, and solutions of nonlinear heat and Schrödinger equations, to name a few. In the field of geometric evolution equations, the singularity models which arise are usually ancient solutions, where the solutions exist all the way back to time minus infinity. Among such ‘long-existing’ solutions (i.e., solutions which exist on an infinite time interval) are the self-similar solutions, which in Ricci flow are called Ricci solitons. In this chapter we shall study some properties and examples of such solutions with a special emphasis on dimension 2 where the solutions are explicit. We also study other long-existing solutions such as the Rosenau solution in Section 4 and homogeneous solutions in Section 7. Finally we consider the Ricci flow on compact Lie groups with bi-invariant metrics (since this is a particularly simple case). The examples in this chapter are useful to keep in mind when studying singularity formation of solutions of the Ricci flow.

Before we discuss Ricci solitons, we list the types of long-existing solutions. Recall that a solution \((M^n, g(t))\) to the Ricci flow is called an ancient solution if it is defined on an interval of the form \((-\infty, \omega)\), where \(\omega \in \mathbb{R} \cup \{\infty\}\) (usually, \(\omega > 0\)). We say that a solution \((M^n, g(t))\) to the Ricci flow is an immortal solution if it is defined on a time interval \(\alpha < t < \infty\).
Finally, if \((M^n, g(t))\) is defined for all \(-\infty < t < \infty\), then we call it an **eternal solution**. From taking limits of dilations of singularities we shall obtain long-existing solutions (see Chapter 8).

1. Gradient Ricci solitons

We say that a quadruple \((M^n, g_0, f_0, \varepsilon)\), where \((M^n, g_0)\) is a Riemannian manifold, \(f_0 : M^n \to \mathbb{R}\) is a function and \(\varepsilon \in \mathbb{R}\), is a **gradient Ricci soliton** if

\[
Rc(g_0) + \nabla g_0 \nabla g_0 f_0 + \frac{\varepsilon}{2} g_0 = 0.
\]

Here \(\nabla g_0\) denotes the covariant derivative with respect to \(g_0\) (the superscript is only for emphasis). For reasons we shall see below, we say that \(g_0\) is **expanding**, **shrinking**, or **steady**, if \(\varepsilon > 0\), \(\varepsilon < 0\), or \(\varepsilon = 0\), respectively.

We say that the gradient soliton is **complete** if \((M^n, g_0)\) is complete and the vector field \(\text{grad}_{g_0} f_0\) is complete. We call the function \(f_0\) the **potential function**.

The following result gives a **canonical form** for the associated time-dependent version of a gradient Ricci soliton.

**Theorem 4.1** (Gradient Ricci solitons). If \((M^n, g_0, f_0, \varepsilon)\) is a complete gradient Ricci soliton, then there exist a solution \(g(t)\) of the Ricci flow with \(g(0) = g_0\), diffeomorphisms \(\varphi(t)\) with \(\varphi(0) = \text{id}_{M^n}\), functions \(f(t)\) with \(f(0) = f_0\) defined for all \(t\) with

\[
\tau(t) \equiv \varepsilon t + 1 > 0,
\]

such that the following hold.

1. \(\varphi(t) : M^n \to M^n\) is the 1-parameter family of diffeomorphisms generated by \(X(t) \equiv \frac{1}{\tau(t)} \text{grad}_{g_0} f_0\). That is,

\[
\frac{\partial}{\partial t} \varphi(t)(x) = \frac{1}{\tau(t)} \left(\text{grad}_{g_0} f_0\right)(\varphi(t)(x)).
\]

2. \(g(t)\) is the pull back by \(\varphi(t)\) of \(g_0\) up to the scale factor \(\tau(t)\):

\[
g(t) = \tau(t) \varphi(t)^* g_0.
\]

3. \(f(t)\) is the pull back by \(\varphi(t)\) of \(f_0\):

\[
f(t) = f_0 \circ \varphi(t) = \varphi(t)^* (f_0).
\]

Moreover,

\[
Rc(g(t)) + \nabla g(t) \nabla g(t) f(t) + \frac{\varepsilon}{2\tau} g(t) = 0
\]
and

\[
\begin{align*}
\frac{\partial f}{\partial t}(t) &= \left| \text{grad}_{g(t)} f(t) \right|^2_{g(t)}.
\end{align*}
\]

**Caveat:** Unless \( \varepsilon = 0 \), i.e., \( \tau(t) \equiv 1 \), the 1-parameter family \( \varphi(t) \) is not a group.

**Proof.** Define \( \tau(t) = \varepsilon t + 1 \). Since the vector field \( \text{grad}_{g_0} f_0 \) is complete, there exists a 1-parameter family of diffeomorphisms \( \varphi(t) : M^n \to M^n \) generated by the vector fields \( \frac{1}{\tau(t)} \text{grad}_{g_0} f_0 \) defined for all \( t \) such that \( \tau(t) > 0 \). Then define \( f(t) = f_0 \circ \varphi(t) \) and \( g(t) = \tau(t) \varphi(t)^* g_0 \). We compute

\[
\frac{\partial}{\partial t} \bigg|_{t=t_0} g(t) = \frac{\varepsilon}{\tau(t_0)} g(t_0) + \tau(t_0) \frac{\partial}{\partial t} \bigg|_{t=t_0} (\varphi(t)^* g_0).
\]

Using Remark 1.24, we have

\[
\tau(t_0) \frac{\partial}{\partial t} \bigg|_{t=t_0} (\varphi(t)^* g_0) = \tau(t_0) \mathcal{L}_{(\varphi(t_0)^{-1})^*} \frac{\partial}{\partial t} \bigg|_{t=t_0} \varphi(t)^* g_0 = \mathcal{L}_{\text{grad}_{g(t_0)} f(t_0)} g(t_0),
\]

which holds since

\[
\frac{\partial}{\partial t} \bigg|_{t=t_0} \varphi(t) = \frac{1}{\tau(t_0)} \text{grad}_{g_0} f_0 = \varphi(t_0)^* (\text{grad}_{g(t_0)} f(t_0)).
\]

Hence (evaluating at \( t \) instead of \( t_0 \))

\[
\frac{\partial}{\partial t} g(t) = \frac{\varepsilon}{\tau(t)} g(t) + \mathcal{L}_{\text{grad}_{g(t)} f(t)} g(t).
\]

Now using Exercise 1.23, we find that

\[
-2 \text{Rc}(g(t)) = \varphi(t)^* \left( -2 \text{Rc}(g_0) \right) = \varphi(t)^* \left( \varepsilon g_0 + \mathcal{L}_{\text{grad}_{g_0} f_0} g_0 \right)
\]

\[
= \frac{\varepsilon}{\tau(t)} g(t) + \mathcal{L}_{\text{grad}_{g(t)} f(t)} g(t).
\]

Hence

\[
\frac{\partial}{\partial t} g(t) = \frac{\varepsilon}{\tau(t)} g(t) + \mathcal{L}_{\text{grad}_{g(t)} f(t)} g(t) = -2 \text{Rc}(g(t)).
\]

Finally we calculate

\[
\frac{\partial f}{\partial t}(x,t) = \left( \frac{\partial}{\partial t} \varphi(t) \right) (f_0)(x) = \frac{1}{\tau(t)} \left| \text{grad}_{g_0} f_0 \right|^2 (\varphi(t)(x))
\]

\[
= \left| \text{grad}_{g(t)} f(t) \right|^2_{g(t)} (x).
\]

\[\square\]
Note that, as mentioned above, \( g(t) \) is expanding, shrinking, or steady, if \( \varepsilon > 0, \varepsilon < 0, \) or \( \varepsilon = 0 \), respectively. In local coordinates we may write equation (4.7) as

\[
\frac{\partial}{\partial t} g_{ij} = -2 R_{ij},
\]

\[
S^\varepsilon_{ij} \overset{\triangle}{=} R_{ij} + \nabla_i \nabla_j f + \frac{\varepsilon}{\tau} g_{ij} = 0.
\]

**Remark 4.2.** If \( \varepsilon = 0 \), then \( g(t) \) is defined for all \( t \in (-\infty, \infty) \); if \( \varepsilon > 0 \), then \( g(t) \) is defined for \( t \in (-1/\varepsilon, \infty) \); and if \( \varepsilon < 0 \), then \( g(t) \) is defined for \( t \in (-\infty, 1/|\varepsilon|) \). The solution is *eternal, immortal* or *ancient*, respectively.

By dilating the solution, except when \( g(t) \) is steady, we can choose \( |\varepsilon| \) to be any positive real number. The above theorem, by requiring (4.2), puts the gradient Ricci soliton in canonical form.

To both motivate the discovery of various monotonicity formulae and study the geometry of gradient Ricci solitons, we derive some equations for gradient Ricci solitons which follow from (4.5). Taking the trace of (4.5) yields

\[
R + \Delta f + \frac{n \varepsilon}{2\tau} = 0.
\]

(4.9)

Subtracting (4.9) from the RHS of (4.6), we obtain

\[
\frac{\partial f}{\partial t} = -\Delta f - R + |\nabla f|^2 - \frac{n \varepsilon}{2\tau}.
\]

(4.10)

This equation is a **backward heat-type equation**.

**Remark 4.3.** Equation (4.10) is stipulated for a general solution of Ricci flow in Perelman’s entropy formula (see (5.14)).

