1. Modular arithmetic.

(a) Let \( m \in \mathbb{N} \). Prove: If \( a_1 \equiv a_2 \mod m \) and \( b_1 \equiv b_2 \mod m \), then \( a_1 + b_1 \equiv a_2 + b_2 \mod m \).

**Ans:** Suppose \( a_1 \equiv a_2 \mod m \) and \( b_1 \equiv b_2 \mod m \). Then there exist \( k, \ell \in \mathbb{Z} \) such that
\[
 a_1 - a_2 = km \quad \text{and} \quad b_1 - b_2 = \ell m.
\]
We compute
\[
 (a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2) = km + \ell m = (k + \ell) m.
\]
Thus \( a_1 + b_1 \equiv a_2 + b_2 \mod m \).

(b) Prove, using the definition of congruence and using properties of division, that \( 15a \equiv 15b \mod 39 \) if and only if \( a \equiv b \mod 13 \). **Ans:** Since we are asked to use the definition of congruence and using properties of division, we proceed as follows. By definition,
\[
 15a \equiv 15b \mod 39
\]
if and only if
\[
 39 \text{ divides } 15(a - b)
\]
if and only if
\[
 13 \text{ divides } (a - b).
\]
Since \( \gcd(13, 5) = 1 \), this is true if and only if
\[
 13 \text{ divides } (a - b),
\]
which is true if and only if
\[
 a \equiv b \mod 13
\]

(c) Find all solutions to the equation \( 6x \equiv 21 \mod 15 \). And, how many solutions are there modulo 15? **Ans:** Since 3 divides 6, 21, and 15, this equation is equivalent to
\[
 2x \equiv 7 \mod 5.
\]
Since \( 12 = 7 + 5 \), this is equivalent to
\[
 2x \equiv 12 \mod 5.
\]
Since \( \gcd(2, 5) = 1 \), this is equivalent to
\[
 x \equiv 6 \mod 5.
\]
That is, the set of solutions is \( [6]_5 = [1]_5 = \{1 + 5k \mid k \in \mathbb{Z} \} \). There are \( \gcd(6, 15) = 3 \) solutions modulo 15. For example, 1, 6, 11 is a complete set of solutions to \( 6x \equiv 21 \mod 15 \) modulo 15.