Chapter 1. The Real and Complex Number Systems

1. Prove that no rational numbers $r$ will satisfy $r^2 = 5$.

2. Let $S$ be an ordered set and $E \subseteq S$.
   (a) Is $E$ also an ordered set with the same order in $S$?
   (b) If $E$ is bounded above in $S$, is there always a least upper bound for $E$?

3. Let $S$ be an ordered set that has the least-upper-bound property. Prove that it also has the greatest-lower-bound property.

4. If $F$ is an ordered field, $x \in F$ with $x > 0$, and $n \in \mathbb{N}$. Does there always exist an $y \in F$ such that $y^n = x$?

5. Is it true that any infinite set contains a subset that is infinitely countable? Why?

6. Prove that no order can be defined in the complex field that turns it into an ordered field.

7. Let $E_1$ and $E_2$ be two nonempty subsets of $\mathbb{R}$. Assume both of them are bounded above. Define $a_1 = \sup E_1$ and $a_2 = \sup E_2$. Define
   $$E_1 + E_2 = \{ x_1 + x_2 : x_1 \in E_1 \text{ and } x_2 \in E_2 \}.$$ 
   Prove the following:
   (1) $E_1 + E_2$ is bounded above in $\mathbb{R}$;
   (2) $\sup(E_1 + E_2) = \sup E_1 + \sup E_2$.

8. Define an order of the complex field that has no least-upper-bound property.

9. Let $x, y \in \mathbb{R}^k$. Prove that $|x \cdot y| \leq |x| |y|$. When the equality holds true?

10. Prove for any $x, y \in \mathbb{R}^k$ that $|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$.

11. State and prove the Cauchy–Schwarz inequality for two vectors $u$ and $v$ in $\mathbb{R}^k$.

12. Review all the homework problems.

Chapter 2. Basic Topology

1. Is the set of all rational numbers countable? How about the set of all irrational numbers?
2. Let \( \mathbb{N} \) denote the set of all natural numbers. Let \( \mathcal{P}(\mathbb{N}) \) denote the set of all subsets of \( \mathbb{N} \). Prove that \( \mathcal{P}(\mathbb{N}) \) is uncountable.

3. Is it true that any infinite set must contain infinitely many subsets each of which is infinitely countable?

4. Let \( \mathbb{N} \) denote the set of all natural numbers. Prove the following:
   
   (a) For any \( k \in \mathbb{N} \), the set \( \mathbb{N}^k := \{(n_1, \ldots, n_k) : n_1, \ldots, n_k \in \mathbb{N} \} \) is countable.
   
   (b) The set \( \mathbb{N}^\infty := \{(n_1, n_2, \ldots) : n_j \in \mathbb{N}, j = 1, 2, \ldots \} \) is uncountable.

5. Let \((X, \rho)\) be a metric space. Define \( d : X \times X \to \mathbb{R} \) by \( d(x, y) = \rho(x, y)/(1 + \rho(x, y)) \) for all \( x, y \in X \). Prove that \((X, d)\) is also a metric space.

6. Consider the metric space \( \mathbb{R} \) of all real numbers. For each of the sets \( A = (0, 1), B = [0, 1), C = (-\infty, 1), D = (-\infty, 1] \cup [2, 10], \) and \( F = \mathbb{Q} \cap (0, 1) \):
   
   (a) Determine if it is open, closed, both, or neither;
   
   (b) Find the corresponding set of limit points;
   
   (c) Find the corresponding closure.

7. Can a nonempty, proper subset of a metric space be both open and closed?

8. Let \( E \) be a subset of a metric space \( X \). Is it true that \( E'' = E' \)? Is it true that \( \bar{E} = \bar{E} \)?

9. Let \( X \) be a metric space. Let \( A \) and \( B \) be two subsets of \( X \) such that \( A \subseteq B \). Prove that \( A' \subseteq B' \) and \( \overline{A} \subseteq \overline{B} \).

10. Let \( X \) be a metric space. Let \( A \) and \( B \) be two subsets of \( X \). Is it true that \( (A \cup B)' = A' \cup B' \)? Is it true that \( \overline{A \cup B} = \overline{A} \cup \overline{B} \)? Let all \( A_n \) \( (n = 1, 2, \ldots) \) be subsets of \( X \). Is it true that \( (\bigcup_{n=1}^\infty A_n)' = \bigcup_{n=1}^\infty A_n' \)? Is it true that \( \bigcap_{n=1}^\infty \overline{A_n} = \bigcup_{n=1}^\infty A_n \)?

11. Prove that in a metric space any compact subset is closed and bounded.

12. True or false:
   
   (1) If \( E \) is a subset of a metric space, then \( \overline{E} = \overline{E} \).
   
   (2) If \( E_n \) \( (n = 1, 2, \ldots) \) are open subsets of a metric space, then \( \bigcap_{n=1}^\infty E_n \) is still open.
   
   (3) If \( E \) is a subset of a metric space, then \( E'' = E' \).

13. Prove that the union of finitely many compact subsets of a metric space is still compact.

14. Let \( X \) be a metric space, \( K \) a compact subset of \( X \), and \( x \in X \setminus K \). Prove that there exist two disjoint open subsets \( G_K \) and \( G_x \) of \( X \) such that \( K \subseteq G_K \) and \( x \in G_x \).

15. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function and \( A = \{x \in \mathbb{R} : f(x) > 0\} \). Prove that \( A \) is open in \( \mathbb{R} \).
16. Let $X$ be a metric space. Let $K_1$ and $K_2$ be two disjoint compact subsets of $X$. Prove that there exist two disjoint open subsets $G_1$ and $G_2$ of $X$ such that $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$.