Next, let’s take the divergence of equation (4.5). By the contracted second Bianchi identity and commuting derivatives, we get in all dimensions:

\[
0 = \nabla_j \left( R_{ij} + \nabla_i \nabla_j f + \frac{\varepsilon}{2\tau} g_{ij} \right) = \frac{1}{2} \nabla_i R + \nabla_i \Delta f - R_{ijjk} \nabla_k f
\]

(4.11)

\[
= -\frac{1}{2} \nabla_i R + R_{ik} \nabla_k f.
\]

Substituting (4.5) into (4.11), we get

\[
0 = \nabla_i R + \nabla_i |\nabla f|^2 + \frac{\varepsilon}{\tau} \nabla_i f.
\]

(4.12)

Hence we have

\[
R + |\nabla f|^2 + \frac{\varepsilon}{\tau} f = C(t)
\]

(4.13)
is constant in the space variables. By taking two times (4.9) and subtracting (4.13), we find that
\begin{equation}
V_\varepsilon = \tau \left( R + 2 \Delta f - |\nabla f|^2 \right) - \varepsilon (f - n)
\end{equation}
is equal to $-\tau C(t)$, which is a function only of $t$.

Another useful equation is obtained from taking the divergence of (4.11):
\begin{equation}
0 = \Delta R + 2 |Rc|^2 + 2 \lambda R - \langle \nabla R, \nabla f \rangle.
\end{equation}
We may also see this last equation by observing that
\begin{equation}
\partial_t g = L \nabla f g + 2 \lambda g
\end{equation}
implies that for a solution to the Ricci flow
\begin{equation}
\Delta R + 2 |Rc|^2 = \partial_t R = L \nabla f R - 2 \lambda R = \langle \nabla R, \nabla f \rangle - 2 \lambda R.
\end{equation}
(We obtain the second equality from the diffeomorphism invariance of the scalar curvature and the fact that $R$ scales like the inverse of the metric.)

2. Gaussian and cylinder solitons

2.1. Gaussian soliton. Euclidean space $\mathbb{R}^n$ with the standard flat metric $g_{\text{can}}$ may seem like an uninteresting solution to the Ricci flow since it is stationary. However, one of the reasons it is interesting is that the metrics are invariant under scaling: for any constant $c > 0$, $cg_{\text{can}}$ is isometric to $g_{\text{can}}$. In particular, $cg_{\text{can}} = \varphi^* g_{\text{can}}$, where $\varphi = \sqrt{c} \text{id}_{\mathbb{R}^n}$. Because of this, we may think of $(\mathbb{R}^n, g_{\text{can}})$ not only as a steady gradient Ricci soliton but also as either an expanding or a shrinking gradient Ricci soliton.

Let $\tau : \mathcal{I} \rightarrow (0, \infty)$ be a smooth function on an interval and define
\begin{equation}
\varphi(t) = \tau(t)^{-1/2} \text{id}_{\mathbb{R}^n}
\end{equation}
for $t \in \mathcal{I}$. Then
\begin{equation}
g(t) \equiv g_{\text{can}} = \tau(t) \varphi(t)^* g(0).
\end{equation}
We call the stationary flat solution $g_{\text{can}}$ the Gaussian soliton, when viewed this way. Choose $\tau(t) = \varepsilon t + 1$ where $\varepsilon \in \mathbb{R}$, as in (4.2), so that
\begin{equation}
\varphi(t) = \tau(t)^{-1/2} \text{id}_{\mathbb{R}^n}
\end{equation}
is a 1-parameter family of diffeomorphisms. If we define\footnote{The adjective ‘Gaussian’ in the term ‘Gaussian soliton’ refers to the form of $f$. Compare with the Euclidean heat kernel (A.1).}
\begin{equation}
f(x, t) = -\frac{\varepsilon |x|^2}{4\tau(t)},
\end{equation}
then $f(t) = f(0) \circ \varphi(t)$, $\text{grad}_{g(0)} f(0) = -\frac{\varepsilon}{2} x$ and
\begin{equation}
\frac{d}{dt} \varphi(t)(x) = -\frac{\varepsilon}{2} \tau(t)^{-3/2} x = \frac{1}{\tau(t)} \left( \text{grad}_{g(0)} f(0) \right) (\varphi(t)(x)).
\end{equation}
At time $t$ we calculate $(\text{Rc} (g(t)) \equiv 0)$

$$2 \text{Rc} (g(t)) + \frac{\varepsilon}{\tau (t)} g(t) + \mathcal{L}_{\text{grad}_g(t) f(t)} g(t) = \frac{\varepsilon}{\tau (t)} g_{\text{can}} - \frac{\varepsilon}{4\tau (t)} \mathcal{L}_{\text{grad}_{g_{\text{can}}} |x|^2 g_{\text{can}}} = 0,$$

which agrees with what we know from the proof of Theorem 4.1.

**Remark 4.4.** Considering $\mathbb{R}^n$ as a shrinking soliton, so that $\varepsilon = -1$, we have

$$(4.18) \quad f(x, t) = \frac{|x|^2}{4\tau}.$$  

The Gaussian soliton is a good model to keep in mind when considering shrinking and expanding gradient Ricci solitons. It shows that it is possible for a solution to the Ricci flow to be a gradient Ricci soliton which is either shrinking, steady, or expanding, depending on how one defines the potential function $f$.

**2.2. Cylinder shrinking soliton.** Consider the product of the shrinking sphere with a line: \((S^{n-1} \times \mathbb{R}, g(t)), t \in (-\infty, 0), n \geq 3, \) where

$$g(t) = 2 (n - 2) |t| g_{S^{n-1}} + dr^2.$$  

Its Ricci tensor is given by

$$\text{Rc} (g(t)) = (n - 2) g_{S^{n-1}} = \frac{1}{2 |t|} g(t) - \frac{1}{2 |t|} dr^2.$$  

If we let

$$(4.19) \quad f(\theta, r, t) \equiv \frac{r^2}{4 |t|}, \quad \theta \in S^{n-1}, r \in \mathbb{R}, t < 0,$$

we then have

$$\text{Rc} (g(t)) + \nabla \nabla f(t) + \frac{1}{2t} g(t) = 0.$$  

Hence $g(t)$ is a shrinking gradient Ricci soliton.

**Exercise 4.5.** Show that \((S^n \times \mathbb{R}^k, g(t)) , t \in (-\infty, 0), n \geq 2, \) where

$$g(t) = 2 (n - 1) |t| g_{S^n} + g_{\mathbb{R}^k},$$

is a shrinking gradient soliton with

$$f(\theta, x, t) \equiv \frac{|x|^2}{4 |t|}, \quad \theta \in S^n, x \in \mathbb{R}, t < 0.$$  

In a sense, the steady (stationary) Euclidean metric is turned into a shrinking soliton by taking the product with a shrinking sphere.
Note that in the above example
\[ \nabla f = \frac{x}{2|t|}, \quad |\nabla f|^2 = \frac{|x|^2}{4t^2}. \]
In particular, \( \nabla f \) is radial and pointing outward. In contrast, by (4.23), for the cigar soliton \( \nabla f = -2x \) is pointing inward.

The cylinder soliton is important since in dimension 3 it models neck-pinching. Note that the form of the potential function in (4.19) motivates some of the estimates in the proof in Section 6 of Chapter 9 on the nonexistence of noncompact 3-dimensional shrinking solitons with positive sectional curvature.

3. Cigar steady soliton

Hamilton’s cigar soliton is the complete Riemannian surface \((\mathbb{R}^2, g_\Sigma)\), where
\begin{equation}
(4.20)
g_\Sigma \doteq dx^2 + dy^2 \div \frac{1}{1 + x^2 + y^2},
\end{equation}
where \( dx^2 \doteq dx \otimes dx \). As a solution to the Ricci flow, its time-dependent version is
\begin{equation}
(4.21)
g_\Sigma (t) \doteq \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}.
\end{equation}
To see that \( g_\Sigma (t) \) is a solution to the Ricci flow, we leave it to the reader to check that \( u (x, y, t) \doteq -\log \left( e^{4t} + x^2 + y^2 \right) \) satisfies equation (2.5) with \( h = dx^2 + dy^2; \ \frac{\partial u}{\partial t} = \Delta \log u \), where \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Euclidean Laplacian. From the change of variables \( \tilde{x} = e^{-2t}x \) and \( \tilde{y} = e^{-2t}y \), we see that \( g_\Sigma (t) = \frac{dx^2 + dy^2}{1 + x^2 + y^2} \) is isometric to \( g_\Sigma = g_\Sigma (0) \). That is, if we define the 1-parameter group of (conformal) diffeomorphisms \( \varphi_t : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( \varphi_t (x, y) \doteq (e^{-2t}x, e^{-2t}y) \), then
\[ g_\Sigma (t) = \varphi^* \Sigma (0). \]
Hence \( g_\Sigma (t) \) is a steady Ricci soliton. Geometrically a domain \( \Omega \subset \mathbb{R}^2 \) with respect to the metric \( g_\Sigma (0) \) corresponds to the domain \( \varphi_t^{-1} (\Omega) = e^{2t} \Omega \subset \mathbb{R}^2 \) with respect to the metric \( g_\Sigma (t) \). That is, \( (\Omega, g_\Sigma (0)) \) is isometric to \( (e^{2t} \Omega, g_\Sigma (t)) \). The vector field generating the 1-parameter group \( \varphi_t \) is \( (-2x, -2y) = \text{grad}_{g_\Sigma} f \), where
\begin{equation}
(4.22)
f (x, y) \doteq -\log \left( 1 + x^2 + y^2 \right).
\end{equation}
Note that
\begin{equation}
(4.23)
\text{grad}_{g_\Sigma (t)} f (t) = (-2x, -2y)
\end{equation}
is independent of time, where
\[ f(x, y, t) \doteq - \log \left( 1 + e^{-4t} \left( x^2 + y^2 \right) \right) = (f \circ \varphi_t)(x, y). \]

