Chapter 1. The Real and Complex Number Systems

1. Prove that no rational numbers \( r \) will satisfy \( r^2 = 5 \).

2. Let \( S \) be an ordered set and \( E \subseteq S \).
   (a) Is \( E \) also an ordered set with the same order in \( S \)?
   (b) If \( E \) is bounded above in \( S \), is there always a least upper bound for \( E \)?

3. Let \( S \) be an ordered set that has the least-upper-bound property. Prove that it also has the greatest-lower-bound property.

4. If \( F \) is an ordered field, \( x \in F \) with \( x > 0 \), and \( n \in \mathbb{N} \). Does there always exist an \( y \in F \) such that \( y^n = x \)?

5. Prove that no order can be defined in the complex field that turns it into an ordered field.

6. Define an order of the complex field that has no least-upper-bound property.

7. Let \( x, y \in \mathbb{R}^k \). Prove that \( |x \cdot y| \leq |x| |y| \). When the equality holds true?

8. Prove for any \( x, y \in \mathbb{R}^k \) that \( |x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2 \).

Chapter 2. Basic Topology

1. Is the set of all rational numbers countable? How about the set of all irrational numbers?

2. Let \( \mathbb{N} \) denote the set of all natural numbers. Let \( \mathcal{P}(\mathbb{N}) \) denote the set of all subsets of \( \mathbb{N} \). Prove that \( \mathcal{P}(\mathbb{N}) \) is uncountable.

3. Is it true that any infinite set must contain infinitely many subsets each of which is infinitely countable?

4. Let \( \mathbb{N} \) denote the set of all natural numbers. Prove the following:
   (a) For any \( k \in \mathbb{N} \), the set \( \mathbb{N}^k := \{(n_1, \ldots, n_k) : n_1, \ldots, n_k \in \mathbb{N}\} \) is countable.
   (b) The set \( \mathbb{N}^\infty := \{(n_1, n_2, \ldots) : n_j \in \mathbb{N}, j = 1, 2, \ldots \} \) is uncountable.
5. Let \((X, \rho)\) be a metric space. Define \(d : X \times X \to \mathbb{R}\) by \(d(x, y) = \rho(x, y)/(1 + \rho(x, y))\) for all \(x, y \in X\). Prove that \((X, d)\) is also a metric space.

6. Consider the metric space \(\mathbb{R}\) of all real numbers. For each of the sets \(A = (0, 1), \ B = [0, 1), \ C = (-\infty, 1), \ D = (-\infty, 1], \ E = (-\infty, 1) \cup [2, 10], \) and \(F = \mathbb{Q} \cap (0, 1)\):
   (a) Determine if it is open, closed, both, or neither;
   (b) Find the corresponding set of limit points;
   (c) Find the corresponding closure.

7. Can a nonempty, proper subset of a metric space be both open and closed?

8. Let \(E\) be a subset of a metric space \(X\). Is it true that \(E'' = E'\)? Is it true that \(\bar{E} = \bar{E}\)?

9. Let \(X\) be a metric space. Let \(A\) and \(B\) be two subsets of \(X\) such that \(A \subseteq B\). Prove that \(A' \subseteq B'\) and \(\overline{A} \subseteq \overline{B}\).

10. Let \(X\) be a metric space. Let \(A\) and \(B\) be two subsets of \(X\). Is it true that \((A \cup B)' = A' \cup B'\)? Is it true that \(\overline{A \cup B} = \overline{A} \cup \overline{B}\)? Let all \(A_n (n = 1, 2, \ldots)\) be subsets of \(X\). Is it true that \((\bigcup_{n=1}^{\infty} A_n)' = \bigcup_{n=1}^{\infty} A_n'\)? Is it true that \(\bigcup_{n=1}^{\infty} \overline{A_n} = \overline{\bigcup_{n=1}^{\infty} A_n}\)?

11. Prove that in a metric space any compact subset is closed and bounded.

12. Let \(X\) be a metric space, \(K\) a compact subset of \(X\), and \(x \in X \setminus K\). Prove that there exist two disjoint open subsets \(G_K\) and \(G_x\) of \(X\) such that \(K \subseteq G_K\) and \(x \in G_x\).

13. Let \(f : \mathbb{R} \to \mathbb{R}\) be a continuous function and \(A = \{x \in \mathbb{R} : f(x) > 0\}\). Prove that \(A\) is open in \(\mathbb{R}\).

14. Let \(X\) be a metric space. Let \(K_1\) and \(K_2\) be two disjoint compact subsets of \(X\). Prove that there exist two disjoint open subsets \(G_1\) and \(G_2\) of \(X\) such that \(K_1 \subseteq G_1\) and \(K_2 \subseteq G_2\).

15. Let \(X\) be a metric space and \(K\) a compact subset of \(X\). Prove that any infinite subset of \(K\) has a limit point in \(K\).

16. Let \(\{K_n\}_{n=1}^{\infty}\) be a sequence of nonempty, decreasing, compact subsets of a metric space \(X\). Prove that \(\cap_{n=1}^{\infty} K_n \neq \emptyset\).

17. Prove that a subset of \(\mathbb{R}^k\) is compact if and only if it is closed and bounded.