Problem 7. Do part of Problem 6 on p. 271: Prove by contradiction that the only solution to the diophantine equation $3x^2 + 4y^2 = 5z^2$ is $(x, y, z) = (0, 0, 0)$ (the trivial solution).

Hint: Suppose there exists a nontrivial solution. Prove that there exists a solution $(x_1, y_1, z_1)$ such that $x_1 \not\equiv 0 \pmod{5}$ or $y_1 \not\equiv 0 \pmod{5}$.

Proof. Suppose that $(x_0, y_0, z_0)$ is a nontrivial solution to the equation $3x^2 + 4y^2 = 5z^2$. This means that $3x_0^2 + 4y_0^2 = 5z_0^2$ and $(x_0, y_0, z_0) \not= (0, 0, 0)$.

Just like we can take the gcd of two integers, we can take the gcd of three integers. Let $g = \gcd(x_0, y_0, z_0)$ be the greatest common divisor of $x_0, y_0, z_0$. Define

$$x_1 = \frac{x_0}{g}, \quad y_1 = \frac{y_0}{g}, \quad z_1 = \frac{z_0}{g}.$$ 

Then $\gcd(x_1, y_1, z_1) = 1$.

Since $3x_0^2 + 4y_0^2 = 5z_0^2$, we have $3\left(\frac{x_0^2}{g^2}\right) + 4\left(\frac{y_0^2}{g^2}\right) = 5\left(\frac{z_0^2}{g^2}\right)$. That is, $3x_1^2 + 4y_1^2 = 5z_1^2$. In other words, $(x_1, y_1, z_1)$ is a solution to the equation $3x^2 + 4y^2 = 5z^2$. Clearly $(x_1, y_1, z_1) \not= (0, 0, 0)$, i.e., is nontrivial.

Claim: $x_1 \not\equiv 0 \pmod{5}$.

Proof by contradiction. Suppose $x_1 \equiv 0 \pmod{5}$. Then 5 divides $x_1$. We have

$$4y_1^2 = 5z_1^2 - 3x_1^2$$

and 5 divides the RHS. So 5 divides $4y_1^2$. Since 5 and 4 are coprime, 5 divides $y_1$. Since 5 is prime, 5 divides $y_1$. Now we go back to the equation $3x_1^2 + 4y_1^2 = 5z_1^2$ and see that since 5 divides $x_1$ and 5 divides $y_1$, there exist integers $m, n$ such that $x_1 = 5m$ and $y_1 = 5n$. Thus

$$5z_1^2 = 3(5m)^2 + 4(5n)^2 = 5^2(3m^2 + 4n^2).$$

Hence

$$z_1^2 = 5(3m^2 + 4n^2).$$

This implies that 5 divides $z_1^2$. Hence 5 divides $z_1$. This and that 5 divides $x_1$ and 5 divides $y_1$ now implies that $\gcd(x_1, y_1, z_1) \geq 5$, a contradiction (to the assumption $x_1 \equiv 0 \pmod{5}$). Hence we have the claimed $x_1 \not\equiv 0 \pmod{5}$.

Note that in an exactly analogous way we can prove that $y_1 \not\equiv 0 \pmod{5}$.

By Problem #1(b), $x_1^2 \equiv 1 \pmod{5}$ or $x_1^2 \equiv 4 \pmod{5}$ since $x_1 \not\equiv 0 \pmod{5}$.

Similarly, we have for $y_1^2 \equiv 1 \pmod{5}$ or $y_1^2 \equiv 4 \pmod{5}$ since $y_1 \not\equiv 0 \pmod{5}$.

Now we have $3x_1^2 \equiv 3 \cdot 1 \equiv 3 \pmod{5}$ or $3x_1^2 \equiv 3 \cdot 4 \equiv 2 \pmod{5}$. Similarly, $4y_1^2 \equiv 4 \cdot 1 \equiv 4 \pmod{5}$ or $4y_1^2 \equiv 4 \cdot 4 \equiv 1 \pmod{5}$. We conclude that

$$3x_1^2 + 4y_1^2$$

is congruent to 3 or 2 plus 4 or 1 mod 5. That is, modulo 5 the possibilities for $3x_1^2 + 4y_1^2$ are

$$3 + 4 \equiv 2, \quad 3 + 1 \equiv 4, \quad 2 + 4 \equiv 1, \quad 2 + 1 \equiv 3 \pmod{5}.$$ 

None of these is $\equiv 0 \pmod{5}$, but $3x_1^2 + 4y_1^2 = 5z_1^2$ is $\equiv 0 \pmod{5}$, a contradiction.

So there are no nontrivial solutions to the equation $3x^2 + 4y^2 = 5z^2$. 

Answer to HW9, #7.