This manifold is also known in the physics literature as Witten’s black hole. In polar coordinates, we may rewrite the cigar metric as
\[ g_\Sigma = \frac{dt^2 + r^2 \, d\theta^2}{1 + r^2}. \]

If we define the new radial distance variable as
\[ s \doteq \text{arcsinh} \, r = \log \left( r + \sqrt{1 + r^2} \right), \]
then we may rewrite \( g_\Sigma \) as
\[ g_\Sigma = ds^2 + \tanh^2 s \, d\theta^2. \]

In general, if a metric takes the form \( g = ds^2 + \phi(s)^2 \, d\theta^2 \), then its scalar curvature is \( R(s, \theta) = -2 \frac{\phi''(s)}{\phi(s)}. \) We can see this from the following moving frames calculation (subsection 7.1 in Chapter 1; see also [304] for a nice exposition of this technique). Let
\[ \omega^1 = ds, \quad \omega^2 = \phi(s) \, d\theta. \]

The connection 1-forms \( \omega^i \) and the curvature 2-forms \( \Omega^j_i \) satisfy the Cartan structure equations (using \( n = 2 \) in the equation for \( \Omega^j_i \))
\[ dw^i = \omega^j \wedge \omega^j_i, \quad \Omega^j_i = d\omega^j_i. \]

From (1.81) we compute
\[ \omega^2_1 = \phi'(s) \, d\theta, \quad \Omega^2_1 = \frac{\phi''(s)}{\phi(s)} \omega^1 \wedge \omega^2, \]
and \( K = 2\Omega^2_1(e_2, e_1) = -\frac{\phi''(s)}{\phi(s)}. \) Thus the scalar curvature of \( g_\Sigma \) is
\[ R_\Sigma = 4 \text{sech}^2 s = \frac{4}{1 + r^2}. \]

From (4.22) we have
\[ f(s) = -2 \log \left( \cosh s \right). \]

We then have
\[ R_{ij} + \nabla_i \nabla_j f = 0, \]
which is the infinitesimal version of the steady Ricci soliton equation for \( g_\Sigma(t) \). We can also verify this directly as follows. The Hessian of \( f \) is given by
\[ \nabla_i \nabla_j f = e_i(e_j(f)) - \omega^k_j(e_i)(e_k(f)). \]
The first term on the RHS is nonzero only if \( i = j = 1 \) whereas the second term on the RHS is nonzero only if \( i = j = 2 \). Since \( \omega_2 (e_2) = - \frac{\phi ' (s)}{\phi (s)} = - \frac{1}{\sinh s \cosh s} \) and \( e_1 (f) = -2 \tanh s \), we have
\[
e_1 (e_1 (f)) = -\omega_2 (e_2) e_1 (f) = -2 \text{sech}^2 s = -\frac{R}{2}
\]
and (4.26) follows.

**Exercise 4.6** (Another form of the cigar). By making the change of variables \( r \equiv M \cosh^2 s \), show that the following is another form of the cigar metric:
\[
(4.28) \quad g = \left( 1 - \frac{M}{r} \right) d\theta^2 + \left( 1 - \frac{M}{r} \right)^{-1} \frac{dr^2}{4r^2},
\]
where \( r > M \).

Now consider the cylinder \( \mathbb{R} \times S^1 (1) \), where \( S^1 (1) \) denotes the circle of radius 1, and let \( x \in \mathbb{R} \) and \( \theta \in S^1 (1) = \mathbb{R} / 2\pi \mathbb{Z} \) be the standard coordinates. Let \( h \) be the flat metric \( h = dx^2 + d\theta^2 \). The **cigar metric on the cylinder** (punctured plane) may be written as
\[
(4.29) \quad g_{\Sigma^2 - 0} = \left( e^{-2x} + 1 \right)^{-1} (dx^2 + d\theta^2).
\]
Defining \( x \equiv \frac{1}{2} \log \left( \frac{r}{M} - 1 \right) > 0 \) so that \( 1 - \frac{M}{r} = \left( e^{-2x} + 1 \right)^{-1} \) and \( dx = \frac{dr}{2(r-M)} \), we see that (4.28) implies (4.29). We can also see (4.29) by solving for \( F \) and \( x \) in the equation
\[
F (x)^2 (dx^2 + d\theta^2) = ds^2 + \tanh^2 s d\theta^2,
\]
where we get \( F (x) dx = ds \) and \( F (x) = \tanh s \) so that \( x = \log (\sinh s) \), \( s > 0 \), and \( F (x)^2 = \frac{e^{2x}}{1 + e^{2x}} = \left( e^{-2x} + 1 \right)^{-1} \) as desired.

**Remark 4.7.** The metric \( g_{\Sigma^2 - 0} \) is incomplete; however, by taking the 1-point compactification of \( \mathbb{R} \times S^1 (1) \) at the end where \( x \to -\infty \), we obtain the complete cigar metric on \( \mathbb{R}^2 \). Note that \( u (x) \equiv \left( e^{-2x} + 1 \right)^{-1} \) satisfies \( \lim_{x \to -\infty} u (x) = 1 \), so at that end the metric is asymptotically cylindrical as we know.

For a proof of the following classification result, see [153].

**Lemma 4.8** (Positively curved 2d gradient steady soliton is a cigar). If \( (\mathbb{R}^2, g(t)) \) is a steady gradient Ricci soliton conformal to the standard metric on \( \mathbb{R}^2 \), then \( (\mathbb{R}^2, g(t)) \) is either the cigar soliton or the flat metric.

If \( (M^2, g) \) is a complete noncompact Riemannian surface with positive curvature, then \( (M^2, g) \) is conformal to the standard metric on \( \mathbb{R}^2 \) (see Huber [310]). Hence we have the following, which is the same as Corollary B.12(ii) and is proved in Section 1 of Appendix B.
Corollary 4.9 (Uniqueness of the cigar). If \((M^2, g(t))\) is a complete steady gradient Ricci soliton with positive curvature, then \((M^2, g(t))\) is the cigar soliton.

In Hamilton’s program for the Ricci flow on 3-manifolds, via dimension reduction, the cigar soliton is a potential singularity model. Perelman’s monotonicity formula(s) rule out this possibility (see Section 5 of Chapter 5).

4. Rosenau solution

Let \((\mathbb{R} \times S^1(2), h)\) denote the flat cylinder, where \(h = dx^2 + d\theta^2\) and \(\theta \in S^1(2) = \mathbb{R}/4\pi \mathbb{Z}\). The Rosenau solution \([471]\) is the solution \(g(t) = u(t) \cdot h\) to the Ricci flow defined for \(t < 0\) by

\[
(4.30) \quad u(x, t) = \frac{\sinh(-t)}{\cosh x + \cosh t}.
\]

One computes that its curvature is

\[
R[g(t)] = -\frac{\Delta_h \log u}{u} = \frac{\cosh t \cdot \cosh x + 1}{\sinh(-t) (\cosh x + \cosh t)}.
\]

From this we easily check that

\[
\frac{\partial}{\partial t} u = -Ru,
\]

so that \(g(t)\) is a solution to the Ricci flow. The metrics \(g(t)\) defined on \(\mathbb{R} \times S^1(2)\) extend to smooth metrics, which we also call \(g(t)\), on the 2-sphere \(S^2\), which is obtained by compactifying \(\mathbb{R} \times S^1(2)\) by adding two points (we call these two points the north and south poles). We can see this as follows. For \(x\) large and negative (similarly when positive), \(g(t)\) is asymptotic to a constant multiple \((2 \sinh(-t))\) of the metric \(e^{x} \left( dx^2 + d\theta^2 \right) = 4 \left( d\bar{x}^2 + \bar{x}^2 d\bar{\theta}^2 \right)\), where \(\bar{x} = e^{x/2}\) and \(\bar{\theta} = \theta / 2\); note that \(\bar{\theta} \in S^1(1) = \mathbb{R}/2\pi \mathbb{Z}\), and when \(x\) is large and negative, \(\bar{x}\) is positive and near 0 (see [163], p. 33 for more details).

Exercise 4.10. Show that

\[
\sup_{S^2} R[g(t)] = \coth(-t).
\]

In particular \(t \mapsto \sup_{S^2} R[g(t)]\) is increasing and \(\lim_{t \to -\infty} \sup_{S^2} R[g(t)] = 1\).

