Math 109. Instructor: Chow
Homework #2 Hints

**Problem 1:** Prove, using logical argument from the definitions, that
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \]
That is, prove that \( x \in A \cup (B \cap C) \) if and only if \( x \in (A \cup B) \cap (A \cup C). \)

**Hint:** You may use the fact that: ‘\( P \lor (Q \land R) \)’ is logically equivalent to ‘\( (P \lor Q) \land (P \lor R) \)’.

**Solution:**
\[ x \in A \cup (B \cap C) \iff x \in A \lor (x \in B \land x \in C) \iff (x \in A \lor x \in B) \land (x \in A \lor x \in C). \]
Here we used the fact that: ‘\( P \lor (Q \land R) \)’ is logically equivalent to ‘\( (P \lor Q) \land (P \lor R) \)’.

**Problem 2:**
(i) Prove: \( \forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+, y > 3x + 2 \).
(ii) Prove: \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, y > x^2 + 9 \).
(iii) **DIS**Prove: \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y < x^2 \).
(iv) Disprove: \( \forall n \in \mathbb{Z}^+, \exists m \in \mathbb{Z}^+, n \leq m^2 \leq n + 39 \).

**Solution:**
(i) Let \( x \) be an arbitrary positive integer. Then let \( y = 3x + 3 \). Clearly \( y > 3x + 2 \), and \( y \) is also a positive integer. This proves the existence of \( y \).

(ii) Let \( x \) be an arbitrary integer. Then let \( y = x^2 + 10 \). Clearly \( y > x^2 + 9 \), and \( y \) is also an integer. This proves existence of \( y \).

(iii) Want to show, \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y \geq x^2 \).
Let \( x \) be an arbitrary real number, let \( y = x^2 + 1 \). Clearly \( y \in \mathbb{R} \) and \( y \geq x^2 \). This proves the existence of \( y \), and hence disproves the original statement.

(iv) Want to show, \( \exists n \in \mathbb{Z}^+, \forall m \in \mathbb{Z}^+, \text{either } m^2 < n \text{ or } m^2 > n + 39 \).
Let \( n = 1601 \in \mathbb{Z}^+, \forall m \in \mathbb{Z}^+, \text{either } m \leq 40 \text{ or } m \geq 41 \), i.e.\( m^2 \leq 1600 < 1601 = n \) or \( m^2 \geq 1681 > 1640 = n + 39 \). This proves the claim, hence disproves the original statement.

**Problem 3:**
(i) Prove: \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, 2xy = x^3 + 2x^2 \).
(ii) Disprove: \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, 2xy > 1 \).
(iii) Prove: \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (x^2 + 1) y = x^3 + 2x^2 \).
(iv) Prove: \( \forall x \in \mathbb{R} - \mathbb{Q}, \exists y \in \mathbb{R} - \mathbb{Q}, xy = 1 \).
Solution:

(i) Let \( x = 0 \in \mathbb{R} \), then \( \forall y \in \mathbb{R}, \ 2xy = 0 = x^3 + 2x^2 \), this proves the existence of \( x \).

(ii) Want to show, \( \forall x \in \mathbb{R}, \ \exists y \in \mathbb{R}, \ 2xy \leq 1 \).
Let \( x \) be an arbitrary real number, Let \( y = 0 \), then \( 2xy = 0 \leq 1 \). This proves the existence of \( y \) and hence disproves the original statement.

(iii) Let \( x \) be an arbitrary real number, then \( x^2 + 1 \geq 1 > 0 \), in particular \( x^2 + 1 \neq 0 \). Let \( y = \frac{x^3 + 2x^2}{x^2 + 1} \), then \( y \in \mathbb{R} \) and \( y(x^2 + 1) = \frac{x^3 + 2x^2}{x^2 + 1} \). This proves the existence of \( y \).

(iv) Let \( x \) be an arbitrary irrational number, then \( x \neq 0 \in \mathbb{Q} \). Let \( y = \frac{1}{x} \), then \( y \in \mathbb{R} \) and \( xy = 1 \). Need only to show \( y \in \mathbb{Q} - \mathbb{Q} \).
Suppose this is not true, i.e. \( y \in \mathbb{Q} \). Then \( y = \frac{m}{n} \) for some nonzero integers \( m, n \) (notice here \( y \neq 0 \) since \( y = \frac{1}{x} \)), it follows that \( x = \frac{n}{m} \in \mathbb{Q} \), contradicting that \( x \in \mathbb{R} - \mathbb{Q} \), this proves \( y \in \mathbb{R} - \mathbb{Q} \) and hence the statement.