17. Let $X$ be a metric space and $K$ a compact subset of $X$. Prove that any infinite subset of $K$ has a limit point in $K$.

18. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of nonempty, decreasing, compact subsets of a metric space $X$. Prove that $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

19. Prove that a subset of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

20. What is the definition of a perfect set in a metric space?

21. What is the definition of the Cantor set? What major properties does the Cantor set have?

22. What is the definition of connected set in a metric space?

23. Is this true that any nonempty, connected subset of $\mathbb{R}^1$ is an interval?

24. Review all the homework problems.

**Chapter 3. Numerical Sequences and Series**

1. Suppose $a_n > 0 \ (n = 1, 2, \ldots)$ and $\lim_{n \to \infty} a_n = a$. Prove by definition that $\lim_{n \to \infty} \sqrt{a_n} = \sqrt{a}$.

2. Prove by definition the Squeeze Theorem: If $a_n \leq b_n \leq c_n \ (n = 1, 2, \ldots)$, and $\lim_{n \to \infty} a_n = d$ and $\lim_{n \to \infty} c_n = d$ for some $d \in \mathbb{R}$, then $\{b_n\}$ converges and $\lim_{n \to \infty} b_n = d$.

3. If $a_n \to a$ then $(\sum_{k=1}^{n} a_n)/n \to a$. Give an example to show that the converse may not be true.

4. True or false (prove it the answer is true and give a counter example other):
   
   (a) In a metric space, any convergent sequence is a Cauchy sequence. But a Cauchy sequence may not be a convergent sequence.
   
   (b) In a metric space, any Cauchy sequence is bounded.
   
   (c) In a metric space, any Cauchy sequence that contains a convergent subsequence is itself a convergent sequence.

5. Let $a_n \in \mathbb{R} \ (n = 1, 2, \ldots)$. Prove that $\{a_n\}$ converges if and only if $\lim \inf_{n \to \infty} a_n = \lim \sup_{n \to \infty} a_n$.

6. Prove by definition that $n^k/2^n \to 0$ as $n \to \infty$ for any $k \in \mathbb{R}$.
7. Describe the definition of the partial sums, convergence, and absolute convergence of a series.

8. Let \( a_n > 0 \) (\( n = 1, 2, \ldots \)) and \( S_n = \sum_{k=1}^{n} a_k \) (\( n = 1, 2, \ldots \)). Prove that the series \( \sum_{n=1}^{\infty} a_n \) converges if and only if the sequence \( \{S_n\} \) is bounded.

9. Why the number \( e \) is not rational?

10. Prove the Root Test for the convergence or divergence of a series \( \sum_{n=1}^{\infty} a_n \).

11. Prove the following variation of the Ratio Test for positive series: Let \( a_n > 0 \) (\( n = 1, 2, \ldots \)). Suppose \( \lim_{n \to \infty} a_{n+1}/a_n = \alpha \). Then the series \( \sum_{n=1}^{\infty} a_n \) converges if \( \alpha < 1 \) and diverges if \( \alpha > 1 \).

12. Determine if the following series converge or not:
   \( \sum_{n=1}^{\infty} \frac{n+8}{3n^2+10} \), \( \sum_{n=1}^{\infty} \frac{n+8}{3n^2+10} \), \( \sum_{n=1}^{\infty} \frac{(-1)^n(n+8)}{3n^2+10} \).

13. Find the radius of convergence for the following power series:
   \( \sum_{n=0}^{\infty} n^2 z^n \), \( \sum_{n=0}^{\infty} (-1)^n (10 + e^n) z^n \), \( \sum_{n=0}^{\infty} \frac{\log(1+n)}{n^2} z^n \).

14. Prove the Root Test.

15. Prove the Summation by Parts formula.

16. Prove Theorem 3.42.

17. If \( \sum_{n=0}^{\infty} a_n = A \) converges absolutely and \( \sum_{n=0}^{\infty} b_n = B \) converges, then \( \sum_{n=0}^{\infty} c_n = AB \) converges, where \( c_n = \sum_{k=0}^{n} a_kb_{n-k} \).

18. Suppose \( \sum_{n=1}^{\infty} a_n \) converges. Let \( \{a_{n_k}\} \) be a subsequence of \( \{a_n\} \). Does \( \sum_{k=1}^{\infty} a_{n_k} \) converge?

19. What is a rearrangement of a given series? Does it always converge if the original one converges?

20. Review all the homework problems.

---

Chapter 4. Continuity

1. Prove that the limit \( \lim_{x \to p} f(x) \) is unique if it exists.

2. True or false: A continuous function always maps an open set to an open set. Justify your answer.

3. True or false: A uniformly continuous function is always bounded. Justify your answer.
4. True or false: If $X$ is a metric space equipped with the discrete metric $d$ (i.e., $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$) and $Y$ is any metric space, then any mapping $f : X \to Y$ is uniformly continuous.

5. Let $X$ and $Y$ be two metric spaces. Prove that $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in $X$ provided that $V \subseteq Y$ is open in $Y$.

6. Let $X$ be a compact metric space and $Y$ a metric space. Let $f : X \to Y$ be continuous.
   (1) Prove that $f$ is bounded.
   (2) Prove that $f$ is uniformly continuous.
   (3) Give counter examples to show that $f$ may not be bounded or uniformly continuous if $X$ is not compact.

7. Prove that a continuous mapping from a metric space to another metric space preserves the connectedness of a subset.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonic function. Prove that for any $x \in \mathbb{R}$ both $f(x-)$ and $f(x+)$ exist.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonic function. Prove that the set of discontinuities of $f$ is at most countable.

10. Review all the homework problems.