Exercise 4.11 (Rosenau tends to round at singularity time). Show that since \(\lim_{t \to 0} \text{csch}(-t) u(x, t) = \frac{1}{\cosh x + 1}\), the limit of \(\text{csch}(-t) g(t)\) as \(t \to 0\) is the round 2-sphere with scalar curvature 1 (radius \(\sqrt{2}\)).
We now take a limit of the Rosenau solution as $t \to -\infty$ to see that we can get the cigar soliton as a (backward) limit (we can also get the cylinder). Note that the Cheeger-Gromov-type compactness theorem (see Theorem 6.35) yields convergence only after pulling back by diffeomorphisms. Fortunately, in the case of the Rosenau solution, the diffeomorphisms sufficient to obtain convergence are translations. In particular, consider

$$u(x + t, t) = (- \cosh x \coth t - \sinh x - \coth t)^{-1},$$

so that

$$\lim_{t \to -\infty} u(x + t, t) = (\cosh x - \sinh x + 1)^{-1} = (e^{-x} + 1)^{-1}.$$

Let $\phi_t : \mathbb{R} \times S^1(2) \to \mathbb{R} \times S^1(2)$ be defined by $\phi_t(x, \theta) = (x + t, \theta)$. Then $(\phi_t^* g)(x, t) = u(x + t, t) h$. Hence

$$\lim_{t \to -\infty} (\phi_t^* g)(x, t) = (e^{-x} + 1)^{-1} h(x).$$

(4.31)

Similarly, for the diffeomorphisms $\psi_t(x, \theta) = (x - t, \theta)$ we have

$$\lim_{t \to -\infty} (\psi_t^* g)(x, t) = (e^x + 1)^{-1} h(x).$$

(4.32)

Making the change of variables $\tilde{x} = x/2$ and $\tilde{\theta} = \theta/2$ in (4.31), we have

$$\lim_{t \to -\infty} (\phi_t^* g)(x, t) = (e^{-x} + 1)^{-1} h(x) = 4 \left( e^{-2\tilde{x}} + 1 \right)^{-1} \left( d\tilde{x}^2 + d\tilde{\theta}^2 \right),$$

(4.33)

where $\tilde{x} \in \mathbb{R}$ and $\tilde{\theta} \in S^1(1)$. Similarly, we could do the same with (4.32). Thus we obtain the cigar soliton (compare (4.33) with (4.29)) as a backward limit in two essentially different ways, corresponding to dilating about points close enough to either the north or south pole. It is interesting that we did not need to rescale (dilate) to obtain the cigar. Note that we can also obtain the cylinder as a backward limit since $\lim_{t \to -\infty} u(x, t) = 1$, so that

$$\lim_{t \to -\infty} g(x, t) = h(x)$$

for all $x \in \mathbb{R}$. This corresponds to dilating about points close enough to the center circle (equator) $\{(x, \theta) : x = 0\}$, and in particular, dilating about points on the center circle as above.

**Remark 4.12.** In the above we are taking the limit of a 1-parameter family of metrics (as $t \to -\infty$) to a single Ricci soliton metric. We leave it as an exercise for the reader to take the analogous limit of a 1-parameter family of solutions of Ricci flow to a soliton solution of the Ricci flow. **In the Cheeger-Gromov compactness theorem one takes a limit of a sequence of solutions.**
Exercise 4.13. Let \( x_0(t) \in \mathbb{R} \) be defined for \( t \leq t_0 \) for some \( t_0 \). Consider the family of functions
\[
\tilde{u}(x, t) \coloneqq R_{\text{max}}(t) u(x + x_0(t), t) = \left( \frac{\cosh x \cosh(x_0(t)) + \sinh x \sinh(x_0(t))}{\cosh t} + 1 \right)^{-1}.
\]
Suppose that \( \lim_{t \to -\infty} \frac{\cosh(x_0(t))}{\cosh t} \doteq A \in [0, \infty) \) exists. Show that
\[
\lim_{t \to -\infty} \frac{\sinh(x_0(t))}{\cosh t}
\]
equals \( -A \).

(1) If \( A = 0 \), show that \( \lim_{t \to -\infty} \tilde{u}(x, t) = 1 \), so we get the cylinder as a backward limit.

(2) If \( A \in (0, \infty) \), show that
\[
\lim_{t \to -\infty} \tilde{u}(x, t) = (A (\cosh x - \sinh x) + 1)^{-1}.
\]

(3) If \( A = \infty \), find an appropriate function \( c(t) \) so that \( c(t) \tilde{u}(x, t) \) has a limit as \( t \to -\infty \). Show that this limit is a cigar.

In the dimension reduction of a 3-dimensional singularity model, one obtains an ancient solution on a surface. For singularity models arising from finite time singularities, Perelman’s no local collapsing theorem (e.g., Corollary 5.55) rules out the Rosenau solution. However, for singularity models arising from infinite time singularities, the Rosenau solution has not been ruled out.

5. An expanding soliton

Recall that we say that a solution \((M^n, g(t))\) to the Ricci flow is an immortal solution if it is defined on a time interval \( \alpha < t < \infty \). In dimension 3 there are a number of immortal solutions which are \textit{locally homogeneous} and whose curvatures decay like \( 1/t \) (see [320], Chapter 1 of [163], or Section 7). In this section we consider another example of an immortal solution, an expanding soliton. In dimension 2, there are explicit nontrivial (that is, nonconstant curvature) examples of expanding Ricci solitons (see Appendix A of Gutperle, Headrick, Minwalla and Schomerus [272]). In particular, consider rotationally symmetric metrics \( g(t) \) on \( \mathbb{R}^2 \) of the form
\[
(4.34) \quad g(t) = t \left( F(r)^2 dr^2 + r^2 d\theta^2 \right),
\]
where \( F : [0, \infty) \to (0, \infty) \) is a positive function to be determined by the expanding Ricci soliton condition and where \( r \in (0, \infty) \) and \( \theta \in \mathbb{R}/(2\pi F(0)) \). That is, each metric \( g(t) \) is first defined on \( (0, \infty) \times S^1(F(0)) \cong \mathbb{R}^2 - \{0\} \).
We define $\theta$ in this range to ensure that the cone angle at the origin is $2\pi$ so that the $g(t)$ indeed define smooth metrics on $\mathbb{R}^2$ by extending smoothly over the origin ($r \to 0$). Note that these metrics $g(t) = tg(1)$ are homothetically expanding for $t > 0$. One can compute that the Gauss curvatures (1/2 the scalar curvature) are given by (see Exercise 1.188)

$$K[g(t)] = \frac{1}{t} \frac{F'(r)}{r F(r)^2}. \tag{4.35}$$

Indeed, an orthonormal coframe is given by

$$\omega^1 = \sqrt{t} F(r) \, dr, \quad \omega^2 = \sqrt{r} d\theta,$$

and one has

$$\omega^2 = \frac{1}{F(r)} d\theta,$$

$$\Omega_1^2 = d\omega_1^2 = -\frac{1}{t} \frac{F'(r)}{r F(r)} \omega^1 \wedge \omega^2.$$

Let $X(t)$ be the vector fields on $\mathbb{R}^2$ defined by

$$X(t) \equiv \frac{r}{t F(r)} \frac{\partial}{\partial r} = \frac{1}{t} X(1). \tag{4.36}$$

Strictly speaking, $X(t)$ is defined on $\mathbb{R}^2 - \{0\}$; however, it will extend smoothly to $\mathbb{R}^2$. Note that for a radial function $f(r)$, its gradient with respect to $g(t)$ is given by

$$\text{grad}_{g(t)} f = \frac{1}{t F(r)} f'(r) \frac{\partial}{\partial r}.$$ 

Hence

$$X(t) = \text{grad}_{g(t)} f \quad \text{where} \quad f'(r) = r F(r). \tag{4.37}$$

We look for $F$ such that $g(t)$ is a solution to the modified Ricci flow

$$\frac{\partial}{\partial t} g(t) = -R_{g(t)} g(t) + L_{X(t)} g(t). \tag{4.38}$$

The dual 1-forms to $X(t)$ are $X(t)^j \equiv r F(r) \, dr = \frac{r}{\sqrt{t}} \omega^1$, which is time-independent. Recall that, in general, from the first structure equations

$$(\nabla_V \omega^j)(e_i) = -\omega^j(V)$$

for any vector $V$, so that $\nabla \omega^j = -\omega^j \otimes \omega^i$. Hence, using this and $\omega^1 = -\frac{1}{\sqrt{t F(r)}} \omega^2$, we find that

$$L_{X(t)} g(t) = 2 \text{Sym} (\nabla X(t)^j) = \frac{2}{\sqrt{t}} \text{Sym} (dr \otimes \omega^1 + r \nabla \omega^1) = \frac{2}{t F(r)} g(t).$$

Thus (4.38) is equivalent to the equation

$$F'(r) = r F(r)^2 \left( 1 - \frac{F(r)}{2} \right). \tag{4.39}$$
4. Ricci Solitons and Special Solutions

(Note that $F(r) = 2$ is the flat Euclidean solution, but this is not the one we are interested in.) Now from (4.37) and (4.39)

$$f'(r) = rF(r) = \frac{F'(r)}{F(r)\left(1 - \frac{F'(r)}{2}\right)}.$$ Integrating this, we have

$$f(r) = -\log\left(\frac{2}{F(r)} - 1\right).$$

**Remark 4.14.**

1. Equation (4.38) is equivalent to the static equation

$$g(1) = -Rg(1)g(1) + \mathcal{L}_{\mathcal{X}(1)}g(1).$$

2. The metrics $\psi(t)^*g(t)$ satisfy Ricci flow if the 1-parameter family of diffeomorphisms $\psi(t)$ satisfies

$$\frac{d}{dt} \big|_{t=t_0} (\psi(t) \circ \psi^{-1}(t_0)) = -\psi(t_0)^* \mathcal{X}(t_0).$$