Problem 4: Let \( n \in \mathbb{Z} \).
Let \( P(n) \) be the statement: \( \exists q \in \mathbb{Z} \) such that \( n = 5q + 3 \).
Let \( Q(n) \) be the statement: \( \exists p \in \mathbb{Z} \) such that \( n^2 = 5p + 4 \).

Prove that \( P(n) \) implies \( Q(n) \).

Solution: \( P(n) \) is true \( \Rightarrow \exists q \in \mathbb{Z}, \) such that \( n = 5q + 3 \Rightarrow n^2 = 25q^2 + 30q + 9 = 5(5q^2 + 6q + 1) + 4 \Rightarrow \exists p = 5q^2 + 6q + 1, \) such that \( n^2 = 5p + 4 \Rightarrow Q(n) \) is true.

Problem 5: Let \( f : [a, b] \to \mathbb{R} \) be a differentiable function. The mean value theorem says that there exists \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

Use the mean value theorem to prove that the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \frac{1}{2}x^3 + 3x^2 + 10x \) is strictly increasing. (Recall that strictly increasing means that for any \( a < b, \ f(a) < f(b) \).

Solution: Let \( a, b \) be two arbitrary real numbers such that \( a < b \), mean value theorem implies that, for some \( c \in (a, b) \) \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

Thus \( f(b) > f(a) \) and \( f \) is strictly increasing.

Problem 6: We say that \( \lim_{x \to \infty} f(x) = \infty \) if for any \( M \in \mathbb{R} \) there exists \( N \in \mathbb{R} \) such that if \( x \geq N \), then \( f(x) \geq M \).

The intermediate value theorem says that if \( f : [a, b] \to \mathbb{R} \) is a continuous function and if \( y \) is between \( f(a) \) and \( f(b) \), then there exists \( x \in (a, b) \) such that \( f(x) = y \).
(a) Define, analogously to the above, what it means for \( \lim_{x \to -\infty} f(x) = -\infty \).

(b) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function with \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = \infty \). **Prove:** If \( y \in \mathbb{R} \), then there exists \( x \in \mathbb{R} \) such that \( f(x) = y \).

**Solution:**

(a) For any \( M \in \mathbb{R} \) there exists \( N \in \mathbb{R} \) such that if \( x \leq N \), then \( f(x) \leq M \).

(b) Let \( y \) be an arbitrary real number, let \( N_1, N_2 \) be such that \( f(x) \leq y - 1 \) whenever \( x \leq N_1 \), and \( f(x) \geq y + 1 \) whenever \( x \geq N_2 \), by definition of \( \lim_{x \to \pm \infty} f(x) = \pm \infty \). Claim that \( N_1 < N_2 \). Suppose this is not true, then \( \exists x_0 \), such that \( N_2 \leq x_0 \leq N_1 \), hence \( f(x_0) \leq y - 1 \) and \( f(x_0) \geq y + 1 \), a contradiction. Applying Intermediate Value Theorem to \( a = N_1 \), \( b = N_2 \) and \( y \in (y - 1, y + 1) \subseteq (f(N_1), f(N_2)) \), the conclusion follows, \( \exists x \in (N_1, N_2) \subseteq \mathbb{R} \), such that \( f(x) = y \).

**Problem 7:** Let \( \mathbb{I} \) denote the irrational numbers. Define the function

\[
 f : \mathbb{I} \times \mathbb{I} \to \mathbb{R} \times \mathbb{R}
\]

by

\[
 f(x, y) = (x + y, x^2 + y^2).
\]

(i) Does there exist \((x, y) \in \mathbb{I} \times \mathbb{I}\) such that \( f(x, y) \in \mathbb{I} \times \{1\}\)?

(ii) Does there exist \((x, y) \in \mathbb{I} \times \mathbb{I}\) such that \( f(x, y) \in \mathbb{Q} \times \mathbb{I}\)?

**Solution:**

(i) Yes. Let \( x = y = \sqrt{2} \), then \( f(x, y) = (x + y, x^2 + y^2) = (2, 2) \in \mathbb{I} \times 1 \).

(ii) Yes. Let \( x = \sqrt{2} \), \( y = 1 - \sqrt{2} \), then \( f(x, y) = (x + y, x^2 + y^2) = (1, 5 - 2\sqrt{2}) \in \mathbb{Q} \times \mathbb{I} \).