Solving the separable ODE (4.39), we obtain

$$h(r) + \log h(r) = -r^2 + C,$$

where $h(r) \equiv \frac{2}{F(r)} - 1$. Here we have made the assumption that $0 < F < 2$ so that $h > 0$. Note that by (4.35) and (4.39) we have

$$K[g(t)] = \frac{1}{F(r)} \left(\frac{1}{2} - \frac{1}{F(r)}\right) > 0.$$}

Recall that the **product log** (or Lambert-W) function $W: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the inverse of the function $w(x) \equiv xe^x$. Taking the exponential of both sides of (4.41), we have

$$h(r)e^{h(r)} = h(0)e^{h(0)}e^{-r^2},$$

so that

$$h(r) = W\left(h(0)e^{h(0)-r^2}\right).$$

In terms of $F$, this equation says that

$$F(r) = \frac{2}{W\left(\left(\frac{2}{F(0)} - 1\right)\exp\left(\frac{2}{F(0)} - 1 - r^2\right)\right) + 1}.$$ The **cone angle** of $g(t)$ at infinity (which is independent of $t$) is

$$\text{ConeAngle} = \frac{2\pi F(0)}{\lim_{r \to \infty} F(r)} = \pi F(0).$$
Hence the range of possible cone angles at infinity is \((0, 2\pi)\) since \(F(0) \in (0, 2)\). Now (4.42) and (4.43) imply
\[
K[g(t)] = \frac{1}{2t} W\left(\left(\frac{2}{F(0)} - 1\right) \exp\left(\frac{2}{F(0)} - 1 - r^2\right)\right).
\]
Finally we note from (4.40) that we have
\[
f(r) = -\log h(r) = -\log W\left(h(0)e^{h(0)}e^{-r^2}\right).
\]

**Exercise 4.15** (Exponential curvature decay of expanding 2d Ricci soliton). Show that each of the metrics \(g(t)\) is asymptotic at infinity to a flat cone and the curvature decays exponentially as a function of the distance to the origin.

**Hint:** \(\lim_{x \to 0^+} \frac{W(x)}{x} = 1\).

**Exercise 4.16.** Consider a sequence of points \(\{x_i\}\) with
\[
\lim_{i \to \infty} d_{g(1)}(x_i, O) = \infty,
\]
where \(O\) is the origin. Show (or convince yourself) that the Gromov-Hausdorff limit of the pointed sequence of Riemannian surfaces\(^2\)
\[(\mathbb{R}^2, R(x_i, 1) g(1), x_i)\]
is a (flat) cone with base-point at the vertex. Explain why the cone is flat even though the scalar curvatures of the metrics \(R(x_i, 1) g(1)\) at \(x_i\) are all equal to 1.

**Hint:** What is the Gromov-Hausdorff limit of the sequence
\[(\mathbb{R}^2, R(x_i, 1) g(1), O)\]?

In higher dimensions there are expanding **Kähler-Ricci solitons** on \(\mathbb{C}^n\) due to H.-D. Cao [75] (compare also Feldman, Ilmanen and Knopf [221]).

Expanding gradient solitons motivate the definition of the matrix Harnack quadratic (see Section 4 of Chapter 10).

### 6. Bryant soliton

Let \(g_{S^{n-1}}\) denote the standard metric on the unit \((n - 1)\)-sphere. We search for warped product steady Ricci solitons on \((0, \infty) \times S^{n-1}\) which extend to Ricci solitons on \(\mathbb{R}^n\) by a 1-point compactification of one end. We call the compactifying point the **origin** \(O\). In particular, consider metrics of the form
\[
(4.44) \quad g = dr^2 + \phi(r)^2 g_{S^{n-1}}.
\]

\(^2\)See for example [63] for the definition of Gromov-Hausdorff limit.
From $\text{Rc}(g_{S^{n-1}}) = (n-2)g_{S^{n-1}}$ and a standard formula for the Ricci tensor of a \textbf{warped product metric} (see Exercise 1.124 or [51], Proposition 9.106), we have

$$\text{Rc}(g) = -(n-1)\frac{\phi''}{\phi} dr^2 + \left((n-2)(1-(\phi')^2) - \phi\phi''\right)g_{S^{n-1}}.$$ 

One can also see this from the formulas for the sectional curvatures. By (1.143) the sectional curvature of a plane passing through the radial vector $\frac{\partial}{\partial r}$ is $K_{\text{rad}} = -\frac{\phi''(r)}{\phi(r)}$, and the sectional curvature of a plane $P_{\text{sph}}$ perpendicular to $\frac{\partial}{\partial r}$ is $K_{\text{sph}} = \frac{1-\phi'(r)^2}{\phi(r)^2}$. We can also calculate that the Hessian of a function $f$ is given by

$$\nabla\nabla f = f''(r)dr^2 + \phi\phi'f'g_{S^{n-1}}.$$ 

The steady Ricci soliton equation $\text{Rc}(g) + \nabla\nabla f = 0$ becomes the following system of two second-order ODE:

$$f'' = (n-1)\frac{\phi''}{\phi}, \quad \phi\phi'f' = -(n-2)(1-(\phi')^2) + \phi\phi''.$$ 

Making the substitutions

$x = \phi', \quad y = \phi f' + (n-1)\phi', \quad dt = \frac{dr}{\phi},$

we obtain the following system of first-order ODE:

$$(4.45) \quad \frac{dx}{dt} = x(x-y) + n-2, \quad \frac{dy}{dt} = x(y - (n-1)x).$$

The constant solutions are $(1, n-1)$ and $(-1, -(n-1))$. For $g$ and $f$ to extend smoothly over the origin, we need $\phi(0) = 0$, $\phi'(0) = 1$ and $f'(0) = 0$. Since then $\lim_{r\to0} t = -\infty$, we have $\lim_{t\to-\infty} (x(t), y(t)) = (1, n-1)$. Linearizing (4.45) at $(1, n-1)$, we have

$$\frac{dX}{dt} = (3-n)X - Y, \quad \frac{dY}{dt} = -(n-1)X + Y.$$ 

The eigenvalues of the matrix $\begin{pmatrix} 3-n & -1 \\ -(n-1) & 1 \end{pmatrix}$ are 2 and $2-n$, so we have a saddle point at $(1, n-1)$. Hence there are two solutions with $\lim_{t\to-\infty} (x(t), y(t)) = (1, n-1)$. For one of these solutions we have

$$(x(t), y(t)) \to (0, \infty)$$

as $t$ increases. It can be shown (see Chapter 1 of [153]) that this solution defines a complete Ricci soliton $g$ on $\mathbb{R}^n$ of the form (4.44), called the Bryant
7. Homogeneous solutions

This solution has positive curvature operator. From the formulas for the sectional curvatures (1.143) and from

\[ C^{-1} r^{1/2} \leq \phi (r) \leq C r^{1/2}, \quad \phi' (r) = O \left( r^{-1/2} \right), \quad \phi'' (r) = O \left( r^{-3/2} \right), \]

we have

(4.46) \[ K_{\text{rad}} = O \left( r^{-2} \right), \quad K_{\text{sph}} = O \left( r^{-1} \right). \]

That is, we have the following.

**Theorem 4.17** (Bryant soliton). For all \( n \geq 3 \), there exists a unique (up to homothety) rotationally symmetric complete, steady, gradient Ricci soliton metric on \( \mathbb{R}^n \) with positive curvature operator. The eigenspaces of the curvature operator consist of 2-forms that are the wedge product of two 1-forms. The corresponding planes are either tangent to the spheres, in which case the sectional curvatures decay inverse linearly in distance to the origin, or they pass through the radial direction, in which case the sectional curvatures decay inverse quadratically.

The volume of the ball of radius \( s \) centered at the origin is

\[ \text{Vol} (B(O, s)) = n \omega_n \int_0^s \phi(r)^{n-1} dr. \]

For \( s \geq 1 \) we have

\[ \text{Vol} (B(O, s)) \approx \int_0^s r^{n-1} \frac{n-1}{2} dr \approx s \left( \frac{n+1}{2} \right). \]

For example, if \( n = 3 \), then the volume of balls grows quadratically in the radius.

In [324], Ivey constructed Ricci solitons on doubly warped products which generalize the Bryant soliton.

The Bryant soliton should appear as a singularity model corresponding to a degenerate neckpinch (see Section 2 in Chapter 9).

7. Homogeneous solutions

We say that a Riemannian manifold \((M^n, g)\) is (globally) **homogeneous** if for every \( x, y \in M^n \) there exists an isometry \( \iota : M^n \to M^n \) with \( \iota (x) = y \). A nice class of homogeneous manifolds is Lie groups with left-invariant metrics.

**7.1. 3-dimensional unimodular Lie groups.** Suppose that \( G \) is a 3-dimensional **unimodular** (i.e., its volume form is bi-invariant) Lie group with a left-invariant metric \( g \). Then there exists a left-invariant frame field
\( \{ f_i \}_{i=1}^3 \) with dual coframe field \( \{ \eta^i \}_{i=1}^3 \) such that there are positive constants \( A, B, C \) such that the metric is diagonal:

\[
g = A\eta^1 \otimes \eta^1 + B\eta^2 \otimes \eta^2 + C\eta^3 \otimes \eta^3
\]

and the Lie brackets are of the form

\[
[f_i, f_j] = \epsilon^k_{ij} f_k,
\]

where \( \epsilon^k_{ij} \in \{ 2, 0, -2 \} \) and \( \epsilon^k_{ij} = 0 \) unless \( i, j, k \) are distinct. Such a frame field is called a \textbf{Milnor frame} \cite{412}. Let \( \lambda \equiv \epsilon^1_{23}, \mu \equiv \epsilon^2_{31}, \nu \equiv \epsilon^3_{12} \). The frame field \( \{ e_i \}_{i=1}^3 \) defined by

\[
e_1 \equiv A^{-1/2} f_1, \quad e_2 \equiv B^{-1/2} f_2, \quad e_3 \equiv C^{-1/2} f_3
\]

is orthonormal.

Formula (4.47) implies

\[
[e_i, e_j] = \frac{\lambda_k \epsilon^k_{ij}}{(\lambda_i \lambda_j \lambda_k)^{1/2}} e_k = \frac{\lambda_k \epsilon^k_{ij}}{(\lambda_1 \lambda_2 \lambda_3)^{1/2}} e_k,
\]

where \( \lambda_1 = A, \lambda_2 = B \) and \( \lambda_3 = C \). By (1.174) the components of the Levi-Civita connection are

\[
\langle \nabla e_i e_j, e_k \rangle = \frac{1}{2} (\langle [e_i, e_j], e_k \rangle - \langle [e_i, e_k], e_j \rangle - \langle [e_j, e_k], e_i \rangle)
\]

\[
= \frac{1}{2} \frac{1}{(\lambda_1 \lambda_2 \lambda_3)^{1/2}} \left( \lambda_k \epsilon^k_{ij} - \lambda_j \epsilon^j_{ik} - \lambda_i \epsilon^i_{jk} \right).
\]

Substituting this into (1.176) and since \( \nabla e_j e_j = 0 \), we have

\[
\langle \text{Rm}(e_i, e_j)e_j, e_i \rangle = \langle \nabla e_i, e_j, \nabla e_j, e_i \rangle - \langle \nabla e_j, e_j, \nabla e_i, e_i \rangle - \langle \nabla [e_i, e_j]e_j, e_i \rangle
\]

\[
= \frac{1}{4\lambda_1 \lambda_2 \lambda_3} \left( \lambda_k \epsilon^k_{ij} - \lambda_j \epsilon^j_{ik} - \lambda_i \epsilon^i_{jk} \right) \left( \lambda_k \epsilon^k_{ji} - \lambda_i \epsilon^i_{jk} - \lambda_j \epsilon^j_{ik} \right)
\]

\[
- \frac{1}{2\lambda_1 \lambda_2 \lambda_3} \lambda_k \epsilon^k_{ij} \left( \lambda_i \epsilon^i_{kj} - \lambda_j \epsilon^j_{ki} - \lambda_k \epsilon^k_{ij} \right)
\]

\[
= \frac{1}{4\lambda_1 \lambda_2 \lambda_3} \left( \left( \lambda_i \epsilon^i_{kj} - \lambda_j \epsilon^j_{ki} \right)^2 - \left( \lambda_k \epsilon^k_{ij} \right)^2 \right)
\]

\[
+ \frac{2}{4\lambda_1 \lambda_2 \lambda_3} \lambda_k \epsilon^k_{ij} \left( \lambda_i \epsilon^i_{kj} + \lambda_j \epsilon^j_{ki} - \lambda_k \epsilon^k_{ij} \right)
\]

where \( \{ i, j, k \} \) is a permutation of \( \{ 1, 2, 3 \} \) and we have used the anti-symmetry of \( \epsilon^k_{ij} \) in \( i \) and \( j \). Thus the sectional curvatures

\[ K (e_i \wedge e_j) = \]
\( \langle \text{Rm}(e_i, e_j)e_j, e_i \rangle \) are given by

\[
K(e_2 \wedge e_3) = \frac{(\mu B - \nu C)^2}{4ABC} + \lambda \frac{2\mu B + 2\nu C - 3\lambda A}{4BC},
\]
\[
K(e_3 \wedge e_1) = \frac{(\nu C - \lambda A)^2}{4ABC} + \mu \frac{2\nu C + 2\lambda A - 3\mu B}{4AC},
\]
\[
K(e_1 \wedge e_2) = \frac{(\lambda A - \mu B)^2}{4ABC} + \nu \frac{2\lambda A + 2\mu B - 3\nu C}{4AB},
\]

and the \( \langle \text{Rm}(e_k, e_i)e_j, e_k \rangle = 0 \) for any \( i \neq j \) and \( k \). From this we can easily derive that the Ricci tensor is diagonal and is given by

\[
\text{Rc}(e_1, e_1) = \frac{(\lambda A)^2 - (\mu B - \nu C)^2}{2ABC},
\]
\[
\text{Rc}(e_2, e_2) = \frac{(\mu B)^2 - (\nu C - \lambda A)^2}{2ABC},
\]
\[
\text{Rc}(e_3, e_3) = \frac{(\nu C)^2 - (\lambda A - \mu B)^2}{2ABC}.
\]

Hence the Ricci flow equation is equivalent to the following system:

\[
\frac{dA}{dt} = \frac{(\mu B - \nu C)^2 - (\lambda A)^2}{BC},
\]
\[
\frac{dB}{dt} = \frac{(\nu C - \lambda A)^2 - (\mu B)^2}{AC},
\]
\[
\frac{dC}{dt} = \frac{(\lambda A - \mu B)^2 - (\nu C)^2}{AB}.
\]

We note that the normalized Ricci flow (3.3) is

\[
\frac{dA}{dt} = -4(\lambda A)^2 + 2(\mu B)^2 + 2(\nu C)^2 - 4\mu B \cdot \nu C + 2\nu C \cdot \lambda A + 2\lambda A \cdot \mu B \quad \frac{3BC}{3BC},
\]
\[
\frac{dB}{dt} = 2(\lambda A)^2 - 4(\mu B)^2 + 2(\nu C)^2 + 2\mu B \cdot \nu C - 4\nu C \cdot \lambda A + 2\lambda A \cdot \mu B \quad \frac{3AC}{3AC},
\]
\[
\frac{dC}{dt} = 2(\lambda A)^2 + 2(\mu B)^2 - 4(\nu C)^2 + 2\mu B \cdot \nu C + 2\nu C \cdot \lambda A - 4\lambda A \cdot \mu B \quad \frac{3AB}{3AB},
\]

since by summing up (4.48)–(4.50), we see that the scalar curvature is

\[
R = -\frac{(\lambda A)^2 - (\mu B)^2 - (\nu C)^2 + 2\mu B \cdot \nu C + 2\nu C \cdot \lambda A + 2\lambda A \cdot \mu B}{2ABC}.
\]
7.2. Two examples: SU(2) and Nil. First we consider the case where the Lie group $G$ is SU(2) so that there exists a frame such that $\lambda = \mu = \nu = -2$. In this case

$$R = 2 \frac{-A^2 - B^2 - C^2 + 2BC + 2CA + 2AB}{ABC}$$

$$= 2 \frac{A^2 + B^2 + C^2 - (B - C)^2 - (A - C)^2 - (A - B)^2}{ABC}.$$  

Note the special case where $B = C$ in which case $R = 2 \frac{4B - A}{B^2}$. In particular, when $B = C < \frac{A}{4}$, we have $R < 0$. Unlike the case of the Ricci flow on surfaces, the scalar curvature does not remain negative on SU(2). In fact we have the following (see pp. 728–729 of [320]).

**Theorem 4.18 (Isenberg-Jackson).** For any left-invariant initial metric $g_0$ on SU(2) there exists a solution $g(t)$ of the normalized Ricci flow defined for all $t \in [0, \infty)$ with $g(0) = g_0$ such that $g(t)$ converges to a constant positive sectional curvature metric as $t \to \infty$.

On SU(2) the normalized Ricci flow equations (4.54)–(4.56) become

$$\frac{dA}{dt} = 4 \frac{-4A^2 + 2B^2 + 2C^2 - 4BC + 2CA + 2AB}{3BC},$$

$$\frac{dB}{dt} = 4 \frac{2A^2 - 4B^2 + 2C^2 + 2BC - 4CA + 2AB}{3AC},$$

$$\frac{dC}{dt} = 4 \frac{2A^2 + 2B^2 - 4C^2 + 2BC + 2CA - 4AB}{3AB}.$$  

Under the normalized flow, $(ABC)(t)$ is independent of time. Hence we may assume without loss of generality that $(ABC)(t) \equiv 8/3$. We then have

$$\frac{dA}{dt} = A \left( A(B + C - 2A) + (B - C)^2 \right),$$

$$\frac{dB}{dt} = B \left( B(A + C - 2B) + (A - C)^2 \right),$$

$$\frac{dC}{dt} = C \left( C(A + B - 2C) + (A - B)^2 \right).$$  

From this we may compute the evolution equations for the difference of the metric components:

$$\frac{d}{dt} (A - C) = 2C^3 - 2A^3 + AB^2 + A^2B - BC^2 - B^2C$$

$$= \left( -2(A^2 + AC + C^2) + B^2 + AB + BC \right) (A - C)$$

and similarly for $\frac{d}{dt} (A - B)$ and $\frac{d}{dt} (B - C)$. We assume without loss of generality that $A(0) \geq B(0) \geq C(0)$.
7. Homogeneous solutions

By the above equations, we have
\[ A(t) \geq B(t) \geq C(t) \]
for all \( t > 0 \). Since \( A + B - 2C \geq 0 \), equation (4.59) implies \( \frac{dC}{dt} \geq 0 \) so that \( C(t) \geq C(0) \).