**Problem 8:** Given \( n \in \mathbb{N} \), let \( \mathbb{N}_n = \{1, 2, \ldots, n\} = \{a \in \mathbb{Z} \mid 1 \leq a \leq n\} \).

Let \( X \) be a finite set. The number of elements in \( X \), called the cardinality of \( X \), is denoted by \( |X| \). We have \( |X| = n \) if and only if there exists a bijection \( f : \mathbb{N}_n \to X \). Answer correctly the following (no need to prove anything).

(i) If \( X \subseteq Y \), then how are \( |X| \) and \( |Y| \) related?

**Ans:** \( |X| \leq |Y| \).

(ii) If \( f : A \to B \) is an injection, then how are \( |A| \) and \( |B| \) related?

**Ans:** \( |A| \leq |B| \).

(iii) If \( g : C \to D \) is a surjection, then how are \( |C| \) and \( |D| \) related?

**Ans:** \( |C| \geq |D| \).

(iv) If \( h : E \to F \) is a bijection, then how are \( |E| \) and \( |F| \) related?

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Ans: $|E| = |F|$.  

Problem 9: Do Problem 18 on p. 118. 

Solution: Let $z \in Z$ be an arbitrary element, by definition of surjection, $\exists y \in Y$, such that $g(y) = z$, and $\exists x \in X$, such that $f(x) = y$. It follows that $g \circ f(x) = g(f(x)) = g(y) = z$, and hence $g \circ f$ is surjective.  

Problem 10: Do Problem 20 on p. 118. 

Solution: 

(i) Let $y \in \overrightarrow{f}(A_1)$. By definition, $\exists x \in A_1$, such that $f(x) = y$. It then follows that $x \in A_1 \subseteq A_2$ and $y = f(x) \in \overrightarrow{f}(A_2)$. Thus $\overrightarrow{f}(A_1) \subseteq \overrightarrow{f}(A_2)$, this completes the proof. 

An counterexample for the converse: Let $f : \mathbb{R} \to \mathbb{R}$ be a constant function, i.e. $f(x) \equiv 0$. Let $A_1 = [0, 1]$, $A_2 = [3, 4]$. Obviously, $\overrightarrow{f}(A_1) = \{0\} = \overrightarrow{f}(A_2)$, thus $\overrightarrow{f}(A_1) \subseteq \overrightarrow{f}(A_2)$. However $A_1 \subseteq A_2$ is not true.

A simple condition on $f$ to make the converse come true is that $f$ is injective: 

Suppose $\overrightarrow{f}(A_1) \subseteq \overrightarrow{f}(A_2)$ and $f$ is injective, let $x \in A_1$, then $f(x) \in \overrightarrow{f}(A_1) \subseteq \overrightarrow{f}(A_2)$. Thus $\exists x_1 \in A_2$ such that $f(x_1) = f(x)$. By injectivity, $x = x_1 \in A_2$, it follows that $A_1 \subseteq A_2$. 

(ii) Let $y \in \overrightarrow{f}(A_1 \cap A_2)$, then $\exists x \in A_1 \cap A_2$, such that $f(x) = y$. It follows from $x \in A_1$ and $x \in A_2$ that $f(x) \in \overrightarrow{f}(A_1)$ and $f(x) \in \overrightarrow{f}(A_2)$, i.e. $y \in \overrightarrow{f}(A_1) \cap \overrightarrow{f}(A_2)$. Therefore, $\overrightarrow{f}(A_1 \cap A_2) \subseteq \overrightarrow{f}(A_1) \cap \overrightarrow{f}(A_2)$. 

Take the example in (i). $\overrightarrow{f}(A_1 \cap A_2) = \overrightarrow{f}(\emptyset) = \emptyset$, while $\overrightarrow{f}(A_1) \cap \overrightarrow{f}(A_2) = \{0\} \cap \{0\} = \{0\}$. In this case, the equality does not hold. 

(iii) $y \in \overrightarrow{f}(A_1 \cup A_2) \iff \exists x \in A_1 \cup A_2, f(x) = y \iff \exists x \in A_1, f(x) = y$ or $\exists x \in A_2, f(x) = y \iff \exists y \in \overrightarrow{f}(A_1)$ or $\overrightarrow{f}(A_2) \iff y \in \overrightarrow{f}(A_1 \cup A_2)$. 

Remark: The original problems #11 and #12 on the inclusion-exclusion principle for 3 sets, have been moved to the 4th HW assignment.