Now we may estimate the factor on the RHS of (4.60):
\[
-2 (A^2 + AC + C^2) + B^2 + AB + BC
= -2C^2 - (A^2 - B^2) - AC - (A + C)(A - B)
\leq -2C^2 \leq -2C(0)^2.
\]
Thus
\[
\frac{d}{dt} (A - C) \leq -2C(0)^2 (A - C).
\]
We conclude that
\[
A(t) - C(t) \leq (A(0) - C(0)) e^{-2C(0)^2t}
\]
for all \( t > 0 \). That is, \( A - C \) decays exponentially to zero. Since \( A \geq B \geq C \) and since we have normalized the volume so that \( ABC \equiv \frac{8}{3} \), we conclude that \( A(t), B(t), C(t) \) exponentially converge to \( A_\infty = B_\infty = C_\infty \div \frac{2}{\sqrt{3}} \).
That is, \( g(t) \) exponentially converges as \( t \to \infty \) to the constant sectional curvature metric
\[
g_\infty \div \left( \frac{2}{\sqrt{3}} \right) (\eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2 + \eta^3 \otimes \eta^3).
\]

Second we consider the case where \( G \) is the Heisenberg group (Nil) of upper-triangular \( 3 \times 3 \) real matrices. In this case there is a frame where \( \lambda = -2 \) and \( \mu = \nu = 0 \). Then for Nil the Ricci flow equations (4.51)–(4.53) are equivalent to
\[
\frac{d}{dt} \log A = \frac{d}{dt} \log B = \frac{d}{dt} \log C = 4 \frac{A}{BC}.
\]
We immediately see that \( A \) is decreasing whereas \( B \) and \( C \) are increasing. In fact \( B/C, AB \) and \( AC \) are independent of time. We compute
\[
\frac{d}{dt} \log \left( \frac{A}{BC} \right) = -12 \frac{A}{BC},
\]
so that
\[
\frac{A}{BC}(t) = \frac{1}{12} \left( \frac{B_0C_0}{12A_0} + t \right)^{-1},
\]
where \( A_0 \div A(0), B_0 \div B(0), C_0 \div C(0) \). Thus one can explicitly solve (4.61) to get
\[
\frac{A_0}{A(t)} = \frac{B(t)}{B_0} = \frac{C(t)}{C_0} = \left( 1 + \frac{12A_0}{B_0C_0}t \right)^{1/3} \leq \text{const} \cdot (t + 1)^{1/3}.
\]
The sectional curvatures

\[ K(e_2 \wedge e_3) = -\frac{3A}{BC} = -3 \left( \frac{B_0 C_0}{A_0} + 12t \right)^{-1}, \]

(4.63a)

\[ K(e_3 \wedge e_1) = \frac{A}{BC} = \left( \frac{B_0 C_0}{A_0} + 12t \right)^{-1}, \]

(4.63b)

\[ K(e_1 \wedge e_2) = \frac{A}{BC} = \left( \frac{B_0 C_0}{A_0} + 12t \right)^{-1}. \]

(4.63c)

satisfy \(|\text{sect}(g(t))| \leq \text{const} \cdot t^{-1}\). Note that the scalar curvature is negative as it must be for the solution to exist for all time. If we consider a compact quotient of the Heisenberg group, such as \(G/\mathbb{Z}^3\), the diameters satisfy \(\text{diam}(g(t)) \leq \text{const} \cdot (t + 1)^{1/6}\). Hence

\[ |\text{sect}(g(t))| \text{ diam}(g(t))^2 \leq \text{const} \cdot t^{-2/3} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \]

(4.64)

Thus the solution \textbf{collapses} as time tends to infinity. In fact, (4.64) says that the metrics become more and more \textbf{almost flat}. The only nonzero bracket is \([e_2, e_3] = -2\sqrt{A/BC} e_1 \approx -\frac{1}{\sqrt{3}} t^{-1/2} e_1\), so that the brackets tend to zero when measured in an orthonormal frame.

\textbf{Exercise 4.19.} Another way to compute the curvatures of a left-invariant metric on a Lie group is as follows. Suppose that \(G\) is a Lie group and \(\{f_i\}_{i=1}^n\) is a left-invariant frame field with

\[ [f_i, f_j] = c_{ij}^k f_k \]

and \(g = a_i^\eta \eta^i \otimes \eta^i\), where \(\{\eta^i\}_{i=1}^n\) is the dual coframe to \(\{f_i\}_{i=1}^n\) (i.e., \(\eta^i(f_j) = \delta^i_j\)). We compute

\[ d\eta^i(f_j, f_k) = f_j(\eta^i(f_k)) - f_k(\eta^i(f_j)) - \eta^i([f_j, f_k]) \]

\[ = -c_{jk}^i. \]

Thus

\[ d\eta^i = -c_{jk}^i \eta^j \wedge \eta^k. \]

An orthonormal frame field is \(\{e_i\}\) where \(e_i = a_i^{-1} f_i\) and the dual coframe is \(\{\omega^i\}\) where \(\omega^i = a_i \eta^i\). Since

\[ d\omega^i(e_j, e_k) = -\frac{1}{2} \frac{a_i}{a_j a_k} c_{jk}^i, \]

we have by (1.81) that

\[ \omega^k = \left( d\omega^i(e_j, e_k) + d\omega^j(e_i, e_k) - d\omega^k(e_j, e_i) \right) \omega^i \]

\[ = \left( -\frac{a_i}{a_j a_k} c_{jk}^i - \frac{a_j}{a_i a_k} c_{ik}^j + \frac{a_k}{a_i a_j} c_{ji}^k \right) \omega^i. \]
Then we may use the second Cartan structure equation to compute the curvature 2-form
\[ \Omega^k_i = d\omega^k_i - \omega^j_i \wedge \omega^k_j. \]

Compute \( \langle Rm(e_i, e_j) e_k, e_\ell \rangle \).

7.3. Ricci flow on compact Lie groups with bi-invariant metrics.
Let \((G, g)\) be a compact Lie group with a bi-invariant metric. Recall that a metric is bi-invariant if and only if the induced inner product on the Lie algebra \(\mathfrak{g}\) is Ad-invariant. The universal cover \(\tilde{G}\) is a simply-connected Lie group and \(\tilde{G} \cong G_1 \times \cdots \times G_k \times \mathbb{R}^m\), where each \(G_i\), for \(i = 1, \ldots, k\), is a simple (i.e., all normal subgroups are finite) compact Lie group. The adjoint representation
\[ \text{Ad}\tilde{G} : G_1 \times \cdots \times G_k \times \mathbb{R}^m \hookrightarrow \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_1) \times \cdots \times \text{Hom}(\mathfrak{g}_k, \mathfrak{g}_k) \times \text{gl}(m, \mathbb{R}) \]
is the diagonal representation
\[ \text{Ad}\tilde{G} = \text{Ad}G_1 \times \cdots \times \text{Ad}G_k \times 0, \]
where \(\text{Ad}G_i : G_i \hookrightarrow \text{Hom}(\mathfrak{g}_i, \mathfrak{g}_i)\), for \(i = 1, \ldots, k\), is the adjoint representation and 0 is the trivial representation of \(\mathbb{R}^m\). Since \(\text{Ad}G_i\) is an irreducible representation of a compact group, there is a unique positive definite symmetric \(\text{Ad}G_i\)-invariant bilinear form on \(\mathfrak{g}_i\) up to scaling. Hence, there is a unique bi-invariant metric \(g_i\) on \(G_i\) up to scaling. Moreover, \(g_i\) is Einstein. One way to see this is by evolving \(g_i\) by the Ricci flow, which preserves the bi-invariance of the metric. In particular, \(g_i(t) = C(t)g_i\), which implies that \(\text{Rc}(g_i) = C \cdot g_i\), for some constant \(C\). Alternatively, one may use the fact that the Ricci tensor of a bi-invariant metric on a simple compact Lie group is positive-definite to conclude that it is a constant multiple of the metric.

**Exercise 4.20.** Describe the Ricci flow on a compact Lie group with a bi-invariant initial metric.

8. The isometry group

If \((M^n, g(t))\) is a solution to the Ricci flow on a closed manifold and \(\gamma\) is an isometry of \(M^n\) with respect to the metric \(g(0)\), then \(\gamma\) is also an isometry of \(M^n\) with respect to \(g(t)\) for all \(t > 0\). This follows from the uniqueness of solutions of the Ricci flow with a given initial metric, for if \(\gamma\) is an isometry of \(g(0)\), then \(g(t)\) and \(\gamma^*g(t)\) are both solutions of the Ricci flow with initial metric \(g(0)\). Hence by uniqueness, \(g(t) = \gamma^*g(t)\) for all \(t > 0\).

**Problem 4.21** (A. Fisher). If \((M^n, g(t))\) is a solution to the Ricci flow on a closed manifold, then is \(\text{Isom}(g(t)) = \text{Isom}(g(0))\) for all \(t > 0\)?
By the above remarks, we see that \( \operatorname{Isom}(g(t)) \supset \operatorname{Isom}(g(0)) \). Since we find it hard to believe that a spontaneous symmetry can be created, we conjecture that we actually have equality. Note that the problem easily reduces to a backward uniqueness question:

**Problem 4.22.** If \( g_1(t) \) and \( g_2(t) \) are solutions to the Ricci flow on a closed manifold on a time interval \([0, T]\) and if \( g_1(t') = g_2(t') \) for some \( t' \in (0, T) \), then is \( g_1(t) = g_2(t) \) for all \( t \in [0, t'] \) (and hence for all \( t \in [0, T) \))? Indeed, if \( \gamma \) is an isometry of \( g(t') \), then \( g(t) \) and \( \gamma^*g(t) \) are solutions of the Ricci flow with \( g(t') = \gamma^*g(t') \); an affirmative answer to the latter problem would imply \( g(t) = \gamma^*g(t) \) for all \( t \). In particular, \( \gamma \) would be an isometry of \( g(0) \).

**Remark 4.23.** Apparently it is not known if \( g(t), t \in [0, T) \), is a solution on a closed manifold such that \( g(t) \) has constant sectional curvature for \( t \in [t_0, T) \) where \( t_0 > 0 \), whether or not \( g(t) \) has constant curvature for all \( t \in [0, T) \). Can one show that this is true when \( n = 2 \) or for the Kähler-Ricci flow?

9. Notes and commentary

**Section 1.** Some of the material in this chapter overlaps with [163]. For further discussion of Ricci solitons, we refer the reader to [153]. The Ricci soliton equation appears in equation (2.2.1) on p. 395 of Friedan [228], where the non-Einstein solutions are called quasi-Einstein metrics.

**Section 2.** There are nontrivial (i.e., non-Einstein) shrinking solitons on closed Kähler manifolds due to Koiso [355].

**Section 3.** The analogue to the cigar soliton for the curve shortening flow (or CSF for short) of a plane curve \( \frac{\partial x}{\partial t} = -\kappa \nu \), where \( \kappa \) is the curvature and \( \nu \) is the unit outward normal, is the grim reaper translating (steady) soliton:

\[
y = t + \log \sec x
\]
for \( x \in (-\pi/2, \pi/2) \) and \( t \in \mathbb{R} \). The CSF was first proposed by Mullins [427] to model the motion of idealized grain boundaries. Mullins also discovered the grim reaper solution (a terminology later coined by Matt Grayson). For curves which are graphs of functions \( y = f(x, t) \), the curve shortening flow becomes

\[
\frac{\partial y}{\partial t} = \left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{-1}\frac{\partial^2 y}{\partial x^2}.
\]

From this one easily checks that the curves given by (4.65) are solutions to the CSF. If we let \( \theta \) denote the angle that the unit tangent vector makes
with the $x$-axis, then we have \[ \kappa = \cos \theta. \]

**Exercise 4.24.** Prove (4.66).

**Hint:** Show that \( \kappa = \left(1 + (y_x)^2\right)^{-3/2} y_{xx} \) using \( T = \frac{(1, y_x)}{\sqrt{1 + (y_x)^2}} \).

The only embedded convex plane curves which are shrinking solitons are round circles. On the other hand, there are immersed convex plane curves which are shrinking solitons due to Abresch and Langer [1].

Next we consider steady soliton solutions of the 1-dimensional heat equation of the form \( u(x,t) = F\left(\frac{x}{\sqrt{t}}\right) \). The heat equation leads to the ODE
\[ F''(y) = -\frac{1}{2} y F'(y), \]
where \( y = \frac{x}{\sqrt{t}} \). Thus, for some \( A \in \mathbb{R} \)
\[ F'(y) = A e^{-\frac{1}{2} y^2}. \]
We conclude that
\[ u(x,t) = B + A \sqrt{\pi} \text{erf}\left(\frac{x}{2\sqrt{t}}\right) \]
for some \( A, B \in \mathbb{R} \) and where
\[ \text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-w^2} dw. \]
Note that
\[ u(x,t+s) = u\left(\frac{\sqrt{t}}{\sqrt{t+s}} x, t\right). \]

**Section 4.** The analogue of the Rosenau solution for the CSF is
\[ y = \log \left(\cos x + \sqrt{e^{2t} + \cos^2 x}\right) - t \]
or \(-\infty < t < 0\). This solution can be derived as follows. The evolution of the curvature \( \kappa(\theta,t) \) is given by \( \kappa_t = \kappa^2 \kappa_{\theta\theta} + \kappa^3 \). Searching for a solution of the form \( \kappa(\theta,t) = a(\theta) + b(t) \) leads to the equations
\[ a''(\theta) + 4a(\theta) = 0, \quad (a'(\theta))^2 + 4a(\theta)^2 = 4C^2, \quad b'(t) - 2b(t)^2 = -2C^2, \]
where \( C \) is a constant. Taking \( C = 1 \), we have the particular solution \( a(\theta) = \cos 2\theta \) and \( b(t) = \coth(-2t) \) for \( t < 0 \). Hence the solution satisfies \( \kappa(\theta,t) = \sqrt{\cos 2\theta + \coth(-2t)} \).

From this we can derive (4.67). Note that
\[ \lim_{t \to -\infty} \kappa(\theta,t) = \sqrt{2} |\cos \theta|. \]
This exhibits the fact that the limit as $t \to -\infty$ at either end of the oval is the grim reaper (without rescaling).

In Olwell [444] translating solutions to the Gauss curvature flow are constructed.

**Section 5.** Recall that graph solutions $y = y(x,t)$ of the curve shortening flow satisfy (4.66). We look for solutions of the form

(4.68) \[ y(x,t) = \sqrt{t} F\left(\frac{x}{\sqrt{t}}\right). \]

Such solutions satisfy \[ y(x,c^2 t) = cy\left(\frac{x}{c}, t\right). \]

We compute

\[ y_t = -\frac{x}{2t} F'(\frac{x}{\sqrt{t}}) + \frac{1}{2\sqrt{t}} F\left(\frac{x}{\sqrt{t}}\right), \]
\[ y_x = F'\left(\frac{x}{\sqrt{t}}\right) \]

and

\[ y_{xx} = \frac{1}{\sqrt{t}} F''\left(\frac{x}{\sqrt{t}}\right). \]

Under the assumption (4.68), setting $r \doteq \frac{x}{\sqrt{t}}$, we find that (4.66) is equivalent to

\[ -\frac{r}{2} F'(r) + \frac{1}{2} F(r) = \frac{F''(r)}{F'(r)^2 + 1}. \]

We may rewrite this as the second-order ODE

(4.69) \[ 2F''(r) + r [F'(r)]^2 - [F'(r)]^2 F(r) + rF'(r) - F(r) = 0. \]

**Lemma 4.25.** There exists a solution of (4.69) with boundary values:

\[ \lim_{r \to 0} F(r) = \infty \quad \text{and} \quad \lim_{r \to \infty} F(r) = 0. \]

This solution is called the $90^\circ$ *Brakke wedge* (see [58]).

**Exercise 4.26.** Prove the above lemma.

A reference for some results on 2-dimensional gradient Ricci solitons is L.-F. Wu [567]. See Baird and Danielo [33] for a construction of Ricci solitons.

**Section 7.** There is a nice presentation of the Ricci flow on homogeneous manifolds in Knopf [345], which we have partially followed. See Carfora, Isenberg, and Jackson [91], Guenther, Isenberg, and Knopf [266], Hamilton and Isenberg [294], Isenberg and Jackson [320], Knopf [346] and Knopf and McLeod [350] for further discussion of the Ricci flow of homogeneous 3-manifolds and related *quasi-stability/convergence* questions.
(some in the category of warped products). Besides SU(2) and Nil, some other 3-dimensional homogeneous geometries correspond to Solv, $\widetilde{\text{SL}}(2, \mathbb{R})$, $\widetilde{\text{Isom}}(\mathbb{R}^2)$, and $\mathbb{E}^3$. In the case of Solv and $\widetilde{\text{SL}}(2, \mathbb{R})$, homogeneous solutions exist for all time and $|\text{Rm}| \leq Ct^{-1}$. In the case of $\widetilde{\text{Isom}}(\mathbb{R}^2)$ the solution exists for all time and $|\text{Rm}| \leq Ce^{-ct}$ while for $\mathbb{E}^3$ the solution is flat and hence stationary. These geometries roughly correspond to Thurston’s model geometries but are not in 1-1 correspondence, since the geometries he considers have maximal isotropy groups. See [321] for work on the Ricci flow on 4-dimensional homogeneous spaces. See also Lauret [362] for a study of the Ricci flow on homogeneous nilmanifolds.